Workshop on Topological Solitons

Kink-Meson Inelastic Scattering

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[1] Liu, Hui, Jarah Evslin, and Baiyang Zhang. "Meson production from kink-meson scattering." *Physical Review D* 107.2 (2023): 025012. [arXiv:2211.01794]
[2] Evslin, J., Liu, H. (Anti-)Stokes scattering on kinks. *J. High Energ. Phys.* **2023**, 95 (2023). https://doi.org/10.1007/JHEP03(2023)095 [arXiv:2301.04099]
[3] J. Evslin and H. Liu, "Quantum Reflective Kinks," [arXiv:2210.12725].

Recap of the kinks (refer to Manton&Sutcliffe's book *Topological Solitons*, chapter 5)

In one space dimension, consider the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$$

We call the scalar field ϕ the meson field.
The Euler-Lagrange equation is

$$\partial_\mu\partial^\mu\phi + \frac{dU}{d\phi} = 0$$

The presence of topological solitons relies on the existence of multiple vacua.

Solutions that interpolate between different vacua are generally termed kinks, a nomenclature inspired by the shape the scalar field takes when plotted against x.

Example 1. ϕ^4 kinks

Consider a potential of this form

$$U(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

The full Lagrangian density is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}(\phi^2 - v^2)^2$$

The Euler-Lagrange equation is

$$\partial_\mu\phi\partial^\mu\phi + \lambda(\phi^2 - v^2)\phi = 0$$

The ϕ^4 kink solution

$$\phi(x) = v\tanh[\sqrt{\frac{\lambda}{2}}(x - x_0)]$$

The energy density of the kink is

$$\mathscr{E} = \frac{1}{2}\phi'^2 + \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 = \frac{\lambda}{2}v^4\,\mathrm{sech}^4(\sqrt{\frac{\lambda}{2}}v(x - x_0))$$

The energy (also the rest mass) of the ϕ^4 kink is

$$E = \int_{-\infty}^{\infty}\mathscr{E}dx = \frac{2\sqrt{2}}{3}v^3\sqrt{\lambda}$$

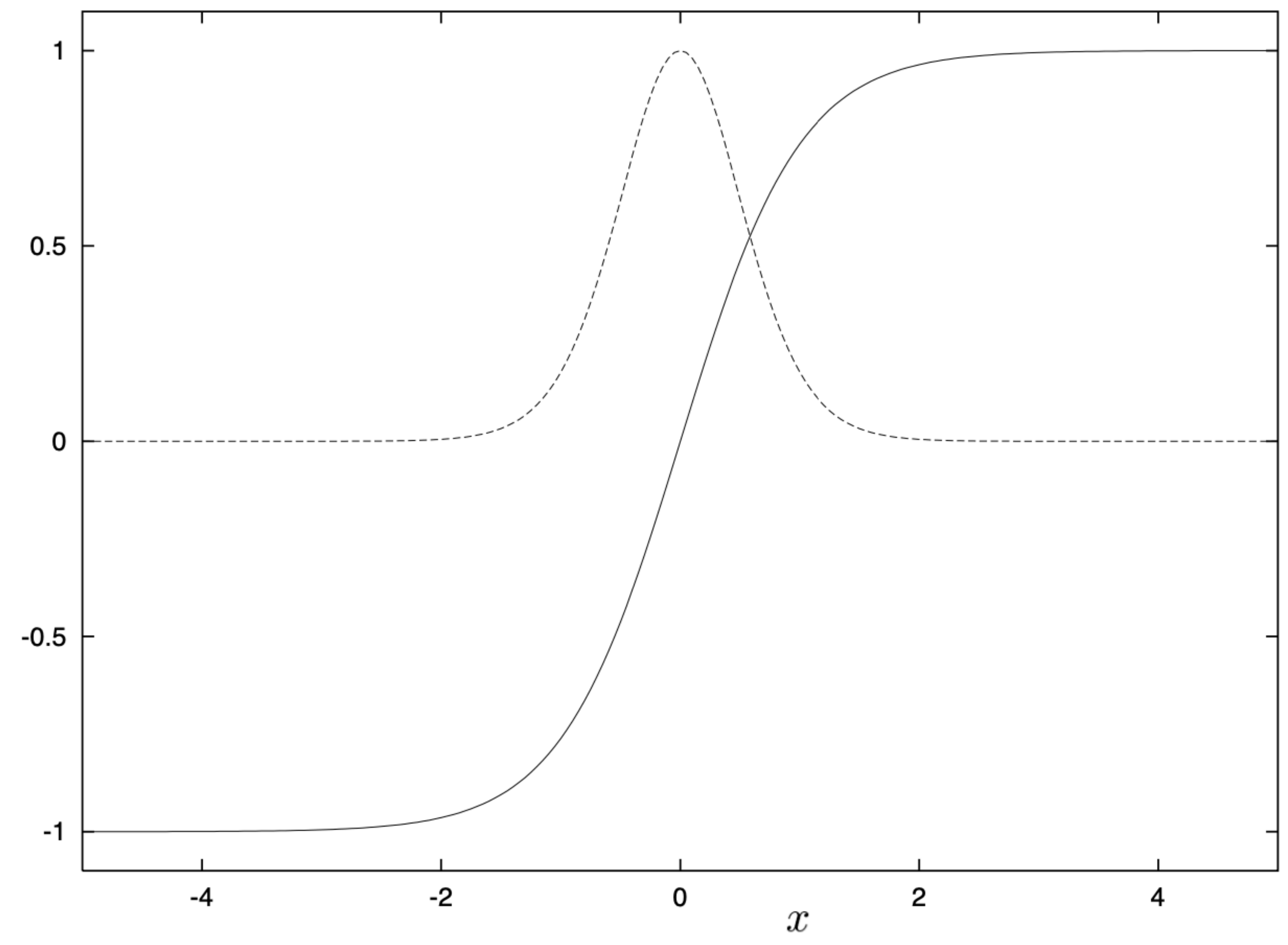


Fig. 5.1. The field $\phi(x)$ of the ϕ^4 kink (solid curve) and its energy density (dashed curve).

from the book *Topological Solitons*

Example 2. Sine-Gordon Solitons

Consider the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{\lambda}\left(1 - \cos(\sqrt{\lambda}\phi)\right)$$

The potential is periodic with degenerate minima at $\phi = \left(2\pi/\sqrt{\lambda}\right)N \equiv Nv$

The Euler-Lagrange equation is

$$\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\phi) = 0$$

The sine-Gordon kink solution interpolating between the vacua at Nv and $(N+1)v$ is

$$\phi(x) = Nv + \frac{2v}{\pi}\tan^{-1}\left[e^{m\left(x - x_0\right)}\right]$$

The energy (also the rest mass) of the sine-Gordon kink is

$$E = \frac{8m}{\lambda}$$

Linearized Soliton Perturbation Theory

- A new formalism developed in recent years by Evslin and Guo [1908.06710, 2012.04912]
- A simple approach to dealing with the problems related to a single kink (+ any number of mesons and impurities), e.g., kink form factors, quantum corrections to the mass of the kink, the quantum state of an excited kink, etc.
- One-kink sector: the Fock space that contains one kink and any number of mesons
- Vacuum sector: the Fock space of mesons without kinks
- Approach: The one-kink sector corresponds intuitively to classical field configurations that closely resemble the classical kink solution $\phi(x) = f(x)$.

- We expect to obtain states in the one-kink sector by handling the difference $\eta(x) = \phi(x) - f(x)$ perturbatively.

- In QFT, the transformation of the corresponding Schrodinger picture fields $\phi(x) \rightarrow \eta(x)$ is achieved with the unitary displacement operator

$$\mathcal{D}_f = \text{Exp}\left[-i\int dx f(x)\pi(x)\right], \qquad \mathcal{D}_f^\dagger \phi(x)\mathcal{D}_f = \phi(x) + f(x).$$

- Seen as an active transformation, D_f adds a kink to a vacuum sector state $|\Omega\rangle$

$$\langle\Omega|\phi(x)|\Omega\rangle = 0 \Rightarrow \langle\Omega|D_f^\dagger\phi(x)D_f|\Omega\rangle = f(x)$$

- We instead choose to apply it as a passive transformation, renaming the coordinate system of the Hilbert space and transforming the operators that act on them:

  We define the *kink frame* as the coordinate system on the Hilbert space in which the ket $|\psi\rangle$ represents the state $D_f|\psi\rangle$ as defined in the usual, *defining frame.*

- What have we gained?

  In the kink frame, states don't have the nonperturbative operator D_f

- The price is that we must transform the operators
- Kink Hamiltonian H'

$$H' = D_f^\dagger H D_f$$

- Kink momentum P'

$$P' = D_f^\dagger P D_f$$

The full quantum-corrected stationary kink states are the eigenvectors of H'. These can be constructed in perturbation theory!

$$H = \int dx : \mathcal{H}(x) :_a, \quad \mathcal{H}(x) = \frac{\pi^2(x)}{2} + \frac{\left(\partial_x \phi(x)\right)^2}{2} + \frac{V(\sqrt{\lambda}\phi(x))}{\lambda}.$$

We may expand H' into terms H'_n which have n factors of $\phi(x)$ and $\pi(x)$ when normal-ordered.

$$H'_0 = Q_0, \qquad H'_1 = 0, \qquad H'_{n>2} = \lambda^{\frac{n}{2}-1} \int dx \frac{V^{(n)}(\sqrt{\lambda} f(x))}{n!} : \phi^n(x) :_a .$$

The kink's normal modes $\mathfrak{g}(x)$ are the constant frequency solutions of the classical equations of motion corresponding to H_2'

$$V^{(2)}(\sqrt{\lambda}f(x))\mathfrak{g}(x)=\omega^2\mathfrak{g}(x)+\mathfrak{g}''(x),\qquad \phi(x,t)=e^{-i\omega t}\mathfrak{g}(x).$$

There are three kinds of normal mode. The first is the real zero-mode $\mathfrak{g}_B(x)$ which has zero frequency $\omega_B=0$. Next, there are complex continuum modes $\mathfrak{g}_k(x)$ with frequencies $\omega_k=\sqrt{m^2+k^2}$. Finally, some kinks enjoy discrete, real shape modes $\mathfrak{g}_S(x)$ with $0<\omega_S<m$. We will fix their normalization via the conditions $\mathfrak{g}_k^*=\mathfrak{g}_{-k}$ and

$$\int dx|\mathfrak{g}_B(x)|^2=1,\ \int dx\mathfrak{g}_{k_1}(x)\mathfrak{g}_{k_2}^*(x)=2\pi\delta(k_1-k_2),\ \int dx\mathfrak{g}_{S_1}(x)\mathfrak{g}_{S_2}^*(x)=\delta_{S_1S_2}.$$

We decompose the fields into creation/annihilation operators for normal modes

$$\phi(x) = \phi_0 \mathfrak{g}_B(x) + \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\left(B_k^{\ddagger} + \frac{B_{-k}}{2\omega_k}\right)\mathfrak{g}_k(x), \qquad B_k^{\ddagger} = \frac{B_k^{\dagger}}{(2\omega_k)}, \qquad B_S^{\ddagger} = \frac{B_S^{\dagger}}{(2\omega_S)}$$

$$\pi(x) = \pi_0 \mathfrak{g}_B(x) + i \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\left(\omega_k B_k^{\ddagger} - \frac{B_{-k}}{2}\right)\mathfrak{g}_k(x), \qquad B_{-S} = B_S, \qquad \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} = \int \frac{dk}{2\pi} + \sum_S .$$

$$\left[\phi_0, \pi_0\right] = i, \quad \left[B_{S_1}, B_{S_2}^{\ddagger}\right] = \delta_{S_1 S_2}, \quad \left[B_{k_1}, B_{k_2}^{\ddagger}\right] = 2\pi\delta\left(k_1 - k_2\right)$$

The miracle: The leading order kink Hamiltonian is a sum of QHOs for the normal modes + a free QM particle Hamiltonian for the center of mass.

$$H_2' = Q_1 + H_{\text{free}}, \quad H_{\text{free}} = \frac{\pi_0^2}{2} + \sum_S \omega_S B_S^{\ddagger} B_S + \int \frac{dk}{2\pi}\omega_k B_k^{\ddagger} B_k.$$

- The spectrum of H_2' is constructed from the harmonic oscillator spectra
- $|0\rangle_0$ is the vacuum of H_2'

$$\pi_0|0\rangle_0 = \boldsymbol{B}_k|0\rangle_0 = \boldsymbol{B}_S|0\rangle_0 = 0.$$

$$|k\rangle_0 = \boldsymbol{B}_k^\ddagger|0\rangle_0, \qquad |kk'\rangle_0 = \boldsymbol{B}_k^\ddagger \boldsymbol{B}_{k'}^\ddagger|0\rangle_0, \qquad |Sk\rangle_0 = \boldsymbol{B}_S^\ddagger \boldsymbol{B}_k^\ddagger|0\rangle_0.$$

- Starting with these, the full kink Hamiltonian H' spectrum can be obtained using the standard perturbation theory using the interactions H_n'

Remarks on Linearized Soliton Perturbation Theory

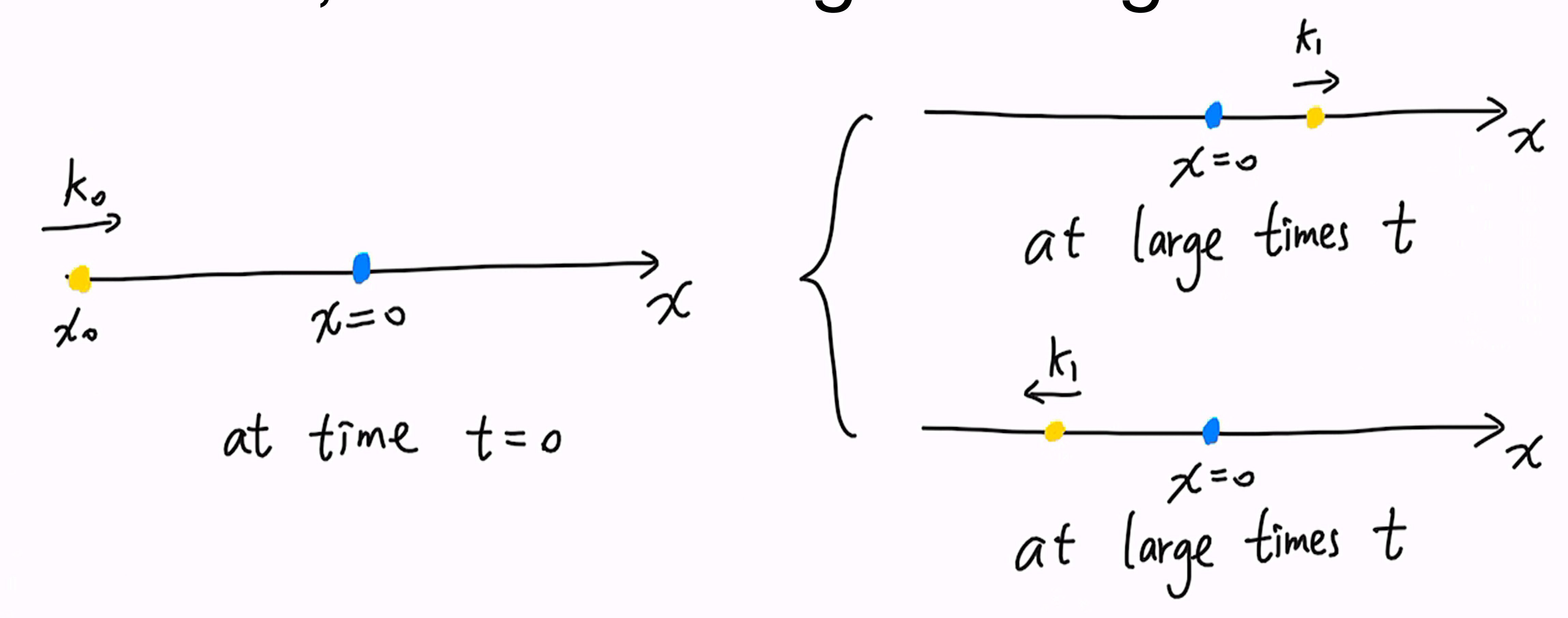
- A base point must be chosen in the moduli space of classical solutions
- So we will lose manifest translation invariance
- However, we can define the reduced inner product [1] for kink states to restore the translation invariance
- Less powerful when dealing with problems with more than one kink
- So in the rest of the talk, we are going to treat the scattering of one kink and one meson

[1] Evslin, J., Liu, H. A reduced inner product for kink states. *J. High Energ. Phys.* **2023**, 70 (2023). https://doi.org/10.1007/JHEP03(2023)070 [arXiv:2212.10344]

Kink-Meson Scattering: At order λ^0

- Only H_0', H_1', H_2' need to be considered at this order.

- When a meson hits a kink, it can either go through the kink or be reflected by the kink.



- According to our calculation, the probabilities of the meson being reflected by the kink or going through the kink correspond to the reflection coefficient and the transmission coefficient when a particle scatters through a symmetric barrier or well in QM.

Kink-Meson Scattering: At order $\sqrt{\lambda}$

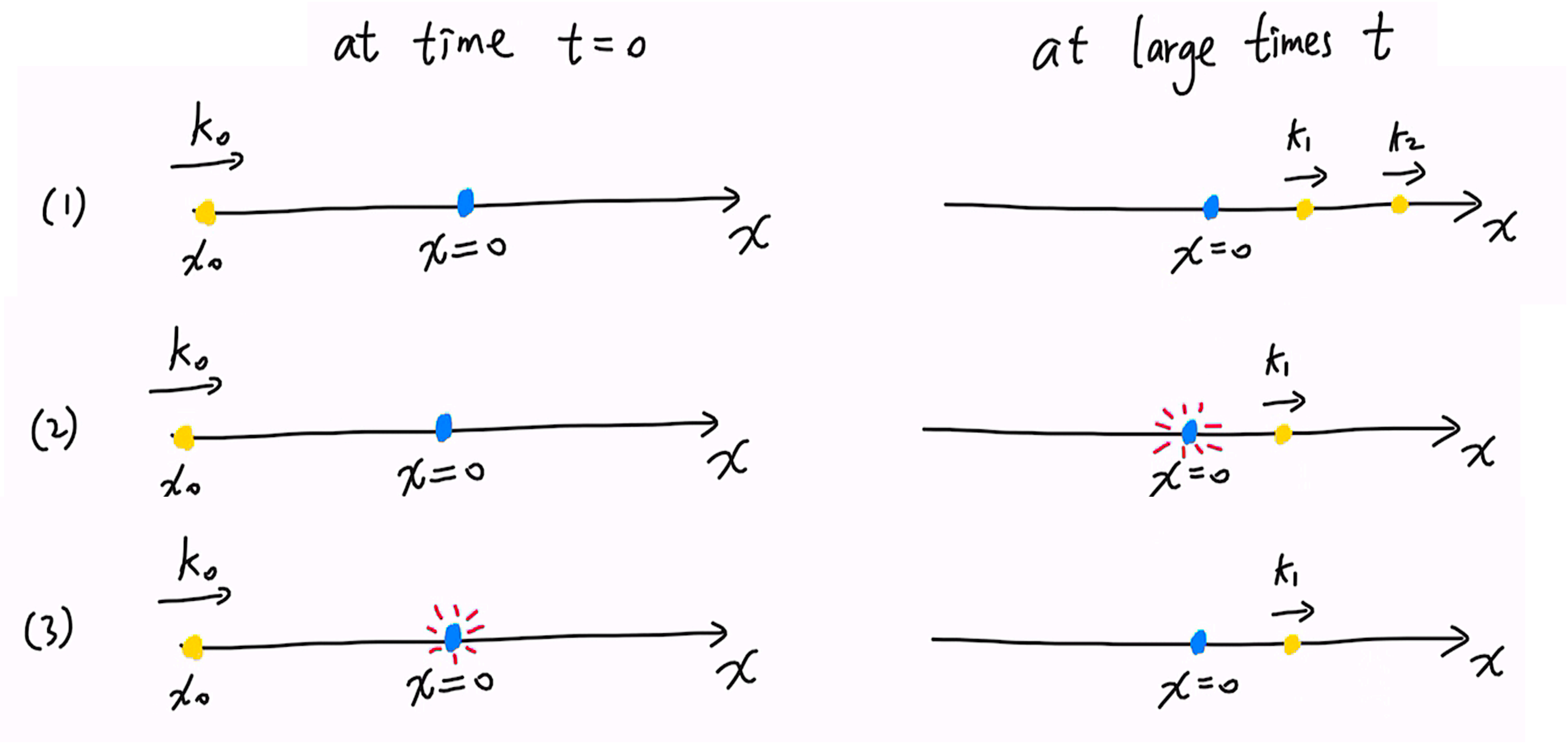
- Now H_3' needs to be considered

$$H_3' = \sqrt{\lambda} \int dx \frac{V^{(3)}(\sqrt{\lambda} f(x))}{3!} : \phi^3(x) :_a$$

and it leads us to the more interesting inelastic scattering situations.

1. meson splitting (at tree level, one meson scatters off a kink and splits into two mesons, we call this process “meson multiplication”)
2. excite the kink’s shape modes and Raman spectroscopy can be performed (at tree level, one meson scatters off a kink and then kink’s shape modes are excited, we call this process “Stokes scattering”)
3. de-excite the shape modes (at tree level, one meson scatters off an excited kink and then the kink is de-excited, we call this process “anti-Stokes scattering”)

meson (yellow) kink (blue) outgoing mesons can also go backwards for all the processes

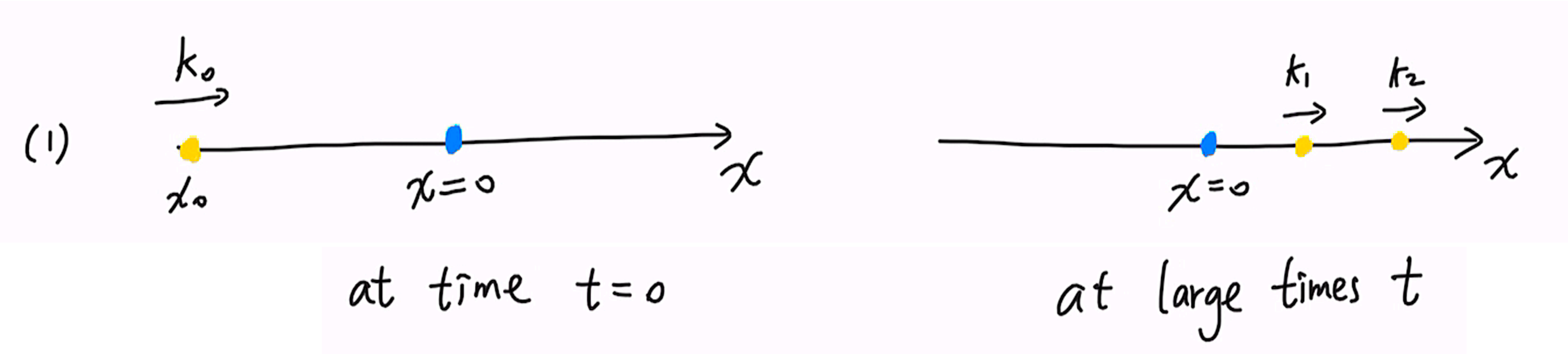


meson multiplication

Stokes scattering

anti-Stokes scattering

Meson Multiplication



Wave packet description of the initial state

$$\Phi(x) = \mathrm{Exp}\left[-\frac{(x-x_0)^2}{4\sigma^2} + ixk_0\right], \quad x_0 \ll -\frac{1}{m}, \quad \frac{1}{k_0}, \frac{1}{m} \ll \sigma \ll |x_0| .$$

$$|\Phi\rangle_0 = \int dx \Phi(x) |x\rangle_0 = \int \frac{dk}{2\pi} \alpha_k |k\rangle_0 , \quad |k\rangle_0 = B_k^{\ddagger} |0\rangle_0, \quad |x\rangle_0 = \int \frac{dk}{2\pi} \mathfrak{g}_k(x) |k\rangle_0 .$$

Steps

- 1. Evolve the wave packet with H', this is at order $\sqrt{\lambda}$
- 2. Project the state at large times t to a two-meson state
- 3. Integrate over the momenta of the final mesons to get the total probability of meson multiplication.

Remember the leading order interaction is $H_3' = \sqrt{\lambda} \int dx \frac{V^{(3)}(\sqrt{\lambda} f(x))}{3!} : \phi^3(x) :_a$

At order $\sqrt{\lambda}$, the only term in H_3' that contributes to meson multiplication is

$$\boldsymbol{H_I} = \frac{\sqrt{\lambda}}{4} \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} V_{-k_1 k_2 k_3} \frac{1}{\omega_{k_1}} B^{\ddagger}_{k_2} B^{\ddagger}_{k_3} B_{k_1}$$

$$V_{-k_1 k_2 k_3} = \int dx V^{(3)}(\sqrt{\lambda} f(x)) \mathfrak{g}_{-k_1}(x) \mathfrak{g}_{k_2}(x) \mathfrak{g}_{k_3}(x).$$

$\boldsymbol{H_I}$
converts a one-meson state into a two-meson state

$$\boldsymbol{H_I} |k_1\rangle_0 = \frac{\sqrt{\lambda}}{4\omega_{k_1}} \int \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} V_{-k_1 k_2 k_3} |k_2 k_3\rangle_0 .$$

Meson Multiplication in the case of Sine-Gordon Soliton

$$V(\sqrt{\lambda}\phi(x)) = m^2\left(1 - \cos(\sqrt{\lambda}\phi(x)\right)$$

$$V_{k_1k_2k_3} = \frac{\pi i\sqrt{\lambda}}{4}\text{sign}(k_1k_2k_3)\text{sech}\left(\frac{\pi(k_1+k_2+k_3)}{2m}\right) \times\frac{(\omega_{k_1}+\omega_{k_2}+\omega_{k_3})(\omega_{k_1}+\omega_{k_2}-\omega_{k_3})(\omega_{k_1}+\omega_{k_3}-\omega_{k_2})(\omega_{k_2}+\omega_{k_3}-\omega_{k_1})}{\omega_{k_1}\omega_{k_2}\omega_{k_3}}.$$

$$\tilde{V}_{-k_Ik_2k_3} = 0$$

- This implies that the differential probability (and so total probability) vanishes.
- This is to be expected, the integrability of the sine-Gordon model implies that the number of mesons is conserved and so meson multiplication does not occur.

Let’s see the nontrivial case of the ϕ^4 kink

The analytical results can be found in our papers

Here I just show the plots with numerical results

The probability of Meson Multiplication kink + meson —> kink + 2 mesons

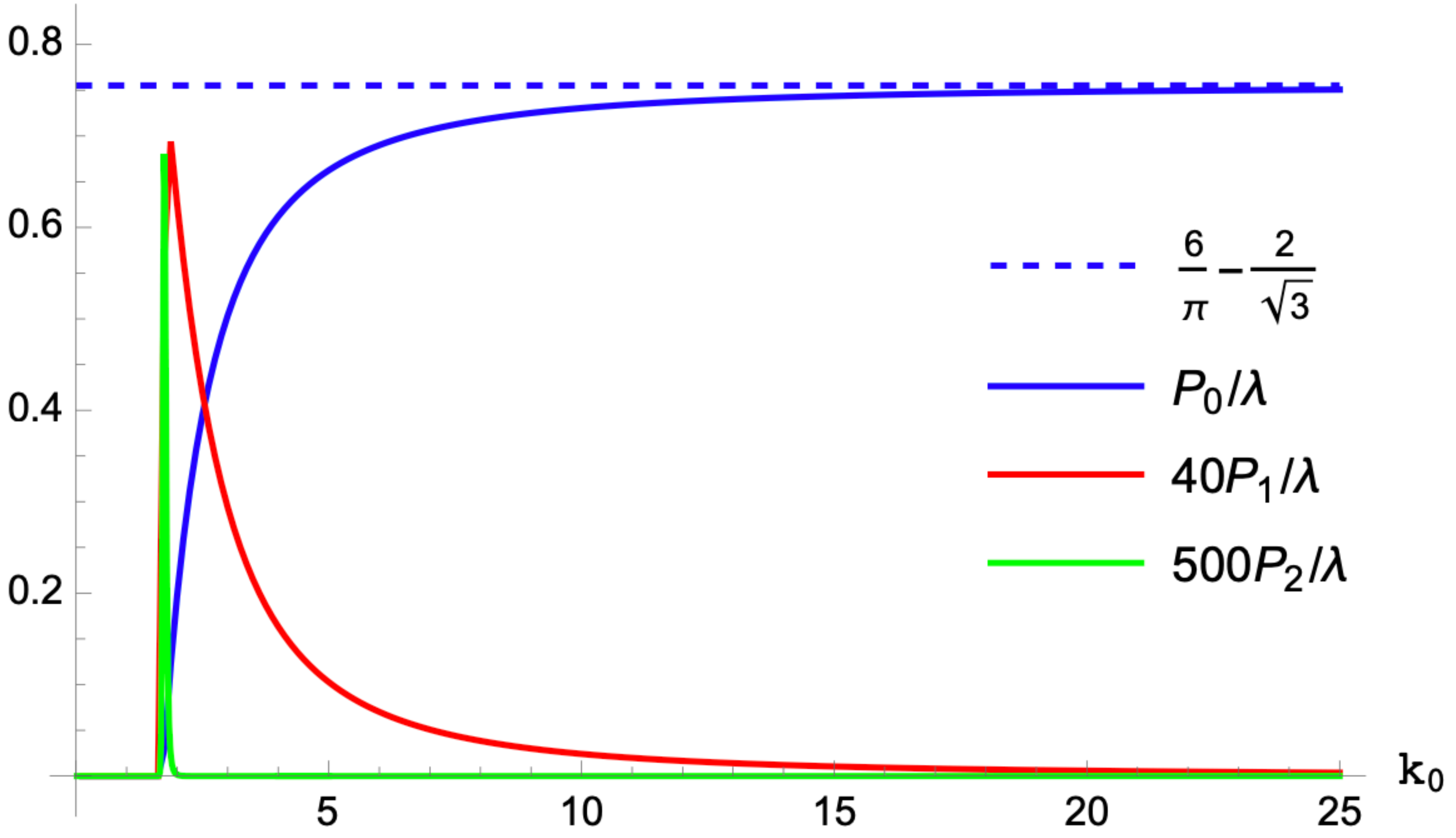


图 3-3 n 个出射介子的动量为负的概率 P_n。为使它们在图中可见，我们对它们按 $1/\lambda$ 以及图例中给出的其他比例因子缩放。

Figure 3-3 The probability P_n that n of the momenta of the outgoing mesons are negative. These are all rescaled by $1/\lambda$ and also by other factors, given in the legend, to make them visible in the plot.

The probability of Stokes Scattering

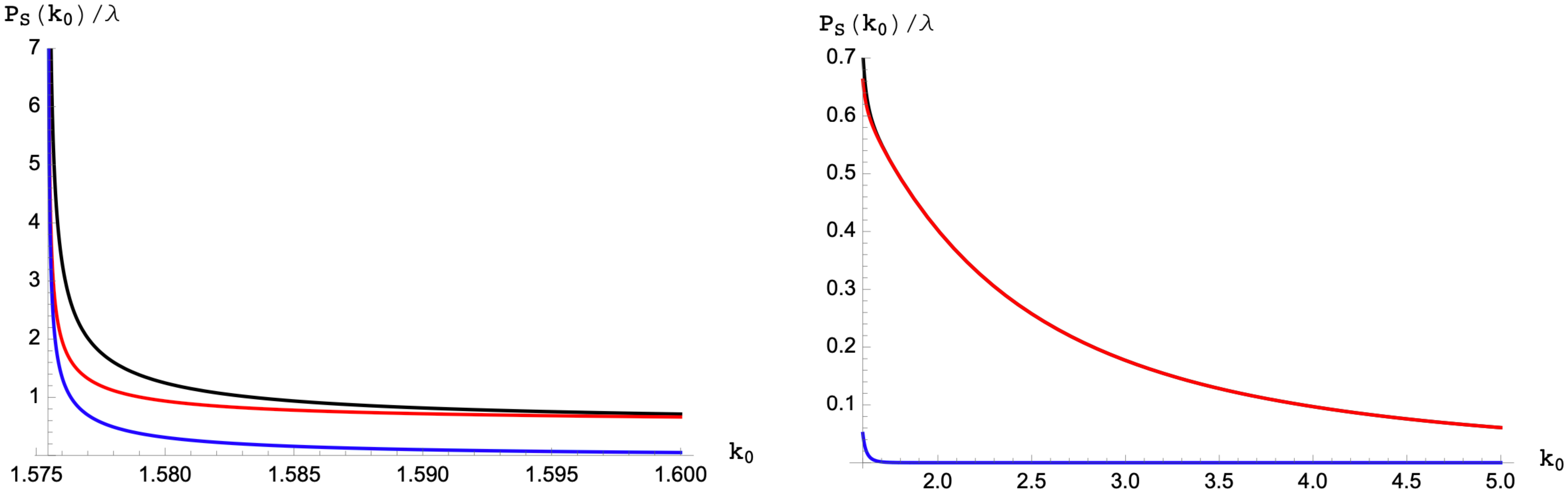


图 4-1 向前斯托克斯散射（红色）、向后斯托克斯散射（蓝色）和总的斯托克斯散射（黑色）概率 $P_S(k_0)$，$m = 1$。

Figure 4-1 The forward (red), backward (blue) and total (black) probabilities $P_S(k_0)$ of Stokes scattering, with $m = 1$.

Anti-Stokes Scattering

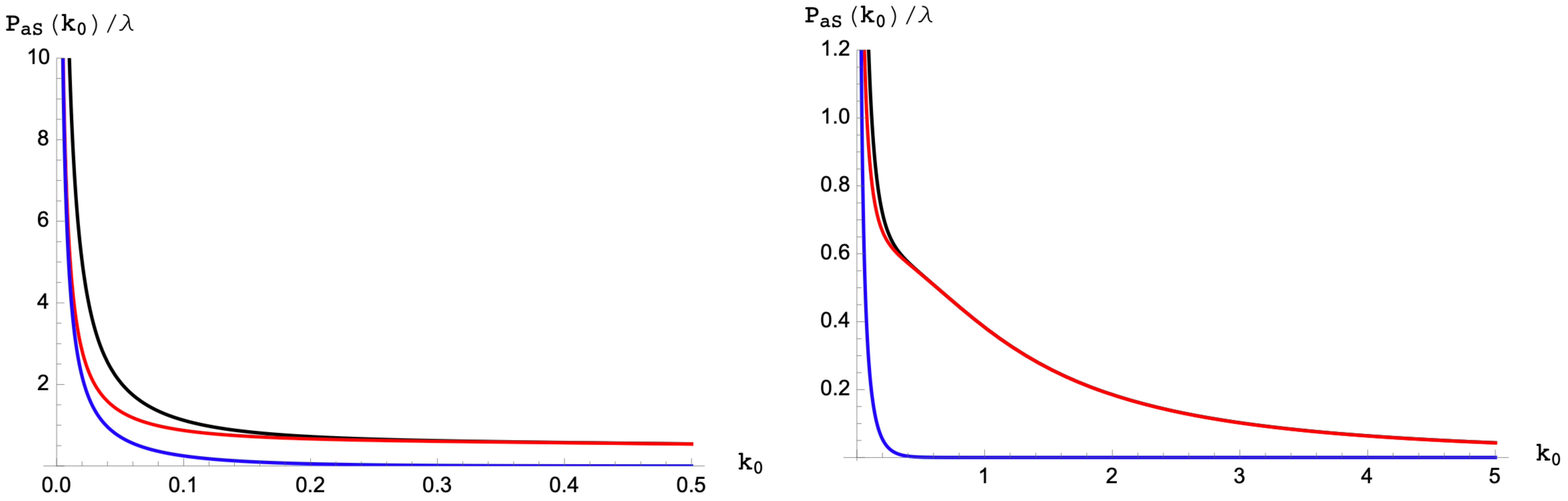


图 4-2 向前反斯托克斯散射（红色）、向后反斯托克斯散射（蓝色）和总的反斯托克斯散射（黑色）概率 $P_{aS}(k_0)$，$m = 1$。

Figure 4-2 The forward (red), backward (blue) and total (black) probabilities $P_{aS}(k_0)$ of anti-Stokes scattering, with $m = 1$.

All the three kink-meson inelastic scattering processes at leading order

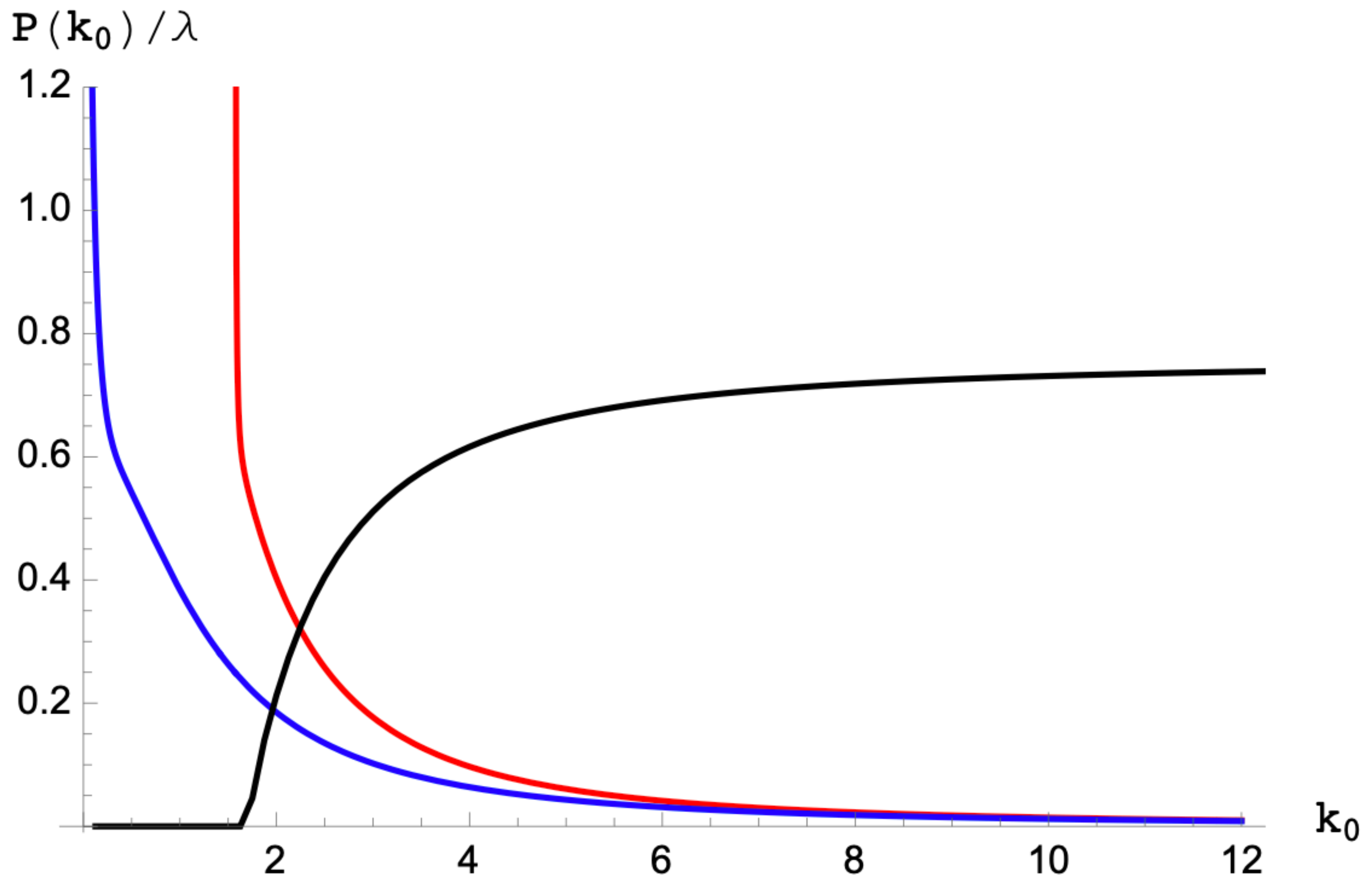


图 4-3 第 3 章中介子倍增（黑色），斯托克斯散射（红色）和反斯托克斯散射（蓝色）的总概率比较。

Figure 4-3 The total probability of meson multiplication (black) from Chapter 3, plotted against the probability of Stokes (red) and anti-Stokes (blue) scattering.

Meson multiplication dominates at high energies, while (anti-)Stokes scattering probabilities become very large at low energies.

Remarks

- At order $\sqrt{\lambda}$, the inelastic scattering of a quantum kink and fundamental meson is now fully understood.
- First, in meson multiplication, the meson may split in two.
- Second, if the kink is in its ground state, then when the meson interacts it may excite a shape mode.
- Finally, if a shape mode is initially excited, then when the meson interacts it may de-excite the shape mode.

What’s next?

- We would like to study kink-meson elastic scattering
- This process is of order λ
- Here we hope to discover an unstable resonance corresponding to the twice-excited shape mode
- Also higher order corrections of the initial and final states must be considered.

Thank you for your listening!

$$P_{\mathrm{tot}} = \frac{1}{2}\int dk_2 dk_3 P_{\mathrm{diff}}(k_2, k_3) = \frac{\lambda\sigma\omega_{k_0}}{16\sqrt{2}\pi^{3/2}}\int dk_2 dk_3 \frac{\left|\tilde{V}_{-k_I k_2 k_3}\right|^2}{\omega_{k_2}\omega_{k_3}k_I^2} e^{-2\sigma^2\left(k_I - k_0\right)^2}.$$