Current algebras and differential graded manifolds

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§1. Introduction

Poisson geometry

Poisson bracket — Siméone Poisson 1809

bilinear + skew + Leibniz rule + Jacobi identity

Modern Poisson geometry

Lie algebra, Poisson bracket, symplectic manifold, associative algebra, L_∞-algebra, algebroids, mechanics, quantization, noncommutative geometry, coquecigrue problem

Poisson geometry is a Tool of math, not a Field of math. We make tools, do not solve problems.

A current algebra

is a Lie algebra structure on a mapping space constructed from a Poisson bracket.

We discuss possible general forms of **current algebras**.

Alekseev-Strobl '05, Bonelli-Zabzine '05, Ekstrand-Zabzine '09, Ekstrand '11, NI-Koizumi '11, NI-Xu '13, Bessho-Heller-NI-Watamura '15, Arvanitakis '21, Hayami '23

Physics

Find new theories and new physics

Math

Groupoidification program

Plan of Talk

1. Alekseev-Strobl current algebras

2. Super symplectic geometry

3. Generalizations to higher dimensional current algebras

§2. Generalized current algebras on loop space

Alekseev, Strobl '04

Consider a mapping space $LM = \mathrm{Map}(S^1, M)$ ($X_2 = S^1 \times \boldsymbol{R}$)

Local coordinate σ on S^1

$x^i(\sigma)$: $S^1 \to M$,

$p_i(\sigma)$: canonical momentum , ($p_i(\sigma)\mathrm{d}\sigma \otimes \mathrm{d}x^i \in \Omega^1(S^1, x^*T^*M)$)

$$\int_{S^1} x^*\omega_{can} = \int_{S^1} \mathrm{d}\sigma \ \delta x^i \wedge \delta p_i$$

$$\{x^i, x^j\}_{PB} = 0, \quad \{x^i, p_j\}_{PB} = \delta^i{}_j \delta(\sigma - \sigma'), \quad \{p_i, p_j\}_{PB} = 0.$$

The Poisson bracket can be deformed by a closed 3-form H on M

as

$$\boldsymbol{\omega} = \int_{S^1} x^*(\omega_{can} + H) = \int_{S^1} \mathrm{d}\sigma\, \delta x^i \wedge \delta p_i + \int_{S^1} \mathrm{d}\sigma\, \frac{1}{2} H_{ijk}(x) \partial_\sigma x^i \delta x^j \wedge \delta x^k.$$

$$\{x^i(\sigma), x^j(\sigma')\}_{PB} = 0, \quad \{x^i(\sigma), p_j(\sigma')\}_{PB} = \delta^i{}_j \delta(\sigma - \sigma'),$$
$$\{p_i(\sigma), p_j(\sigma')\}_{PB} = -H_{ijk}(x) \partial_\sigma x^k(\sigma) \delta(\sigma - \sigma').$$

Assumption

Suppose that currents do not depend on a metric on S^1 and M.

Ansatz of currents:

$$J_{0(f)}(\sigma) = x^*f = f(x(\sigma)),$$
$$J_{1(X+\alpha)}(\sigma)= x^*\alpha + \langle x^*X\ ,p\rangle = \alpha_i(x(\sigma))\partial_\sigma x^i(\sigma) + X^i(x(\sigma))p_i(\sigma),$$

$f(x) \in C^\infty(M)$, $X + \alpha = X^i(x)\partial_i + \alpha_i(x)dx^i \in \Gamma(TM \oplus T^*M)$.

$$\begin{aligned}
\{J_{0(f)}(\sigma), J'_{0(g)}(\sigma')\}_{PB} &= 0, \\
\{J_{1(X+\alpha)}(\sigma), J'_{0(g)}(\sigma')\}_{PB} &= -\rho(X+\alpha)(\sigma)J'_{0(g)}(x(\sigma))\delta(\sigma-\sigma'), \\
\{J_{1(X+\alpha)}(\sigma), J'_{1(Y+\beta)}(\sigma')\}_{PB} &= -J_{1([X+\alpha,Y+\beta]_D)}(\sigma)\delta(\sigma-\sigma') \\
&\quad +\langle X+\alpha, Y+\beta\rangle(\sigma')\partial_\sigma\delta(\sigma-\sigma'),
\end{aligned}$$

where $X, Y \in \Gamma(TM)$, $\alpha, \beta \in \Gamma(T^*M)$,

$$\rho(X + \alpha) = X = X^i(x)\frac{\partial}{\partial x^i},$$
$$\langle X + \alpha, Y + \beta \rangle = \iota_X \beta + \iota_Y \alpha = X^i \beta_i + Y^i \alpha_i,$$
$$[X + \alpha, Y + \beta]_D = [X, Y] + L_X \beta - \iota_Y d\alpha + \iota_X \iota_Y H.$$

§2-1. Examples

Kac-Moody algebra (G/G WZW model)

$M = \{\mathrm{pt}\}$

$$J^L = -\frac{k}{4\pi} g^{-1}\partial_\sigma g + p \qquad J^R = \frac{k}{4\pi}\partial_\sigma g g^{-1} + gpg^{-1}$$

String sigma model

If $V \in \mathfrak{X}(M)$ be a Killing-vector field on M,

$$J = V^i(x(\sigma))p_i(\sigma)$$

Poisson sigma model

Let $x : \Sigma_2 \to M$ and $A \in \Omega^1(\Sigma_2, X^*TM)$.

$$S = \int_{\Sigma_2} \mathrm{d}^2\sigma \left(A_i \mathrm{d}x^i + \frac{1}{2}\pi^{ij}(x) A_i \wedge A_J \right),$$

$$J \;\; = \;\; \alpha_i(\mathrm{d}x^i + \pi^{ij}(x) A_j)$$

where $\alpha_i(x(\sigma)) = \alpha_i$=const.

cf. JT gravity

To do

- Generalizations to higher dimensions.
- Find fundamental structures behind current algebras.

§2-3. Current algebras in two dimensions

Worldvolume: $X_3 = \Sigma_2 \times \boldsymbol{R}$ and a target vector bundle E over M.

We consider $\mathrm{Map}(T^*\Sigma_2, T^*E)$.
$x^i : \Sigma_2 \to M$: 0-form, $q^a \in \Omega^1(\Sigma_2, x^*E)$: 1-form,
$p_i \in \Omega^2(\Sigma_2, x^*TM)$: 2-form
A symplectic form is

$$\boldsymbol{\omega} = \int_{\Sigma_2} (\langle \delta x \, , \delta p \rangle + (\delta q, \delta q) + x^*H)$$

where H is a closed 4-form on M, $\langle - \, , - \rangle$ is the pairing of TM and T^*M and $(-, -)$ is a fiber metric on E.

The Poisson brackets of canonical quantities are

$$\{x^i(\sigma), p_{j\mu\nu}(\sigma')\}_{PB} = \epsilon_{\mu\nu}\delta^i_j \delta^2(\sigma - \sigma'),$$
$$\{q^a_\mu(\sigma), q^b_\nu(\sigma')\}_{PB} = \epsilon_{\mu\nu} k^{ab} \delta^2(\sigma - \sigma'),$$
$$\{p_{i\mu\nu}(\sigma), p_{j\lambda\rho}(\sigma')\}_{PB} = -\frac{1}{2}\epsilon_{\mu\nu}\epsilon_{\lambda\rho} H_{ijkl}(x)\epsilon^{\sigma\tau}\partial_\sigma x^l \partial_\tau x^l \delta^2(\sigma - \sigma').$$

The mass dimensions of each quantities: $\dim[\omega] = 0$, $\dim[\sigma] = -1$, $\dim[\partial] = 1$, $\dim[x^i] = 0$, $\dim[q^a] = 1$, $\dim[p_i] = 2$.

Each current has the homogeneous mass dimension, the following three currents are most general forms of mass dimension zero, one and two, respectively:

$$J_{0(f)}(\sigma) = f(x(\sigma)),$$

$$J_{1\mu(\alpha,u)}(\sigma) = x^*(\alpha + u(q)) = \alpha_i(x(\sigma))\partial_\mu x^i(\sigma) + u_a(x(\sigma))q^a_\mu(\sigma),$$

$$J_{2\mu\nu(G,K,F,B,E)}(\sigma) = x^*G(p) + x^*K(\mathrm{d}q) + x^*F(q,q) + x^*B + x^*E(q)$$

$$= \epsilon_{\mu\nu}\epsilon^{\lambda\rho}\left(\frac{1}{2}G^i(x(\sigma))p_{i\lambda\rho}(\sigma) + K_a(x(\sigma))\partial_\lambda q^a_\rho(\sigma) + \frac{1}{2}F_{ab}(x(\sigma))q^a_\lambda(\sigma)q^b_\rho(\sigma)\right.$$

$$\left. + \frac{1}{2}B_{ij}(x(\sigma))\partial_\lambda x^i(\sigma)\partial_\rho x^j(\sigma) + E_{ai}(x(\sigma))\partial_\lambda x^i(\sigma)q^a_\rho(\sigma)\right).$$

$\alpha \in \Omega^1(M)$, $u \in \Gamma(E^*)$, $G \in \mathfrak{X}(M)$, $K \in \Gamma(E^*)$, $F \in \Gamma(\wedge^2 E^*)$, $B \in \Omega^2(M)$, $E \in \Omega^1(M, E^*)$.

The Poisson brackets of currents are directly calculated:

$$\{J_{0(f)}(\sigma), J_{0(f')}(\sigma')\}_{PB} = \ldots,$$
$$\{J_{1(\alpha,u)}(\sigma), J_{0(f')}(\sigma')\}_{PB} = \ldots,$$
$$\{J_{2(G,K,F,B,E)}(\sigma), J_{0(f')}(\sigma')\}_{PB} = \ldots$$
$$\{J_{1(\alpha,u)}(\sigma), J_{1(\alpha',u')}(\sigma')\}_{PB} = \ldots$$
$$\{J_{2(G,K,F,B,E)}(\sigma), J_{1(\alpha',u')}(\sigma')\}_{PB} = \ldots$$
$$\{J_{2(G,K,F,B,E)}(\sigma), J_{2(G',K',F',B',E')}(\sigma')\}_{PB} = \ldots$$

What is the solution and the algebra?

§3. Algebraic structure of AS current algebra

$$
\begin{aligned}
\{J_{0(f)}(\sigma), J'_{0(g)}(\sigma')\}_{PB} &= 0, \\
\{J_{1(X+\alpha)}(\sigma), J'_{0(g)}(\sigma')\}_{PB} &= -\rho(X+\alpha)(\sigma)J'_{0(g)}(x(\sigma))\delta(\sigma-\sigma'), \\
\{J_{1(X+\alpha)}(\sigma), J'_{1(Y+\beta)}(\sigma')\}_{PB} &= -J_{1([X+\alpha,Y+\beta]_D)}(\sigma)\delta(\sigma-\sigma') \\
&\quad +\langle X+\alpha, Y+\beta\rangle(\sigma')\partial_\sigma\delta(\sigma-\sigma'), \\
\rho(X+\alpha) &= X^i(x)\frac{\partial}{\partial x^i}, \\
\langle X+\alpha, Y+\beta\rangle &= \iota_X\beta + \iota_Y\alpha \\
[X+\alpha, Y+\beta]_D &= [X,Y] + L_X\beta - \iota_Y d\alpha + \iota_X\iota_Y H.
\end{aligned}
$$

The 'algebra' of these operations is the **standard Courant algebroid**. Liu,Weinstein,Xu '97

Definition 1. *Liu, Weinstein, Xu '97, Kosmann-Schwarzbach '07*

Let E be a vector bundle over M equipped with a pseudo-Euclidean inner product $\langle - , - \rangle$, a bundle map $\rho : E \longrightarrow TM$ and a binary bracket $[-,-]_D$ on $\Gamma(E)$. The bundle is called the $\boldsymbol{Courant}$ $\boldsymbol{algebroid}$ *if three conditions are satisfied,*

$$
\begin{aligned}
[e_1, [e_2, e_3]_D]_D &= [[e_1, e_2]_D, e_3]_D + [e_2, [e_1, e_3]_D]_D, \\
\rho(e_1)\langle e_2 , e_3 \rangle &= \langle [e_1, e_2]_D , e_3 \rangle + \langle e_2 , [e_1, e_3]_D \rangle, \\
\rho(e_1)\langle e_2 , e_3 \rangle &= \langle e_1 , [e_2, e_3]_D + [e_3, e_2]_D \rangle,
\end{aligned}
$$

where $e_1, e_2, e_3 \in \Gamma(E)$. (In our case, we take $E = TM \oplus T^*M$.)

Dirac Structure

Definition 2. *If a subbundle L of the Courant algebroid E satisfies*

$$\langle e_1 \,, e_2 \rangle = 0 \ (\textit{isotropic}), \quad [e_1, e_2]_D \in \Gamma(L) \ (\textit{involutive}),$$
$$\mathrm{rank}(L) = \frac{1}{2}\mathrm{rank}(E)$$

for $e_1, e_2 \in \Gamma(L)$,
L is called the $\boldsymbol{Dirac\ structure}$.

If and only if $X + \alpha \in \Gamma(L)$, the term of the 'central charge' in the Poisson bracket in the current algebra is zero.

Rewriting of Courant algebroid

Proposition 1. *Roytenberg '99*

*A **Courant algebroid** structure on E is equivalent to a QP-manifold (differential graded symplectic manifold) of degree 2 on $T^*[2]E[1]$.*

Our idea

- BRST-BFV(BV) formalism of current algebras
- Chevalley-Eilenberg complex of Courant algebroid

§4. Supergeometry

Derived bracket (supergeometric) construction: Lie algebra

Let $\mathfrak{g}$ be a vector space.

1. $V = \mathfrak{g} \oplus \mathfrak{g}^* \simeq T^*\mathfrak{g}$ and $\wedge^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*)$

(b_a, c^a): basis of degree $(1, 1)$, $xy = -yx$ for (b_a, c^a).

2. The odd Poisson bracket $\{c^a, b_b\} = \{b_b, c^a\} = \delta^a_b$ on $C^\infty(V) = T(V)/(xy + yx) = \oplus_{k=0}^\infty V^{\otimes k}/(xy + yx)$

3. A function $\Theta = \frac{1}{2} f^c_{ab} c^a c^b b_c$ gives degree $+1$ vector field $Q = \{\Theta, -\}$.

Impose $Q^2 = 0$ ($\{\Theta, \Theta\} = 0$), which is equivalent to $f_{ab}^{d} f_{dc}^{e} + (abc \text{ cyclic}) = 0$. The *derived bracket*

$$[b_a, b_b] = \{\{b_a, \Theta\}, b_b\} = f_{ab}^{c} b_c$$

gives a Lie bracket. This induces a Lie algebra on $\mathfrak{g}$.

Chevalley-Eilenberg complex

In fact, $C^*_{CE}(\wedge^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*))$

c^a is the basis of $\mathfrak{g}^*$, b_a is the basis of $\mathfrak{g}$

$Q \simeq \mathrm{d}_{CE}$: Chevalley-Eilenberg differential

Graded manifolds

The Courant algebroid has a supergeometric construction.

A **graded manifold** $\mathcal{M} = (M, \mathcal{O}_M)$ on a smooth manifold M is a ringed space which structure sheaf $\mathcal{O}_M$ is $\boldsymbol{Z}$–graded commutative algebras over M, locally isomorphic to $C^\infty(U) \otimes S^\cdot(V)$, where U is a local chart on M, V is a graded vector space and $S^\cdot(V)$ is a free graded commutative ring on V.

Grading is called **degree**. We denote $\mathcal{O}_M = C^\infty(\mathcal{M})$.

If degrees are nonnegative, a graded manifold is called a **N-manifold**.

Definition 3. *A following triple $(\mathcal{M}, \omega, Q)$ is called a QP-manifold (a differential graded symplectic manifold) of degree n if $\mathcal{L}_Q \omega = 0$.*
Schwarz '92

1. $\mathcal{M}$: N-manifold (nonnegatively graded manifold)

2. ω: (P-structure) a graded symplectic form of degree n on $\mathcal{M}$.

3. Q: (Q-structure) (a homological vector field)
A graded vector field of degree $+1$ such that $Q^2 = 0$,

Note: A graded Poisson bracket $\{-, -\}$ of degree $-n$ is induced from ω.

$$\{f, g\} = -(-1)^{(|f|-n)(|g|-n)}\{g, f\},$$
$$\{f, gh\} = \{f, g\}h + (-1)^{(|f|-n)|g|}g\{f, h\},$$
$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|-n)(|g|-n)}\{g, \{f, h\}\}.$$

Note:

If degree $n \neq 0$, there exists a Hamiltonian function (a homological function) $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$ such that $\iota_Q \omega = -\delta\Theta$.

$Q^2 = 0$ is equal to the equation, $\{\Theta, \Theta\} = 0$.

The QP-manifold triple can be described by $(\mathcal{M}, \{-, -\}, \Theta)$.

Derived bracket construction of Courant algebroid

Roytenberg '99

Take $n = 2$.

1. Take a vector bundle E over a smooth manifold M and an N-manifold $\mathcal{M} = T^*[2]E[1]$.

2. canonical graded symplectic form ω_{can} and the induced Poisson bracket $\{-,-\}$.

3. a function Θ of degree 3 on $\mathcal{M}$ satisfying $\{\Theta, \Theta\} = 0$

Functions on graded manifold $\mathcal{O}_M = C^\infty(\mathcal{M})$

We decompose $C^\infty(\mathcal{M}) = \bigoplus_{m \geq 0} C_m(\mathcal{M})$, where $C_m(\mathcal{M})$ is functions of degree m.

Note : $C_0(\mathcal{M}) \oplus C_1(\mathcal{M})$ make a closed subalgebra by the bracket $\{-,-\}$ and the **derived bracket** $\{\{-,\Theta\},-\}$. (Count degree!)

Operations

The 'operations' on E is defined by **graded Poisson brackets** and **derived brackets** on $C_0(\mathcal{M}) \oplus C_1(\mathcal{M})$.

For $f, g \in C_0(\mathcal{M})$, $e, e_1, e_2 \in C_1(\mathcal{M})$, Poisson brackets

$C_0 \times C_0, \quad 0 = \{f, g\}$

$C_1 \times C_0, \quad 0 = \{e, f\}$

$C_1 \times C_1, \quad \langle e_1 \, , e_2 \rangle = \{e_1, e_2\}$ (inner product)

Derived brackets

$C_0 \times C_0 \to 0, \quad 0 = \{\{f, \Theta\}, g\}$

$C_1 \times C_0 \to C_0, \quad \rho(e) f = -\{\{e, \Theta\}, f\}$ (anchor map)

$C_1 \times C_1 \to C_1, \quad [e_1, e_2]_D = -\{\{e_1, \Theta\}, e_2\}$ (Dorfman bracket)

$\{\Theta, \Theta\} = 0$ gives identities on three operations in a Courant algebroid.

Conversely, any Courant algebroid induces a function Θ on $T{*}[2]E[1]$ such that $\{\Theta, \Theta\} = 0$.

Proposition 2. *Roytenberg '99*

A ***Courant algebroid*** *structure on E is equivalent to a QP-manifold (differential graded symplectic manifold) of degree 2 on $T^*[2]E[1]$.*

Standard Courant algebroid $TM \oplus T^*M$

An N-manifold is $\mathcal{M} = T^*[2]T[1]M$.

Local coordinates of $T[1]M$ are (x^i, q^i) of degree $(0, 1)$ and the conjugate coordinates are (ξ_i, p_i) of degree $(2, 1)$.

• A canonical graded symplectic form is of degree 2 (bidegree $(2, 2)$).

$$\omega = \delta x^i \wedge \delta \xi_i + \delta q^i \wedge \delta p_i.$$

$\{-, -\}$ of degree -2

• Q: the Hamiltonian function of degree 3,

$$\Theta = \xi_i q^i + \frac{1}{3!} H_{ijk}(x) q^i q^j q^k.$$

Impose

$$\{\Theta, \Theta\} = 0 \Longleftrightarrow \mathrm{d}H = 0.$$

Θ gives a QP-manifold if H is a closed 3-form.

Functions on graded manifold $\mathcal{O}_M = C^\infty(\mathcal{M})$

We decompose $C^\infty(\mathcal{M}) = \bigoplus_{m \geq 0} C_m(\mathcal{M})$, where $C_m(\mathcal{M})$ is functions of degree m.

degree 0: $f(x) \in C_0(\mathcal{M}) \simeq C^\infty(M)$,

degree 1: $X^i(x)p_i + \alpha_i(x)q^i \in C_1(\mathcal{M}) \leftrightarrow X + \alpha = X^i(x)\partial_i + \alpha_i(x)dx^i \in \Gamma(TM \oplus T^*M)$. i.e. $C_1(\mathcal{M}) \simeq \Gamma(TM \oplus T^*M)$, Poisson brackets

$C_0 \times C_0, \quad 0 = \{f, g\}$

$C_1 \times C_0, \quad 0 = \{e, f\}$

$C_1 \times C_1, \quad \langle e_1 \, , e_2 \rangle = \iota_X \beta + \iota_Y \alpha = \{e_1, e_2\}$ (inner product)

Derived brackets

$C_0 \times C_0 \to 0, \quad 0 = \{\{f, \Theta\}, g\}$

$C_1 \times C_0 \to C_0, \quad \rho(e) f = X f = -\{\{e, \Theta\}, f\}$ (anchor map)

$C_1 \times C_1 \to C_1, \quad [e_1, e_2]_D = [X, Y] + L_X \beta - \iota_Y d\alpha + \iota_X \iota_Y H = -\{\{e_1, \Theta\}, e_2\}$ (Dorfman bracket)

Example 1. *For general n, we call a* ***Lie n-algebroid*** *the structure on the vector bundle E induced by the derived brackets on the corresponding QP manifold of degree n.*

A ***Lie 1-algebroid*** *is a Lie algebroid (a Poisson structure).*

A ***Lie 2-algebroid*** *is the Courant algebroid.*

§5. BRST-BFV formalism of current algebras

NI-Koizumi '11, NI-Xu '13,

Fundamental theorems

Let $(\mathcal{M}, \omega, \Theta)$ be a QP-manifold of degree n. Let $pr : \mathcal{M} \to \mathcal{L}$ be a projection to a Lagrangian graded submanifold $\mathcal{L}$. We define a bilinear bracket on $\mathcal{L}$ as

$$\{f, g\}_{\mathcal{L}} := pr_* \{\{pr^* f, \Theta\}, pr^* g\},$$

for $f, g \in C^\infty(\mathcal{L})$.

Theorem 1. $\{-,-\}_{\mathcal{L}}$ *is a graded Poisson bracket of degree* $-n+1$ *on* $\mathcal{L}$.

Proof. Since the bracket $\{-,-\}$ is of degree $-n$, a bracket $\{-,-\}_{\mathcal{L}}$ is of degree $-n+1$. The derived bracket satisfies,

$$\begin{aligned}\{\{pr^*f,\Theta\},pr^*g\} &= -(-1)^{(|f|-n+1)(|g|-n+1)}\{\{pr^*g,\Theta\},pr^*f\} \\ &\quad -(-1)^{(|f|-n)(|g|-n)}\{\Theta,\{pr^*f,pr^*g\}\},\end{aligned}$$

for any function $f,g \in C^\infty(\mathcal{M})$. $\square$

Theorem 2. *A graded Poisson bracket of degree* 0 *is a normal Poisson bracket.*

Construction of Poisson brackets from supergeometric data Recall supergeometric data of the standard Courant algebroid.

1, $\mathcal{M} = T^*[2]T[1]M$

2, $\omega = \delta x^i \wedge \delta \xi_i + \delta q^i \wedge \delta p_i$

3, $\Theta = \xi_i q^i + \frac{1}{3!} H_{ijk}(x) q^i q^j q^k$.

- **Observation** in the target space computation.

For $(x^i, p_i) \in T^*[1]M$, the derived bracket is of degree -1 and

$$\{\{x^i, \Theta\}, x^j\} = 0,$$
$$\{\{x^i, \Theta\}, p_j\} = \delta^i{}_j,$$
$$\{\{p_i, \Theta\}, p_j\} = -H_{ijk}(x)q^k$$

Poisson brackets

$$\{x^i, x^j\}_{PB} = 0, \quad \{x^i, p_j\}_{PB} = \delta^i{}_j \delta(\sigma - \sigma'),$$
$$\{p_i, p_j\}_{PB} = -H_{ijk}(x)\partial_\sigma x^k \delta(\sigma - \sigma').$$

The simple Lagrangian submanifold $\{\xi_i = q^i = 0\}$ does not work.

Superfields

Next, we consider a mapping space $\mathrm{Map}(T[1]S^1, T^*[2]T[1]M)$.

Take the supermanifold $\mathcal{X} = T[1]S^1$ with local coordinates (σ, θ).

Local coordinates on $\mathrm{Map}(T[1]S^1, T^*[2]T[1]M)$ are superfields,
$\boldsymbol{x}^i(\sigma, \theta) : T[1]S^1 \to M$ of degree 0
$\boldsymbol{q}^i(\sigma, \theta) \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T_x[1]M))$ of degree 1

and canonical conjugates

$\boldsymbol{\xi}_i(\sigma, \theta) \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T^*_x[2]M))$ of degree 2,
$\boldsymbol{p}_i(\sigma, \theta) \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T^*_q[2]T_x[1]M))$ of degree 1.

QP on mapping space (AKSZ Construction)

Alexandrov, Kontsevich, Schwartz, Zaboronsky '97

- Graded symplectic form (BFV bracket) of degree 1

$$\boldsymbol{\omega} = -\delta\vartheta = \int_{\mathcal{X}} x^*\omega = \int_{\mathcal{X}} \mathrm{d}\sigma\mathrm{d}\theta(\delta\boldsymbol{x}^i \wedge \delta\boldsymbol{\xi}_i + \delta\boldsymbol{q}^i \wedge \delta\boldsymbol{p}_i),$$

- Hamiltonian function (BRST charge) of degree 2

$$\begin{aligned} S &= \int_{\mathcal{X}} (x^*\vartheta + x^*\Theta) \\ &= \int_{\mathcal{X}} \mathrm{d}\sigma\mathrm{d}\theta \left(-\boldsymbol{\xi}_i \mathrm{d}\boldsymbol{x}^i + \boldsymbol{p}_i \mathrm{d}\boldsymbol{q}^i\right) + \left(\boldsymbol{\xi}_i \boldsymbol{x}^i + \frac{1}{3!} H_{ijk}(\boldsymbol{x}) \boldsymbol{q}^i \boldsymbol{q}^j \boldsymbol{q}^k\right) \end{aligned}$$

Proposition 3. [Alexandrov-Kontsevich-Schwartz-Zaboronsky '97] *S satisfies $\{S, S\} = 0$, and $Q = \{S, -\}$ is a vector field of degree $+1$ such that $Q^2 = 0$.*

Note

• $(\mathrm{Map}(T[1]X, \mathcal{M}), \boldsymbol{\omega}, \boldsymbol{Q} = \{S, -\})$ is a QP-manifold of degree 1. Therefore, $\{-, -\}$ is of degree -1.

Note

• The derived bracket $\{\{-, S\}, -\}$ is of degree 0, therefore, $pr_*\{\{-, S\}-\}$ gives a normal Poisson bracket.

Twist and Lagrangian Submanifold

Definition 4. *Let $\alpha \in C^\infty(\mathcal{M})$ be a function of degree n. A twist e^{δ_α} is defined by $f' = e^{\delta_\alpha} f = f + \{f, \alpha\} + \frac{1}{2}\{\{f, \alpha\}, \alpha\} + \cdots$. e^{δ_α} is also called twisting.*

The formula: $\{e^{\delta_\alpha} f, e^{\delta_\alpha} g\} = e^{\delta_\alpha}\{f, g\}$.
If $\{\Theta, \Theta\} = 0$, $\{e^{\delta_\alpha}\Theta, e^{\delta_\alpha}\Theta\} = e^{\delta_\alpha}\{\Theta, \Theta\} = 0$ for any twisting.

Twisting

$pr : T^*[2]T^*[1]M \to T^*[1]M$

The derived bracket $pr_*\{\{-, \Theta\}, -\}$ induce the graded Poisson bracket on the Lagrangian submanifold $\mathcal{L}$ spanned by (x^i, p_i).

The Liouville 1-form on the Lagrangian submanifold $\widehat{\mathcal{L}}_0 = \mathrm{Map}(\mathcal{X}, \mathcal{L})$ is

$$\alpha_0 = -\int_{\mathcal{X}} \mathrm{d}\sigma \mathrm{d}\theta\, \boldsymbol{p}_i \mathrm{d}\boldsymbol{x}^i.$$

If we project $\mathcal{M}$ to the canonical Lagrangian submanifold $\widehat{\mathcal{L}}$ defined by $\boldsymbol{\xi}_i = \boldsymbol{q}^i = 0$ after twisting by α_0, we obtain a normal Poisson bracket,

$$\{-,-\}_{PB} = pr_* e^{\delta_{\alpha_0}} \{\{-, S_1\}, -\}.$$

In fact,

$$\{\boldsymbol{x}^i(\sigma,\theta),\boldsymbol{x}^j(\sigma',\theta')\}_{PB}=0,$$
$$\{\boldsymbol{x}^i(\sigma,\theta),\boldsymbol{p}_j(\sigma',\theta')\}_{PB}=-\delta^i{}_j\delta(\sigma-\sigma')\delta(\theta-\theta'),$$
$$\{\boldsymbol{p}_i(\sigma,\theta),\boldsymbol{p}_j(\sigma',\theta')\}_{PB}=-H_{ijk}(\boldsymbol{x}(\sigma,\theta))\mathrm{d}\boldsymbol{x}^k(\sigma,\theta)\delta(\sigma-\sigma')\delta(\theta-\theta').$$

Another construction: Zero locus condition

Arvanitakis '21

$\mathcal{L}$ is defined by the condition $\omega|_{\mathcal{L}}=0$

In our case, $\boldsymbol{q}^i=\mathrm{d}\boldsymbol{x}^i$ and $\boldsymbol{\xi}_i=\mathrm{d}\boldsymbol{p}_i$.

Physical Fields

We expand the superfields to component fields by the local coordinate θ on $T[1]S^1$,

$$\boldsymbol{x}^i(\sigma,\theta) = x^{(0)i}(\sigma) + \theta x^{(1)i}(\sigma), \qquad \boldsymbol{p}_i(\sigma,\theta) = p_i^{(0)}(\sigma) + \theta p_i^{(1)}(\sigma).$$

Here, $|\theta| = 1$, $|\boldsymbol{x}| = 0$ and $|\boldsymbol{p}| = 1$. Degree 0 components in the expansions are physical fields (and degree nonzero components are ghost fields).

Physical fields are $x^i(\sigma) = x^{(0)i}(\sigma)$ and $p_i(\sigma) = p_i^{(1)}(\sigma)$.

The Poisson brackets of the physical canonical quantities are degree zero components:

$$
\begin{aligned}
\{x^i(\sigma), x^j(\sigma')\}_{PB} &= 0, \\
\{x^i(\sigma), p_j(\sigma')\}_{PB} &= \delta^i{}_j \delta(\sigma - \sigma'), \\
\{p_i(\sigma), p_j(\sigma')\}_{PB} &= -H_{ijk}(x)\partial_\sigma x^k(\sigma)\delta(\sigma - \sigma'),
\end{aligned}
$$

which recover AS Poisson brackets.

Construction of currents from supergeometric data

$C^\infty(\mathcal{M}) = \bigoplus_{m\geq 0} C_m(\mathcal{M})$, where $C_m(\mathcal{M})$ is functions of degree m.

We take $C_0 \oplus C_1$ as space of currents.

$j_{0(f)} = f(x) \in C_0(\mathcal{M}) \simeq C^\infty(M)$,

$j_{1(X+\alpha)} = \alpha_i(x) q^i + X^i(x) p_i \in C_1(\mathcal{M})$

$\leftrightarrow \alpha + X = \alpha_i(x) dx^i + X^i(x)\partial_i \in \Gamma(TM \oplus T^*M)$.

The corresponding currents are

$$\boldsymbol{J}_{(0)(f)}(\epsilon_{(1)}) = pr_* e^{\delta_{\alpha_0}} \int_{T[1]S^1} \epsilon_{(1)} x^* j_{(0)(f)} = \int_{T[1]S^1} \mu \epsilon_{(1)}(\sigma, \theta) f(\boldsymbol{x}),$$

$$\boldsymbol{J}_{(1)(X+\alpha)}(\epsilon_{(0)}) = pr_* e^{\delta_{\alpha_0}} \int_{T[1]S^1} \epsilon_{(0)} \mathrm{ev}^* j_{(1)(u,\alpha)}$$

$$= \int_{T[1]S^1} \mu \epsilon_{(0)}(\sigma, \theta)(-\alpha_i(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}^i + X^i(\boldsymbol{x}) \boldsymbol{p}_i),$$

where $\epsilon_{(m)} = \epsilon_{(m)}(\sigma, \theta)$ is a test function of degree m on the super circle $\mathcal{X} = T[1]S^1$. The integrands of degree zero components of

$\boldsymbol{J}_0$ and $\boldsymbol{J}_1$ are

$$J_{(0)(f)}(\sigma) = f(x(\sigma)),$$
$$J_{(1)(X+\alpha)}(\sigma) = \alpha_i(x)\partial_\sigma x^i(\sigma) + X^i(x)p_i(\sigma),$$

which are the correct AS currents.

Righthand side of the current algebra

Current algebra is obtained from two computations of the derived bracket on the mapping space,

$$
\begin{aligned}
& \{\{\boldsymbol{J}_a, S_1 + S_0\}, \boldsymbol{J}_b\}|_{\mathrm{Map}(\mathcal{X},\mathcal{L})}, \\
= \ & \{\{\boldsymbol{J}_a, S_1\}, \boldsymbol{J}_b\}|_{\mathrm{Map}(\mathcal{X},\mathcal{L})} + \{\{\boldsymbol{J}_a, S_0\}, \boldsymbol{J}_b\}|_{\mathrm{Map}(\mathcal{X},\mathcal{L})},
\end{aligned}
$$

and twisting.

We compute the Poisson algebra of these supergeometric currents from the Poisson brackets of canonical quantities $(\boldsymbol{x}^i, \boldsymbol{p}_i)$,

$$
\begin{aligned}
\{\boldsymbol{J}_{0(f)}(\epsilon), \boldsymbol{J}_{0(g)}(\epsilon')\}_{PB} &= 0, \\
\{\boldsymbol{J}_{1(X+\alpha)}(\epsilon), \boldsymbol{J}_{0(g)}(\epsilon')\}_{PB} &= \rho(X+\alpha)\boldsymbol{J}_{0(g)}(\epsilon\epsilon'), \\
\{\boldsymbol{J}_{1(X+\alpha)}(\epsilon), \boldsymbol{J}_{1(Y+\beta)}(\epsilon')\}_{PB} &= \boldsymbol{J}_{1([X+\alpha, Y+\beta]_H)}(\epsilon\epsilon') \\
&+ \int_{T[1]S^1} \mu \, \mathrm{d}\epsilon_{(0)}\epsilon'_{(0)} \langle X+\alpha \, , Y+\beta \rangle(\boldsymbol{x}),
\end{aligned}
$$

where $\boldsymbol{J}'_{0(g)} = \int_{T[1]S^1} \mu\epsilon_{(1)} g(\boldsymbol{x})$, $\boldsymbol{J}'_{1(Y+\beta)} = \int_{T[1]S^1} \mu\epsilon_{(0)}(-\beta_i(\boldsymbol{x})\mathrm{d}\boldsymbol{x}^i + Y^i(\boldsymbol{x})\boldsymbol{p}_i)$. The AS current algebra is given by degree zero components.

§6. General BFV formalism of current algebras

Data

$(\mathcal{X}, D, \mu)$: Here $\mathcal{X} = T[1]\Sigma$, where $\Sigma \times \boldsymbol{R}$ is an n dimensional manifold. D is a differential on $\mathcal{X}$. μ is a D-invariant nondegenerate Berezin measure.
$(\mathcal{M}, \omega, \Theta, \mathcal{L})$: A QP manifold of degree n and a Lagrangian submanifold.

Assume $\{-,-\}_{\mathcal{L}} = pr_*\{\{-,\Theta\},-\}$ is nondegenerate.

We consider the induced symplectic structure $\omega_{\mathcal{L}}$ defined from $\{-,-\}_{\mathcal{L}}$ on $\mathcal{L}$.

Twisting by canonical 1-form on Lagrangian submanifold

Let $\vartheta_{\mathcal{L}}$ be the canonical 1-form for $\omega_{\mathcal{L}}$ such that $\omega_{\mathcal{L}} = -\delta\vartheta_{\mathcal{L}}$.

The function α_0 on the mapping space is

$$\alpha_0 = \int_{\mathcal{X}} x^* \vartheta_s.$$

Poisson bracket

$$\{-,-\}_{PB} = pr^* e^{\delta\alpha_0}\{\{-,S\},-\}.$$

is a graded Poisson bracket of degree 0, therefore, a normal Poisson bracket.

Current functions on target space $\mathcal{M}$

We consider a closed subalgebra of the structure sheaf $C^\infty(\mathcal{M})$ not only under the Poisson bracket $\{-,-\}$, but also under the derived bracket $\{\{-,\Theta\},-\}$:

$$C^{(n-1)}(\mathcal{M}) = \bigoplus_{m=0}^{n-1} C_m(\mathcal{M}) = \{f \in C^\infty(\mathcal{M}) | |f| \leq n-1\}$$

Currents

Pullbacks of $j_m \in C^{(n-1)}(\mathcal{M})$ to $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})$, is a BFV current

$$\boldsymbol{J}_m(\epsilon) = pr_* e^{\delta_{\alpha_0}} \int_{\mathcal{X}} \epsilon\, x^* j_m,$$

where ϵ is a test function on $T[1]\Sigma_{n-1}$ of degree $n-1-m$.

$$\mathcal{CA}^{(n-1)}(\mathcal{X},\mathcal{M}) = \{\boldsymbol{J}_m\}.$$

Supergeometic (BFV) current algebras

The (graded) Lie algebra on the space $\mathcal{CA}^{(n-1)}(\mathcal{X},\mathcal{M})$ is a current algebra.

Theorem 3. *NI-Xu '13,*

For currents $\boldsymbol{J}_{j_1}$ and $\boldsymbol{J}_{j_2}$ associated to current functions j_1, j_2 $\in C^{(n-1)}(\mathcal{M})$ respectively, the commutation relation is given by

$$\{\boldsymbol{J}_{j_1}(\epsilon_1), \boldsymbol{J}_{j_2}(\epsilon_2)\}_{PB} = -pr_* e^{\delta_{\alpha_0}} \left\{ \left\{ \int_{\mathcal{X}} \epsilon_1 \mathrm{ev}^* j_1, S_1 + S_0 \right\}, \int_{\mathcal{X}} \epsilon_2 \mathrm{ev}^* j_2 \right\}$$

$$= -\boldsymbol{J}_{[j_1,j_2]_D}(\epsilon_1\epsilon_2) - pr_* e^{\delta_{\alpha_0}} \int_{\mathcal{X}} (\mathrm{d}\epsilon_1)\epsilon_2 \mathrm{ev}^* \{j_1, j_2\}$$

Physical Currents

For a superfield, introduce the second degree, the **form degree** $\deg f$, which is the order of θ^μ, where $|\theta^\mu| = 1$. $\mathrm{gh}\ f = |f| - \deg f$ is called the **ghost number**.

We define a physical current $J_{cl} = \boldsymbol{J}|_{\mathrm{gh} f = 0}$.

§. Results, Conclusions and Future Outlook

• We construct a general formulation of current algebras based on supergeometry.

• $n-1$ dimensional current algebras have a structure of a Lie n-algebroid (a QP-manifold of degree n).

• Current algebras connected to D-branes have this structure. Besho-Heller-NI-Watamura '15, Arvanitakis '21

• PVA versions based on (higher) Courant-Dorfman algebra is formulated. Ekstrand '11, Hayami '23

Outlook

- Applications to string theory and gauge theories

- Quantization problem

Deformation, Geometric, Path integral $\cdots$ anomaly terms in current algebras.

- Generalizations to currents with higher derivatives of delta functions

- General Poisson vertex algebras and vertex algebras Li '02, De Sole-Kac-Wakimoto '10, Ekstrand-Heluani-Zabzine '11, Hekmati-Mathai '12

Thank you for your attention!

§8. Brane Current Algebras

As one example, we construct a new current algebra of Alekseev-Strobl type on a $2+1$ dimensional manifold $X_3 = \Sigma_2 \times \boldsymbol{R}$.

Target space

A classical phase space: $(x^i(\sigma), q^a(\sigma)) \in \mathrm{Map}(\Sigma_2, T^*M) \oplus \mathrm{Map}(T\Sigma_2, T^*E)$.

1, $T^*[3]\mathcal{L} = T^*[3]T^*[2]E[1]$, a graded symplectic manifold

Take local coordinates (x^i, q^a, p_i) of degree $(0, 1, 2)$ on $T^*[2]E[1]$, and conjugate Darboux coordinates (ξ_i, η^a, χ^i) of degree $(3, 2, 1)$.

2, A graded symplectic structure on $T^*[3]\mathcal{L}$ is

$$\omega_b = \delta x^i \wedge \delta\xi_i - k_{ab}\delta q^a \wedge \delta\eta^B + \delta p_i \wedge \delta\chi^i,$$

where k_{ab} is a fiber metric on E.

3, We take the following function

$$\Theta = \chi^i\xi_i + \frac{1}{2}k_{ab}\eta^a\eta^b + \frac{1}{4!}H_{IJKL}(x)\chi^i\chi^J\chi^K\chi^L.$$

$\{\Theta, \Theta\} = 0$ if $\mathrm{d}H = \mathrm{d}k = 0$.

A Lagrangian submanifold $\mathcal{L} = T^*[2]E[1]$ spanned by (x^i, q^a, p_i).

Lie 3-Algebroid

NI, Uchino '10

Let $F = E \oplus TM$ and $F^* = E \oplus T^*M$.

$C_0(\mathcal{M}) = C^\infty(M)$.
$C_1(\mathcal{M}) = \Gamma(F^*)$ which is spanned by degree one basis q^a, χ^i.
$C_2(\mathcal{M}) = \Gamma(F \oplus \wedge^2 F^*)$ which is spanned by degree two basis $p_i, \eta^a, q^a q^b, q^a \chi^i, \chi^i \chi^J$, etc.

Let $f \in C_0(\mathcal{M})$, $t, t_1, t_2, \cdots \in C_1(\mathcal{M})$ and $s, s_1, s_2, \cdots \in C_2(\mathcal{M})$.

A graded Poisson bracket

induces a natural pairing $\Gamma(F^*) \times \Gamma(F \oplus \wedge^2 F^*) \to \boldsymbol{C}$, and a bilinear form on $\Gamma(F \oplus \wedge^2 F^*) \times \Gamma(F \oplus \wedge^2 F^*) \to \Gamma(F^*)$,

$$\langle t\,, s\rangle \;=\; \{t, s\}, \quad \langle s_1\,, s_2\rangle = \{s_1, s_2\}.$$

Derived brackets

$C_1 \times C_1 \to \boldsymbol{C}$, a bilinear symmetric form on ΓF^*,

$$(t_1, t_2) \;=\; \{\{t_1, \boldsymbol{\Theta}\}, t_2\}.$$

$C_2 \times C_0 \to C_0$, an anchor map $\rho : F \oplus \wedge^2 F^* \to TM$ is

$$\rho(s) f(x) \;=\; -\{\{s, \boldsymbol{\Theta}\}, f(x)\}.$$

$C_2 \times C_1 \to C_1$, a Lie type derivative $L : (F \oplus \wedge^2 F^*) \times F^* \to F^*$ is

$$\mathcal{L}_s t \;\; = \;\; -\{\{s, \Theta\}, t\}.$$

$C_2 \times C_2 \to C_2$, a higher Dorfman bracket on $\Gamma F \oplus \wedge^2 F^*$,

$$[s_1, s_2]_3 \;\; = \;\; -\{\{s_1, \Theta\}, s_2\}.$$

Definition 5. *We impose* $\{\Theta, \Theta\} = 0$. *Then* $(E, \langle - \,, - \rangle, (-, -), \rho, \mathcal{L}, [-, -]_3)$ *is called a Lie* 3*-algebroid.*

Superfields

$(\boldsymbol{x}^i(\sigma,\theta), \boldsymbol{q}^a(\sigma,\theta), \boldsymbol{p}_i(\sigma,\theta))$ and $(\boldsymbol{\xi}_i(\sigma,\theta), \boldsymbol{\eta}^a(\sigma,\theta), \boldsymbol{\chi}^i(\sigma,\theta))$ are corresponding superfields.

Poisson brackets

Derived bracket

$$\begin{aligned}
\{\{x^i, \Theta\}, p_j\} &= \delta^i{}_j, \\
\{\{q^a, \Theta\}, q^b\} &= k^{ab}, \\
\{\{p_i, \Theta\}, p_j\} &= -\frac{1}{2} H_{ijkl}(x) \chi^k \chi^l.
\end{aligned}$$

A canonical 1-form is $\alpha_0 = \int_{\mathcal{X}} \mu \, (\delta \boldsymbol{x}^i \wedge \delta \boldsymbol{p}_i + \frac{1}{2} k_{ab} \delta \boldsymbol{q}^a \wedge \delta \boldsymbol{q}^b)$.

These derive the Poisson brackets for canonical conjugates

$$
\begin{aligned}
\{\boldsymbol{x}^i(\sigma,\theta), \boldsymbol{p}_j(\sigma',\theta')\}_{PB} &= \delta^i{}_j \delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
\{\boldsymbol{q}^a(\sigma,\theta), \boldsymbol{q}^b(\sigma',\theta')\}_{PB} &= k^{ab} \delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
\{\boldsymbol{p}_i(\sigma), \boldsymbol{p}_j(\sigma')\}_{PB} &= -\frac{1}{2} H_{ijkl} \mathrm{d}\boldsymbol{x}^k \mathrm{d}\boldsymbol{x}^l \delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),
\end{aligned}
$$

Currents

Let us consider the space of functions of degree equal to or less than 2. $C_2(T^*[3]T^*[2]E[1]) = \{f \in C^\infty(T^*[3]T^*[2]E[1]) || f| \leq 2\}$. General functions of $C_2(T^*[3]T^*[2]E[1])$ of degree 0, 1 and 2 are

$$\begin{aligned} J_{(0)(f)} &= f(x), \\ J_{(1)(a,u)} &= a_i(x)\chi^i + u_a(x)q^a, \\ J_{(2)(G,K,F,B,E)} &= G^i(x)p_i + K_a(x)\eta^a + \frac{1}{2}F_{ab}(x)q^a q^b \\ &\quad + \frac{1}{2}B_{ij}(x)\chi^i\chi^J + E_{ai}(x)\chi^i q^a. \end{aligned}$$

Next we pullback and twist current functions with canonical 1-form, $\alpha_0 = \int_{T[1]\Sigma_2} \mu \left(-\boldsymbol{p}_i \mathrm{d}\boldsymbol{x}^i + \frac{1}{2} k_{ab} \boldsymbol{q}^a \mathrm{d}\boldsymbol{q}^b \right)$. Currents of degree 0, 1 and 2 are

$$
\begin{aligned}
\boldsymbol{J}_{(0)(f)} &= \int_{T[1]\Sigma_2} \mu \epsilon_{(2)} f(\boldsymbol{x}), \\
\boldsymbol{J}_{(1)(a,u)} &= \int_{T[1]\Sigma_2} \mu \epsilon_{(1)} (a_i(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}^i + u_a(\boldsymbol{x}) \boldsymbol{q}^a), \\
\boldsymbol{J}_{(2)(G,K,F,B,E)}(\sigma, \theta) &= \int_{T[1]\Sigma_2} \mu \epsilon_{(0)} (G^i(\boldsymbol{x}) \boldsymbol{p}_i + K_a(\boldsymbol{x}) \mathrm{d}\boldsymbol{q}^a \\
&+ \frac{1}{2} F_{ab}(\boldsymbol{x}) \boldsymbol{q}^a \boldsymbol{q}^b + \frac{1}{2} B_{ij}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}^i \mathrm{d}\boldsymbol{x}^j + E_{ai}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}^i \boldsymbol{q}^a).
\end{aligned}
$$

The current algebra is given by

$$\{\{\boldsymbol{J}_a, S_1 + S_0\}, \boldsymbol{J}_b\}|_{\mathrm{Map}(\mathcal{X},\mathcal{L})}$$

$$\{\boldsymbol{J}_{(0)(f)}(\epsilon), \boldsymbol{J}_{(0)(f')}(\epsilon')\}_{P.B.} = 0,$$

$$\{\boldsymbol{J}_{(1)(u,a)}(\epsilon), \boldsymbol{J}_{(0)(f')}(\epsilon')\}_{P.B.} = 0,$$

$$\{\boldsymbol{J}_{(2)(G,K,F,H,E)}(\epsilon), \boldsymbol{J}_{(0)(f')}(\epsilon')\}_{P.B.} = -G^I \frac{\partial \boldsymbol{J}_{(0)(f')}}{\partial \boldsymbol{x}^i}(\epsilon\epsilon'),$$

$$\{\boldsymbol{J}_{(1)(u,a)}(\epsilon), \boldsymbol{J}_{(1)(u',a')}(\epsilon')\}_{P.B.} = -\int_{T[1]\Sigma_2} \mu\epsilon_{(1)}\epsilon'_{(1)}\mathrm{ev}^* k^{ab} u_a u'_b,$$

$$\{\boldsymbol{J}_{(2)(G,K,F,B,E)}(\epsilon), \boldsymbol{J}_{(1)(u',a')}(\epsilon')\}_{P.B.}$$

$$= -\boldsymbol{J}_{(1)(\bar{u},\bar{\alpha})}(\epsilon\epsilon') - \int_{T[1]\Sigma_2} \mu(\mathrm{d}\epsilon_{(0)})\epsilon'_{(1)}(G^i\alpha'_i - k^{ab}K_a u'_b),$$

$$\{\boldsymbol{J}_{(2)(G,K,F,B,E)}(\epsilon), \boldsymbol{J}_{(2)(G',K',F',B',E')}(\epsilon')\}_{P.B.} = -\boldsymbol{J}_{(2)(\bar{G},\bar{K},\bar{F},\bar{B},\bar{E})}(\epsilon\epsilon')$$

$$-\int_{T[1]\Sigma_2} \mu(\mathrm{d}\epsilon_{(0)})\epsilon'_{(0)} \left[(G^j B'_{ji} + G'^j B_{ji} + k^{ab}(K_a E'_{bi} + E_{ai}K'_b))\mathrm{d}\boldsymbol{x}^i \right.$$

$$\left. +(G^i E'_{ai} + G'^I E_{ai} + k^{bc}(K_B F'_{ac} + F_{ac}K'_b))\boldsymbol{q}^a\right].$$

Here

$$\bar{\alpha} = (i_G d + d i_G)\alpha' + \langle E - dK, u'\rangle, \quad \bar{u} = i_G du' + \langle F, u'\rangle,$$
$$\bar{G} = [G, G'],$$
$$\bar{K} = i_G dK' - i_{G'} dK + i_{G'} E + \langle F, K'\rangle,$$
$$\bar{F} = i_G dF' - i_{G'} dF + \langle F, F'\rangle,$$
$$\bar{B} = (d i_G + i_G d)B' - i_{G'} dB + \langle E, E'\rangle + \langle K', dE\rangle - \langle dK, E'\rangle + i_{G'} i_G H,$$
$$\bar{E} = (d i_G + i_G d)E' - i_{G'} dE + \langle E, F'\rangle - \langle E', F\rangle + \langle dF, K'\rangle - \langle dK, F'\rangle,$$

where all the terms are evaluated by σ'. Here $[-,-]$ is a Lie bracket on TM, i_G is an interior product with respect to a vector field G and $\langle -,-\rangle$ is the graded bilinear form on the fiber of E with respect to the metric k^{AB}.