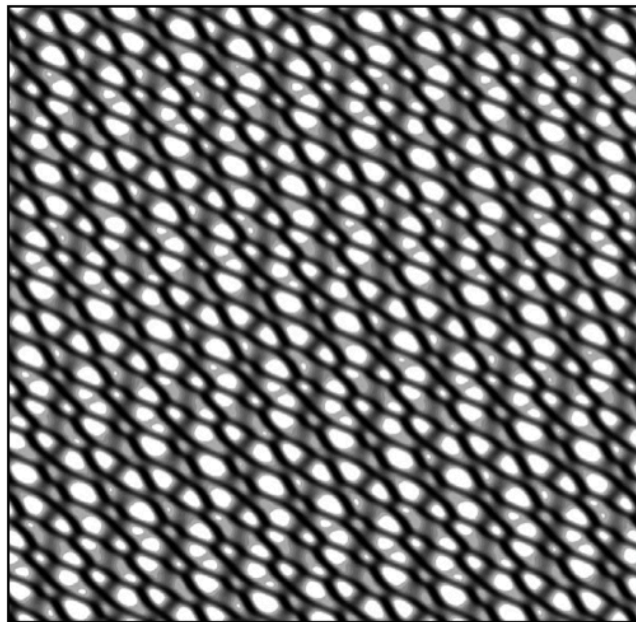


KP solitons and

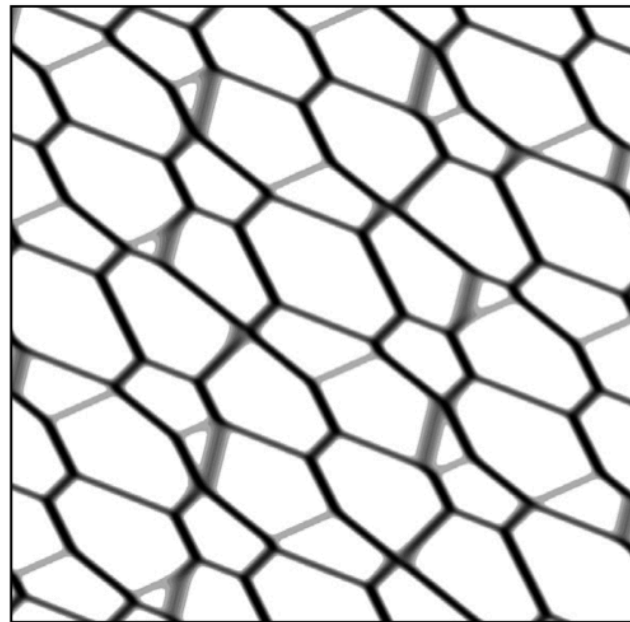
the Riemann theta functions

Yuji Kodama (SDUST & OSU)

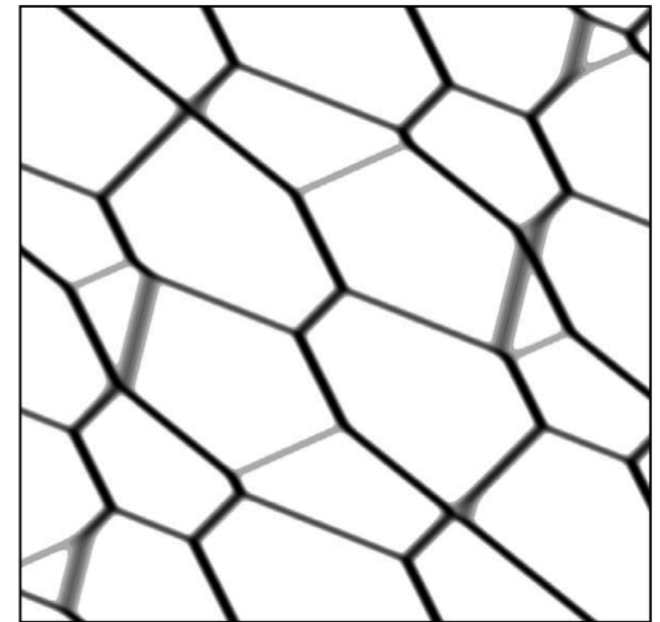
Nagoya, July 20-2023.



(a)

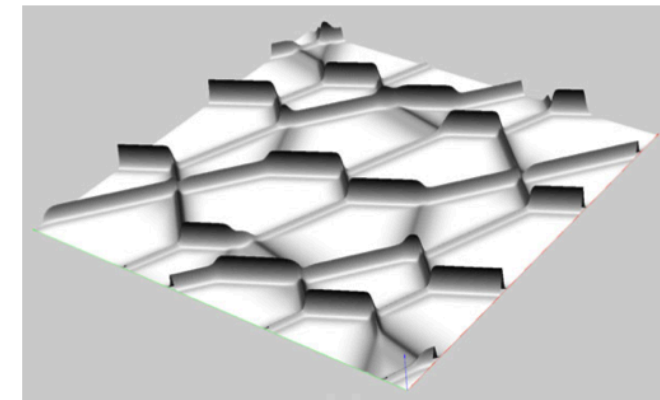
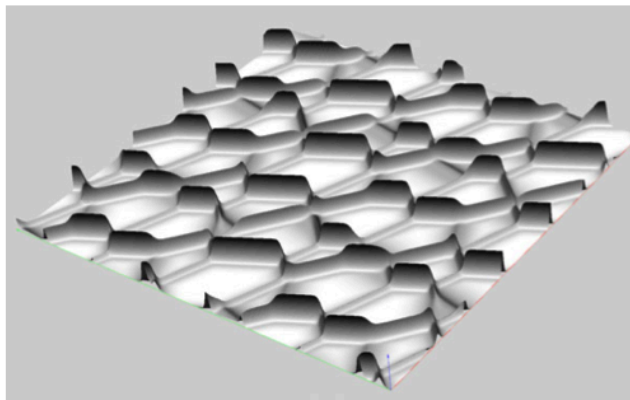
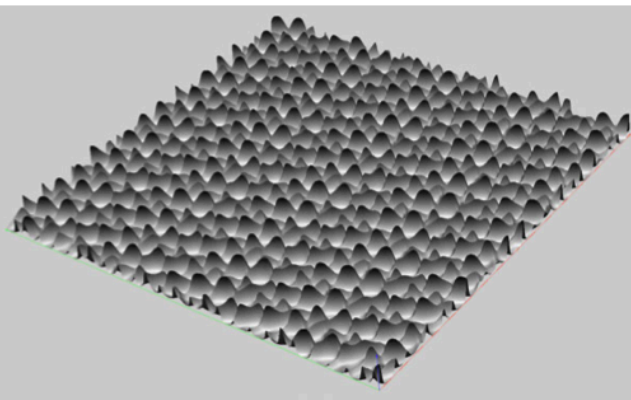


(b)



(c)

Fig. 7. Level lines for the solutions of the KP-II equation for (a) $\varepsilon = 10^{-2}$, (b) $\varepsilon = 10^{-10}$, and (c) $\varepsilon = 10^{-18}$. The horizontal axis is $-60 \leq x \leq 60$, and the vertical axis is $0 \leq y \leq 120$; $t = 0$. The light color corresponds to the lowest values of u , and the dark color, to the highest values of u .



Abenda - Grinevich 2017, $(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$

1. Basic Information of KP solitons

One soliton has a form,

$$U(x, y, t) = \frac{(k_i - k_j)^2}{2} \operatorname{sech}^2 \frac{1}{2} (\xi_i - \xi_j)$$

$$\xi_i(x, y, t) = k_i x + k_i^2 y + k_i^3 t + \xi_i^0$$

where k_i & k_j are arbitrary real conrts.

We call it $[i, j]$ -soliton, represent $\begin{pmatrix} i & j \\ j & i \end{pmatrix}$

Any KP soliton can be written in

$$u(x, y, t) = 2 \partial_x^2 \ln \tau(x, y, t)$$

$$\tau(x, y, t) = \det(AE(x, y, t)).$$

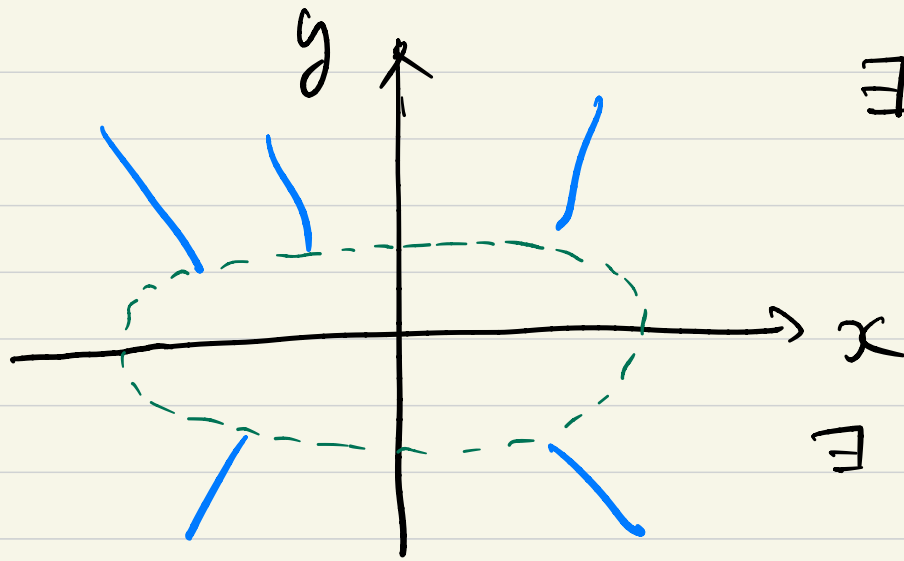
$A \in Gr^{TNN}(N, M)$: $N \times M$ matrix of full rank.

$$E(x, y, t) = \begin{bmatrix} e^{\sum_1} & k_1 e^{\sum_1} & \dots & k_1^{N-1} e^{\sum_1} \\ \vdots & \vdots & & \vdots \\ e^{\sum_M} & k_M e^{\sum_M} & \dots & k_M^{N-1} e^{\sum_M} \end{bmatrix}$$

Lemma: Each $A \in G_r^{TNN}(N, M)$ can be parametrized by a derangement $\pi \in S_M$

Theorem: Each KP soliton has the following properties: Let π be a derangement of S_M .

- if $\pi(i) > i$ (excedence), \exists a soliton of type $[i, \pi(i)]$ in $y \gg 0$
- if $\pi(i) < i$ (anti-excedence), \exists a soliton of type $[\pi(i), i]$ in $y \ll 0$



$\exists N$ solitons in $y \gg 0$

$\exists M-N$ solitons in $y \ll 0$

Interaction patterns consist of

X and Y shapes.

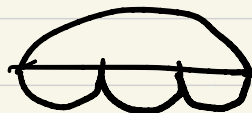
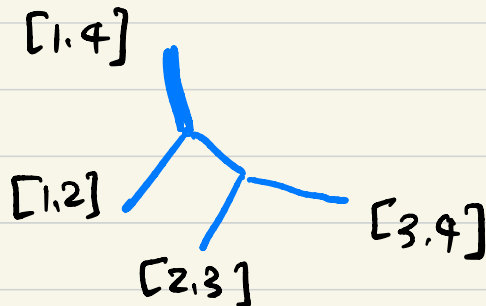
Examples

Gr (N. 4)

- $N = 1$

$$A = (1, *, *, *) ,$$

$$\dim = 3$$



$$\pi = (4123)$$

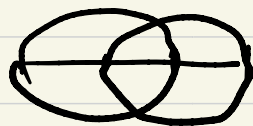
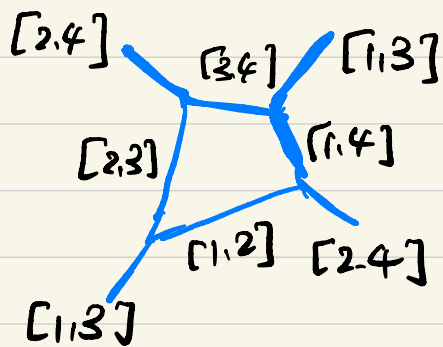
$$(g = 3)$$

- $N = 2$

(a)

$$A = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

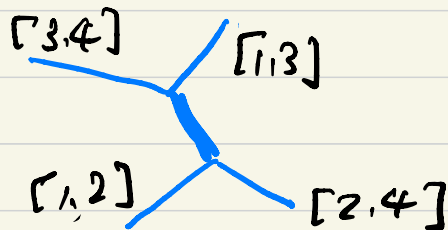
$$\dim = 4$$



$$\pi = (3412)$$

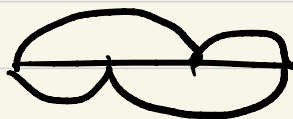
$$(g=3)$$

(b)



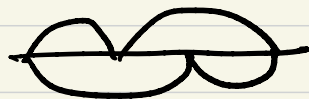
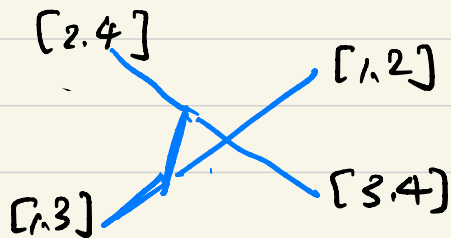
$$A = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \quad \dim = 3$$

$$(g=3)$$



$$\pi = (3142)$$

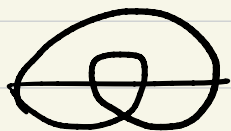
(c)



$$\pi = (2413)$$

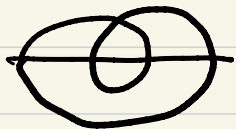
$$\dim = 3 \quad (g=3)$$

~~X~~ (d)



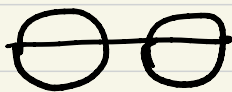
$$\pi = (4312), \quad (g=3)$$

~~X~~ (e)



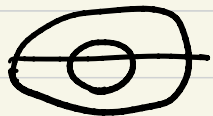
$$\pi = (3421), \quad (g=3)$$

~~X~~ (f)



$$\pi = (2143), \quad (g=2)$$

~~X~~ (g)



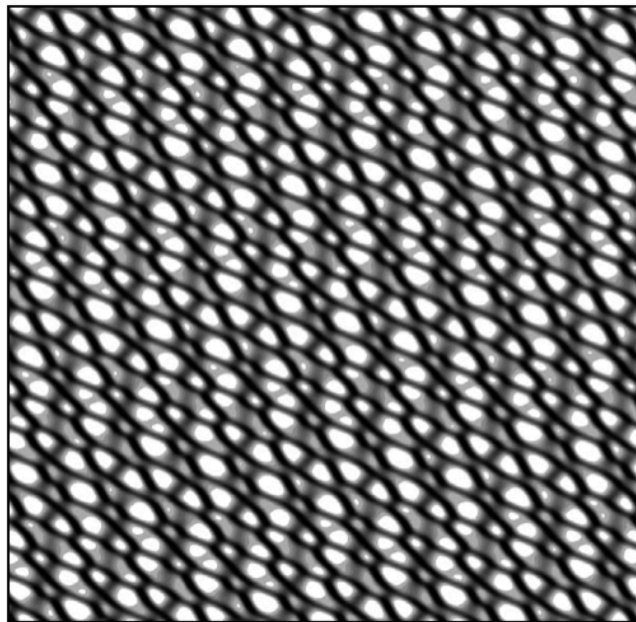
$$\pi = (4321), \quad (g=2)$$

• $N=3$

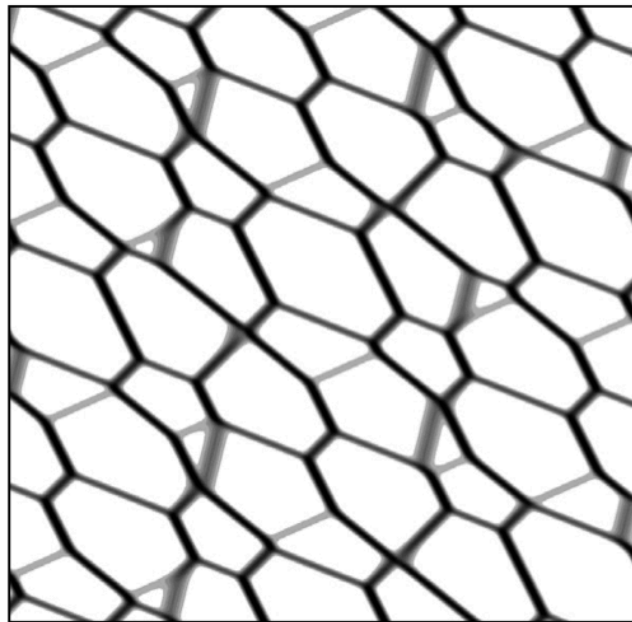


$$\pi = (2341), \quad (g=3)$$

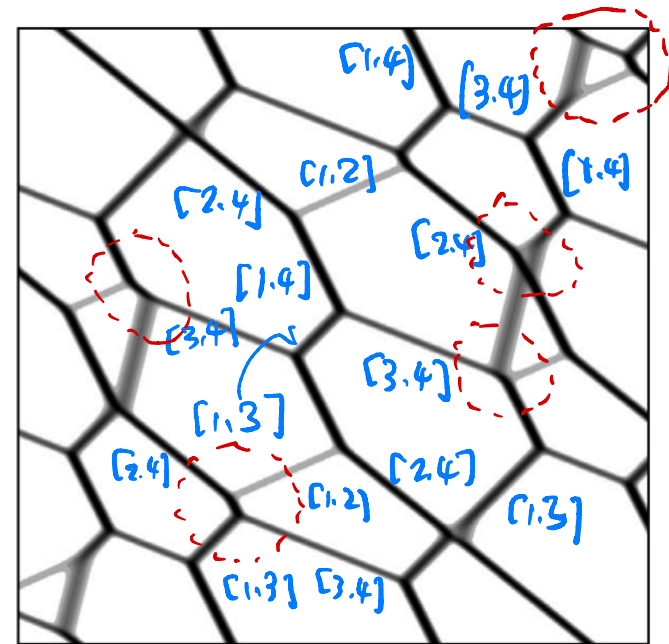




(a)

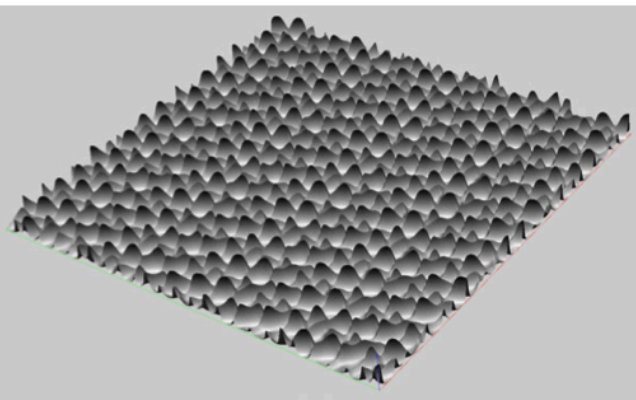


(b)

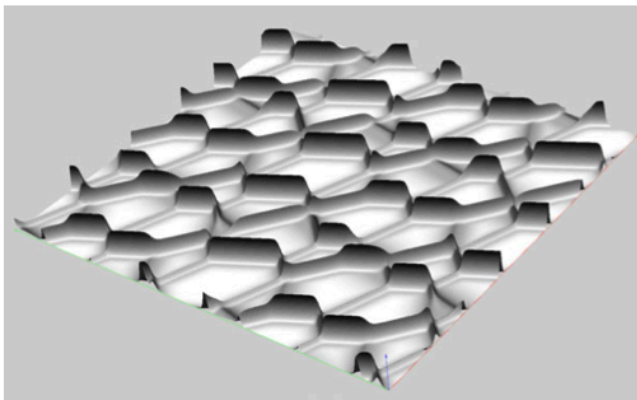


(c)

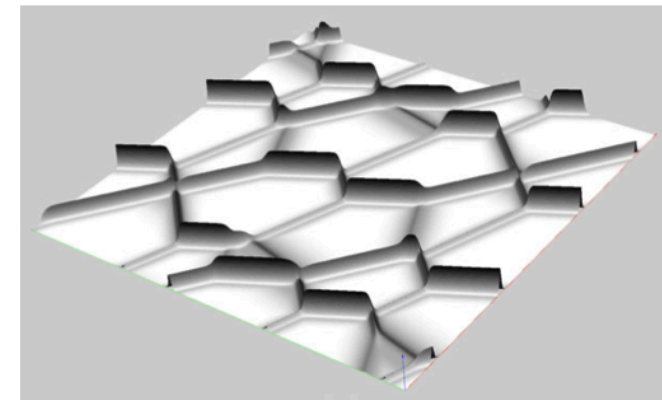
Fig. 7. Level lines for the solutions of the KP-II equation for (a) $\varepsilon = 10^{-2}$, (b) $\varepsilon = 10^{-10}$, and (c) $\varepsilon = 10^{-18}$. The horizontal axis is $-60 \leq x \leq 60$, and the vertical axis is $0 \leq y \leq 120$; $t = 0$. The light color corresponds to the lowest values of u , and the dark color, to the highest values of u .



(a)



(b)



(c)

Fig. 8. 3D plots for the solutions of the KP-II equation. The parameters and colors are the same as in Fig. 7.

Note:

Each KP soliton can be also
obtained by a certain limit of
Riemann Θ -function. (Tropical?)

(cf. Mumford, Abenda-Grinevich)

$$u(x, y, t) = 2 \lambda_x^2 \ln \Theta(x, y, t) + C$$

Tropical limit of Θ -function

(Mumford, 84, Agostini et al 23)

$$\Theta(z; \Omega) = \sum_{m \in \mathbb{Z}^g} \exp 2\pi i \left[\frac{m^T \Omega m}{2} + m^T z \right]$$

$$z \in \mathbb{C}^g$$

g : genus.

$\Omega: g \times g$. symmetric $\text{Im } \Omega > 0$

period matrix $\Omega = A^{-1}B$, $A = (\oint_{a_i} \omega_j)$, $B = (\oint_{b_i} \omega_j)$

Exponent

$$\frac{1}{2} m^T \Omega m + m^T z = \frac{1}{2} \sum_{i=1}^g m_i^2 \Omega_{ii} + \sum_{i < j} m_i m_j \Omega_{ij} + \sum_{i=1}^g m_i z_i$$

Shift

$$z_i \rightarrow z_i - \frac{1}{2} \Omega_{ii} \quad (1 \leq i \leq g)$$

Then

$$\frac{1}{2} \sum_{i=1}^g m_i (m_i - 1) \Omega_{ii} + \sum_{i < j} m_i m_j \Omega_{ij} + \sum_{i=1}^g m_i z_i$$

Now take the limits $\boxed{\Omega_{ii} \rightarrow +i\infty}$

Then only the terms with $m_i \in \{0, 1\}$ remain non-zero.

Thus ∞ -sum of exponential terms in Θ

becomes a finite sum of 2^g terms

\Rightarrow KP soliton (with some conditions)

Examples

- $g=1$, $\Theta(z, \Omega) = 1 + e^{2\pi i z_1}$ $\begin{cases} m_1=0 \\ m_2=1 \end{cases}$

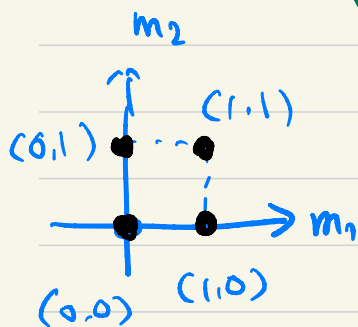
Take $2\pi i z_1 = \varphi_1 = \xi_2 - \xi_1$

where $\xi_i = K_i x + K_i^2 y + K_i^3 t + \xi_i^0$

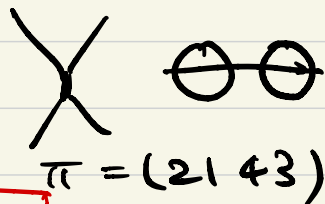
cf. $\tau_{kp} = e^{\xi_1} + e^{\xi_2} = e^{\xi_1} (1 + e^{\varphi_1})$

One-KP soliton.

• $g=2$. $\Theta(z, \Omega) = [1 + e^{\varphi_1} + e^{\varphi_2} + e^{2\pi i \Omega_{12}} e^{\varphi_1 + \varphi_2}]$



(a) $e^{2\pi i \Omega_{12}} = \frac{(k_2 - k_3)(k_1 - k_4)}{(k_2 - k_4)(k_1 - k_3)}$

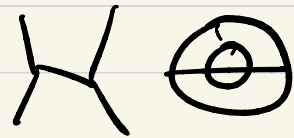


$$\tau_{kp} = (k_1 - k_3) e^{\xi_1 + \xi_3} \Theta(z, \Omega)$$

$G_{\pi}(2,4)$

(b)

$$e^{2\pi i \Omega_{12}} = \frac{(k_1 - k_2)(k_3 - k_4)}{(k_1 - k_4)(k_2 - k_3)}$$



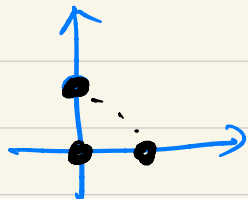
$\pi = (4321)$

$$\tau_{kp} = (k_1 - k_2) e^{\xi_1 + \xi_2} \Theta(z, \Omega)$$

$G_{\pi}(2,4)$

(c) further limit $\Omega_{1,2} \rightarrow +i\infty$

$$\Theta(z, \Omega) = 1 + e^{\Phi_1} + e^{\Phi_2}$$



$$\tau_{kp} = e^{\xi_1} + e^{\xi_2} + e^{\xi_3} = e^{\xi} \Theta(z, \Omega).$$

$G_{\text{or}}(1,3)$

$\curvearrowright \ominus \pi = (312)$

Remark: Dual graph of sditor graph
(Voronoi polytope):

For $g=2$. \triangle or \square (Agostini et al '23)

In general, genus g θ -function has
 the limit after taking $\Omega_{ii} \rightarrow +i\infty$ limits,

$$\begin{aligned} \theta(z, \Omega) &\rightarrow \sum_{m \in \{0,1\}^g} \exp 2\pi i \left[\sum_{i < j} m_i m_j \Omega_{ij} + \sum_{i=1}^g m_i z_i \right] \\ &= 1 + \sum_{k=1}^g e^{\varphi_k} + \sum_{k < l} e^{2\pi i \Omega_{kl}} e^{\varphi_k + \varphi_l} \\ &\quad + \dots + e^{2\pi i \sum_{k < l} \Omega_{kl}} e^{\sum_{k=1}^g \varphi_k} \end{aligned}$$

KP τ -function for $A \in \text{Gr}(N, M)$

$$\tau_{\text{kp}}(x, y, t) = \sum_{I \in \mathcal{M}(A)} \Delta_I(A) E_I(x, y, t)$$

where $\mathcal{M}(A) := \{ I = (i_1, \dots, i_N) \mid \Delta_I(A) \neq 0 \}$

$\Delta_I(A)$: $N \times N$ minor of A

$$E_I(A) = \prod_{1 \leq k < l \leq N} (x_{i_k} - x_{i_l}) \cdot e^{\xi_{i_1} + \dots + \xi_{i_N}}$$

Proposition: τ_{kp} can be written in

$$\tau_{kp} \equiv 1 + \sum_{k=1}^{N(M-N)} \tilde{\Delta}_k \mathcal{C}^{\tilde{\Phi}_k} + \dots$$

where $\tilde{\Phi}_k = \xi_{j_k} - \xi_{i_k} \quad \left\{ \begin{array}{l} j_k = J \setminus I_0 \\ i_k = I_0 \setminus J \end{array} \right.$

I_0 : Lexicographic min of pivot set $= \{\tilde{e}_1, \dots, \tilde{e}_k\}$

$$|J \cap I_0| = N-1.$$

Theorem:

τ_{kp} can be expressed by the Θ -function with appropriate limits, $\Theta(z, \Omega_{kp})$, and

$$2\pi i z_k = \xi_{ik} - \xi_{jk}.$$

Examples:

$$A = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix} \in \text{Gr}^{TNW}(2, 4)$$

$$\begin{aligned} \tau_{kp} = & (k_1 - k_2) e^{\xi_1 + \xi_2} + a(k_1 - k_3) e^{\xi_1 + \xi_3} + b(k_1 - k_4) e^{\xi_1 + \xi_4} \\ & + c(k_2 - k_3) e^{\xi_2 + \xi_3} + d(k_2 - k_4) e^{\xi_2 + \xi_4} + \Delta_{34} (k_2 - k_4) e^{\xi_3 + \xi_4} \end{aligned}$$

$$\equiv \left| + \frac{a(k_1 - k_2)}{k_1 - k_2} e^{\xi_3 - \xi_2} + \frac{b(k_1 - k_4)}{k_1 - k_2} e^{\xi_4 - \xi_2} + \frac{c(k_2 - k_3)}{k_1 - k_2} e^{\xi_4 - \xi_1} \right.$$

$$\left. + d \frac{k_2 - k_4}{k_1 - k_2} e^{\xi_4 - \xi_1} + \Delta_{34} \frac{k_3 - k_4}{k_1 - k_2} e^{\xi_4 - \xi_1 + \xi_3 - \xi_2} \right.$$

$$\begin{cases} \varphi_1 = \xi_3 - \xi_2 + \varphi_1^0 & \varphi_2 = \xi_4 - \xi_2 + \varphi_2^0 \\ \varphi_3 = \xi_3 - \xi_1 + \varphi_3^0 & \varphi_4 = \xi_4 - \xi_1 + \varphi_4^0 \end{cases}$$

$$\begin{aligned} \tau_{kp} = & 1 + e^{\varphi_1} + e^{\varphi_2} + e^{\varphi_3} + e^{\varphi_4} \\ & + \sum_{i,j} e^{\varphi_i + \varphi_j} \end{aligned}$$

with $\varphi_1 + \varphi_4 = \varphi_2 + \varphi_3$

This is a limit of θ of genus 3.

Consider $g=4$ θ -function, and

take limits $\Omega_{ii} \rightarrow +i\infty \quad i=1, \dots, 4.$

Then we have

$$\begin{aligned} \hat{\theta} = & 1 + e^{\varphi_1} + \dots + e^{\varphi_4} \\ & + e^{2\pi i \Omega_{12}} e^{\varphi_1 + \varphi_2} + \dots + e^{2\pi i \Omega_{34}} e^{\varphi_3 + \varphi_4} \\ & + e^{2\pi i \Omega_{12} \Omega_{13} \Omega_{14}} e^{\varphi_1 + \varphi_2 + \varphi_3} \\ & + \dots + e^{2\pi i \varphi_1 + \dots + \varphi_4} \end{aligned}$$

Then take $\Omega_{12}, \Omega_{13}, \Omega_{24}, \Omega_{34} \rightarrow +i\infty$

$$\text{and } \varphi_1 + \varphi_4 = \varphi_2 + \varphi_3.$$

Remark:

\mathcal{L}_{KP} can be written in the form of

$$\text{Gramian, } \mathcal{L}_{KP} = \left| \delta_{ij} + a_i b_j \frac{e^{P_i - Q_j}}{P_i - Q_j} \right|$$

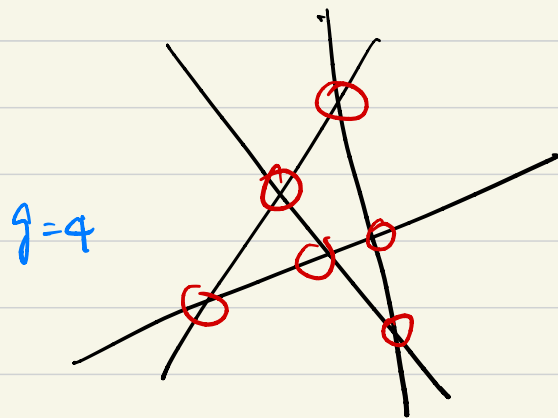
$$P_i = p_i^1 x + p_i^2 y + p_i^3 z, \quad Q_j = q_j^1 x + q_j^2 y + q_j^3 z.$$

Possible models of KP soliton gas

Recall that the limits $\Omega_{ii} \rightarrow +i\infty$ gives

$$\begin{aligned} \mathcal{Z}_{KP} = & 1 + \sum e^{\varphi_k} + \sum_{k < l} e^{2\pi i \Omega_{kl}} e^{\varphi_k + \varphi_l} \\ & + \dots + e^{2\pi i \sum_{k < l} \Omega_{kl}} e^{\sum_i \varphi_k} \end{aligned}$$

The corresponding KP soliton
consists of g line solitons
without resonances, i.e.



Those interaction pts

have "phase shifts".

which are determined by Ω_{ij}

- Then considering Ω_{ij} to be random variables, this KP soliton may be considered to be a KP soliton gas with random phase shifts.
(similar to the KdV soliton gas)

- Also taking random choice of $\Omega_{ij} \rightarrow +i\infty$, one can have random pattern including resonant interaction (Y-shape) in addition to random phases.

- A quasi-periodic (QP) solution gives a set of KP solitons (flag structure).

Since each KP soliton can be parametrised by a permutation, one can (?) consider a probabilistic measure of random permutation (like Schur measure) and find the most likely KP soliton in the solution.
(wave turbulence?)