Quantum Analog of Mishchenko-Fomenko Theorem for $U\mathfrak{gl}_d$

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Noncommutative Integrable Systems

Introduction

What Georgy talked:

1. Classical shift method in the symmetric algebra $S(\mathfrak{g})$.
2. Vinberg's problem and solutions by
   1. Nazarov and Olshanski.
   2. Rybnikov.
   3. Tarasov.
   4. Molev and Yakimova.
3. Definition of the quantum derivations
   1. in coordinates.
   2. by coproduct.
   3. as the symmetrisation of the mapping $\exp(D)$.

Theorem of A. Mishchenko and A. Fomenko

I will explain a quantum analogue of the theorem of A. Mishchenko and A. Fomenko. Let $\mathfrak{g}$ be a complex Lie algebra.

Theorem (A. Mishchenko and A. Fomenko, 1978)

Suppose that ∂_ξ is a constant vector field on the dual space $\mathfrak{g}^*$. We have

$$\{\partial_\xi^m(x), \partial_\xi^n(y)\} = 0$$

for any m and n and for any Poisson central elements x and y of the symmetric algebra $S(\mathfrak{g})$.

Argument Shift Operator ∂_ξ

- The operator ∂_ξ is called the argument shift operator in the direction ξ.
- Why is it called the argument shift? Let $e_1, \dots, e_d$ be a basis of the Lie algebra $\mathfrak{g}$ and $x = x(e_1, \dots, e_d)$ be an element of the symmetric algebra $S\mathfrak{g} = \mathbb{C}\big[e_1, \dots, e_d\big]$. We have

$$\begin{aligned} & x\big(e_1 + t\xi(e_1), \dots, e_d + t\xi(e_d)\big) \\ & = \sum_{n=0}^{\dim x} \frac{t^n \partial_\xi^n x(e_1, \dots, e_n)}{n!} \\ & = \exp(t\partial_\xi) x(e_1, \dots, e_d). \end{aligned}$$

Derivation on $S\mathfrak{gl}(d,\mathbb{C})$

Let us consider the case $\mathfrak{g} = \mathfrak{gl}(d,\mathbb{C})$.

- Let
$$e = \begin{pmatrix} e_1^1 & \dots & e_d^1 \\ \dots & \dots & \dots \\ e_1^d & \dots & e_d^d \end{pmatrix} \in M\big(d, \mathfrak{gl}(d,\mathbb{C})\big),$$
where e_j^i form a linear basis of $\mathfrak{gl}(d,\mathbb{C})$ and satisfy the commutation relation $\left[e_j^i, e_l^k\right] = e_j^k \delta_l^i - \delta_j^k e_l^i$.
- $S\mathfrak{gl}(d,\mathbb{C}) \simeq \mathbb{C}\left[(e_j^i)_{i,j=1}^d\right]$.
- A constant vector field on the dual space is given by
$$\partial_\xi = \mathrm{tr}(\xi\partial), \qquad \partial_j^i = \frac{\partial}{\partial e_i^j} \in \hom S\mathfrak{gl}(d,\mathbb{C}),$$
where ξ is a numerical matrix.

Derivation on $S\mathfrak{gl}(d, \mathbb{C})$

Remark

The derivation

$$S\mathfrak{gl}(d, \mathbb{C}) \to M\big(d, S\mathfrak{gl}(d, \mathbb{C})\big), \qquad x \mapsto \partial x = (\partial^i_j x)^d_{i,j=1}$$

is a unique linear mapping satisfying the following.

1. $\partial \nu = 0$ for any scalar ν.
2. $\partial\, \mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ. $\Longleftrightarrow \partial^i_j e^k_\ell = \delta^i_\ell \delta^k_j$
3. the Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y)$$

for any elements x and y of the symmetric algebra.

Quantum Derivation on $U\mathfrak{gl}(d,\mathbb{C})$

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum derivation

$$U\mathfrak{gl}(d,\mathbb{C}) \to M\big(d, U\mathfrak{gl}(d,\mathbb{C})\big), \qquad x \mapsto \partial x = (\partial^i_j x)^d_{i,j=1}$$

is a unique linear mapping satisfying the following.

1. $\partial \nu = 0$ for any scalar ν.
2. $\partial\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. the quantum Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra.

Quantum M-F Theorem

Theorem (I. and S., 2023)

Suppose that ξ *is a numerical matrix and let* $\partial_\xi = \mathrm{tr}(\xi\partial)$*. We have*

$$\left[\partial_\xi^m(x), \partial_\xi^n(y)\right] = 0$$

for any positive m *and* n *and for any central elements* x *and* y *of the universal enveloping algebra* $U\mathfrak{gl}(d,\mathbb{C})$*.*

Quantum M-F Theorem

- We may assume $\xi = \mathrm{diag}(z_1, \ldots, z_d)$ ($z_1, \ldots, z_d$ are distinct).

$\hat{A}_\xi$

- The quantum M-F algebra in the direction ξ is the centraliser of the set

$$\left\{ e_i^i, \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^{d}$$

(Vinberg and Rybnikov).

Quantum M-F Theorem

We are reduced to proving ($(\operatorname{ad} x)(y) = [x, y]$)

$$(\operatorname{ad} e_i^i)(\partial_\xi^n x) = (\operatorname{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^n x) = 0 \qquad (1)$$

for any positive n and for any central element x. It is sufficient to show

1. $(\operatorname{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi x) = 0.$
2. $\left[\operatorname{ad} e_i^i, \partial_\xi\right] = 0.$ $\Rightarrow 0 = \partial_\xi^n [e_i^i, x] = [e_i^i, \partial_\xi^n x]$
3. $\left[\left[\operatorname{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right] = 0.$

the second half of (1)

Quantum M-F Theorem

Let us assume these three conditions and prove (1). Since x is central, $(\mathrm{ad}\, e_i^i)(x) = 0$ and the first condition is equivalent to

$$\left[\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right](x) = 0.$$

The second and the third conditions imply

$$(\mathrm{ad}\, e_i^i)(\partial_\xi^n x) = \left[\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right](\partial_\xi^n x) = 0 \qquad (2)$$

for any positive n.

Quantum M-F Theorem

We prove the second half of (1)

$$(\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^n x) = 0$$

by induction. The first condition implies the case $n = 1$. Suppose that $n > 1$. We have

$$(\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^n x) = \left[\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right](\partial_\xi^{n-1} x) + \partial_\xi(\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^{n-1} x) = 0$$

(2)

by the induction hypothesis.

Quantum M-F Theorem

Now we are reduced to proving

1. $$(\operatorname{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi x) = 0.$$
2. $$\left[\operatorname{ad} e_i^i, \partial_\xi\right] = 0.$$
3. $$\left[\left[\operatorname{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right] = 0.$$

Quantum M-F Theorem

C: center of $U\mathfrak{gl}_d$
x: central
$\Rightarrow \partial_\xi x = \sim$
exact formula [3]

1. $$\Big(\mathrm{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j}\Big)(\partial_\xi x)=0.$$

The element $\partial_\xi x$ belongs to the module $\mathrm{span}_C\{\mathrm{tr}(\xi e^n)\}_{n=0}^{\infty}$.

$$\begin{aligned}
\Big(\mathrm{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j}\Big)\big(\mathrm{tr}(\xi e^n)\big) &= \sum_{j\neq i}\sum_{k=1}^{d}\frac{z_k}{z_i-z_j}\big[e_i^j e_j^i, (e^n)_k^k\big] \\
&= \sum_{j\neq i}\big(-(e^n)_i^j e_j^i + e_i^j (e^n)_j^i\big) = 0.
\end{aligned}$$

Quantum M-F Theorem

2 $\left[\operatorname{ad} e_i^j, \partial_\xi\right] = 0.$

$$\begin{aligned}\left[\operatorname{ad} e_i^j, \partial_\xi\right](x) &= \left[e_i^j, \partial_\xi x\right] - \partial_\xi\left[e_i^j, x\right] \\ &= -\left[\partial_\xi e_i^j, x\right] - \operatorname{tr}\left(\xi\left[\partial e_i^j, \partial x\right]\right) \\ &= z_i\left((\partial x)_i^j - (\partial x)_i^j\right) = 0.\end{aligned}$$

Quantum M-F Theorem

3 $$\left[\left[\mathrm{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right] = 0.$$

$$\left[\mathrm{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right](x) = \left[\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi x\right] - \partial_\xi\left[\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, x\right]$$

$$= -\left[\partial_\xi\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, x\right] - \mathrm{tr}\left(\xi\left[\partial\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial x\right]\right)$$

since $\partial_\xi\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j} = \sum_{j\neq i}\frac{z_i}{z_i - z_j}$.

Quantum M-F Theorem

Applying this

$$\left[\left[\mathrm{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right](x)$$

$$= -\operatorname{tr}\left(\xi\left[\partial\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial\partial_\xi x\right]\right) + \partial_\xi \operatorname{tr}\left(\xi\left[\partial\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial x\right]\right)$$

$$= \operatorname{tr}\left(\xi\left[\partial\partial_\xi\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j}, \partial x\right]\right)$$

$$+ \sum_{i_1,j_1,i_2,j_2=1}^{d}(z_{i_1}z_{i_2} - z_{j_1}z_{j_2})(\partial_{j_1}^{i_1}\partial_{j_2}^{i_2}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i - z_j})(\partial_{i_1}^{j_1}\partial_{i_2}^{j_2}x),$$

here $(z_{i_1}z_{i_2} - z_{j_1}z_{j_2})\partial_{j_1}^{i_1}\partial_{j_2}^{i_2}(e_i^j e_j^i)$

$$= (z_{i_1}z_{i_2} - z_{j_1}z_{j_2})\left((\partial_{j_1}^{i_1}e_i^j)(\partial_{j_2}^{i_2}e_j^i) + (\partial_{j_2}^{i_2}e_i^j)(\partial_{j_1}^{i_1}e_j^i)\right) = 0.$$

Application: Generators of Quantum M-F Algebras

Let us define an increasing sequence of commutative algebras:

$$C_\xi^{(n)} = \begin{cases} C & n = 0 \\ C_\xi^{(n-1)}\left[\, \partial_\xi^n x : x \text{ is central} \,\right] & n > 0 \end{cases}$$

The algebra $C_\xi^{(2)}$ is generated by the center C and the elements

$$\left\{ \operatorname{tr}(\xi e^n), \sum_{m=0}^{n} \operatorname{tr}\left(\xi e^n f_+^{(n-m-1)}(e) \xi e^m\right), \right.$$
$$\left. \sum_{m=0}^{n} \operatorname{tr}\left(\xi\left(e^n f_+^{(n-m)}(e) + e^{n+1} f_+^{(n-m-1)}(e)\right)\xi e^m\right) \right\}_{n=1}^{\infty},$$

where $f_\pm^{(n)}(x) = \sum_{m=0}^{n+1} \frac{1 \pm (-1)^{n-m}}{2} \binom{n}{m} x^m$.

Application: Generators of Quantum M-F Algebras

The generators are $\mathrm{tr}(\xi e)$, $\mathrm{tr}(\xi e^2)$, ... and

$$\mathrm{tr}(\xi^2 e),$$
$$\mathrm{tr}(2\xi^2 e^2 + \xi e \xi e),$$
$$\mathrm{tr}(\xi^2 e^3 + \xi e \xi e^2),$$
$$\mathrm{tr}(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2),$$
$$\mathrm{tr}(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3),$$
$$\mathrm{tr}(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3),$$
$$\mathrm{tr}(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4), \ldots.$$

$\sum_{i,j}$ [illegible] $\mathrm{tr}(\xi e^i \xi e^j)$ ← not arbitrary to commute

They are mutually commutative.

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