Particles and $p-$adic integrals of Spin $\left(\frac{1}{2}\right)$:Spin Lie group, $\mathcal{R}(\rho, q)-$gamma and $\mathcal{R}(\rho, q)-$beta Functions, Ghost and Applications

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Outline

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Motivation

- The concept of $p-$adic analysis has many applications in Mathematics and Physics, especially, in string theory;for instance the tree level effective action in the world-volume theory on the $D-$brance of Bosonic string which contains a tachyonic mode.
- General $n-$point massless $p-$adic Feynman amplitude was computed by V. A. Smirnov.
- Dragovich. Aref'eva, Frampton and Volovich considered the $p-$adic gravity and the wave function of the Universe.
- Dragovich goes further to research into what he terms as $p-$adic Matter.
- Zabrodin found that the world sheet of the $p-$adic string is a Bruhat-Tits tree.

For more applications of $p-$adics to string theory see; "$p-$adic Strings Then and Now" by Peter G.O. Freund

Motivation

In this talk we shall try to answer the following questions:

(1) Are Fermion Spin Lie groups $p-$adic?

(2) What are the Iwasawa algebras of the $\mathfrak{spin}(\frac{1}{2})$?

(3) Can one define the $p-$adic zeta function for $\mathfrak{spin}(\frac{1}{2})$?

(4) Can one extend the $p-$adic integral (Fermionic and Bosonic due to T. Kim) quantum calculus to $\mathcal{R}(p,q)-$deformation developed by Hounkonnou?

We shall consider some examples at the end of this talk and see how they relate to $p-$adic amplitude string theory.

Motivation

For a comprehensive review see; $p-$adic String Amplitudes and Ads/CFT Correspondence by Edgar Y. L. Hernandez.

Y. Morita's $p-$adic gamma function was used by Hamza Menken and Özge Çolakoglu 2015 to consider a $p-$adic analogue of the classical beta function. Some fundamental properties of the $p-$adic beta function were discovered, as well as some relationships between the classical beta and $p-$adic beta functions at natural number values. Duran and Acikgoz 2019 also extended these results to the $(\rho, q)-$gamma and $(\rho, q)-$beta functions.

p-Adic Spin Lie Group

Spin Lie group

The Lie algebra $\mathfrak{spin}(j)$ of spin particles can be represented by classical matrices, which makes it easier to see their algebraic nature:

$$\mathfrak{spin}(j) = \begin{cases} \text{higgs} & j = 0; \\ \text{fermions} & j = \frac{\mathbb{Z}}{2} \text{ when odd integer spins are considered;} \\ \text{bosons} & j = \mathbb{Z} \text{ when positive integer spins are considered.} \end{cases}$$

The Lie algebra $\mathfrak{sl}(2n, \mathbb{C})$ can represent the fermion spin Lie algebra of elementary particles in quantum physics. As indicated in the mapping below. We define $\frac{\mathbb{Z}}{2}$ as fraction of the form $(\frac{2k+1}{2},\ k = 0, 1, 2, 3, \ldots)$:

$$\mathfrak{sl}(2n, \mathbb{C}) \longrightarrow \mathfrak{spin}\left(\frac{\mathbb{Z}}{2}\right) \longrightarrow \text{fermions.}$$

Spin Lie group

- The Lie group $\mathrm{SL}(2n,\mathbb{C})$ structure can represent the fermion Spin Lie group analogue:

$$\mathrm{SL}(2n,\mathbb{C}) \longrightarrow \mathrm{Spin}(\tfrac{\mathbb{Z}}{2}) \longrightarrow \text{fermions},$$

while the Lie group $\mathrm{SL}(2n+1,\mathbb{C})$ represents the boson Spin Lie group analogue:

$$\mathrm{SL}(2n+1,\mathbb{C}) \longrightarrow \mathrm{Spin}(\mathbb{Z}) \longrightarrow \text{bosons}.$$

- We denote by p a odd-prime number and by $\mathbb{Q}_p$ the field of p-adic numbers with the normalized absolute value $\mid x \mid$, $x \in \mathbb{Q}_p$. The proper quotient rings of $\mathbb{Z}_p$ are the familiar finite rings $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}_p$. $\mathbb{Z}_p$ can be regarded as the inverse limit of $\mathbb{Z}_p/p^n\mathbb{Z}_p$. We also put $\mathbb{Q}_p^* = \mathbb{Q}_p - \{0\}$ as the locally compact

Hamilton spin Lie group

Definition 1 [V. Drinfeld]

Let the map $\mu : G \times G \longrightarrow G$ defined by $\mu(x, y) = xy$ be grouped, if it is Hamiltonian then G is said to have a grouped Hamiltonian structure.

Definition 2 R. Hooke

A Spin Lie group G together with a grouped Hamiltonian structure on it will be called a Hamilton-Spin Lie group.

Example 1

The elementary spin particles Lie groups such as Fermions Spin Lie groups:

$$\text{Spin}\left(\frac{1}{2}\right), \text{Spin}\left(\frac{3}{2}\right), \text{Spin}\left(\frac{5}{2}\right), \text{Spin}\left(\frac{7}{2}\right), \cdots$$

and Bosons $\text{Spin}(1), \text{Spin}(2)$ are all Hamilton-Spin Lie groups.

$p-$adic Spin Lie group

Definition 3

A p-adic Spin Lie group is a Hamilton-Spin Lie group with a fermionic Spin Lie group structure.

p-Adic Spin Lie group

Fermionic p-adic spin Lie algebra

Proposition 1 Y. Morita

The fermionic p-adic spin Lie algebra, $\mathfrak{spin}_{\mathbb{Z}_p}(\frac{1}{2})$ can be generated by

1. the elements

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

2. the commutation relations are given by:

$$[S_z, S_+] = 2\hbar S_+, \quad [S_z, S_-] = -2\hbar S_-, \quad [S_+, S_-] = \hbar S_z.$$

p-Adic spin Lie group

Iwasawa Algebras Lazard, Ray

Proposition 2

For the any $\mathfrak{spin}_{\mathbb{Z}_p}(\frac{1}{2})$ there exists elements $S_+, S_-,$ and S_z which generate the Iwasawa algebras when we set $\hbar = 1$.

The principal congruence subgroup

Ilani

The principal congruence subgroup of $\mathrm{Spin}_{\mathbb{Z}_p}\left(\frac{1}{2}\right)$ where p is an odd prime. We put

$$G_i := \left[\ker \mathrm{Spin}_{\mathbb{Z}_p}\left(\frac{1}{2}\right) \to \mathrm{Spin}_{(\mathbb{Z}_p/p^i\mathbb{Z})}\left(\frac{1}{2}\right)\right]$$
$$= \left\{I + A \mid \det(I+A) = 1 \quad \textit{and} \quad A \in M_2(p^i\mathbb{Z}_p)\right\}.$$

$$L_i := \mathfrak{spin}_{(p^i\mathbb{Z}_p)}\left(\frac{1}{2}\right) = \left\{A \in M_2(p^i\mathbb{Z}_p) \mid \mathrm{Tr}(A) = 0\right\}.$$

If p is an odd prime then for all $i \leq 1$, $G_i = \log(L_i)$; $\quad L_i = \exp(G_i)$ it follows immediately that $G_{i+1} = G_i^p$ and G_i is a uniform pro-p group of dimension 3.

Principal Congruence Subgroup

Now we set $\hbar = 1$ for the $\mathfrak{spin}_{(p^i\mathbb{Z}_p)}\left(\frac{1}{2}\right)$ and choose a basis which contains a base for the Cartan subalgebra, and a base for each of the root spaces such as

1. the elements

$$S_- = p^i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_z = \frac{p^i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_+ = p^i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

2. The commutation relations of such basis are:

$$[S_z, S_+] = 2p^i S_+, \quad [S_z, S_-] = -2p^i S_-, \quad [S_+, S_-] = p^i S_z$$

Igusa Zeta Function

and the mapping

$$(xS_- + yS_z + zS_+) \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a bijection of $\mathfrak{spin}_{(p^i\mathbb{Z}_p)}\left(\frac{1}{2}\right)$ and $(\mathbb{Z}_p)^3$.

Theorem 1 Denef, Sautoy, Ilani

Let $\mathcal{L}$ be a 3-dimensional $\mathbb{Z}_p$ - fermionic spin Lie algebra. Then there is a ternary quadratic form $f(x) \in \mathbb{Z}_p[x_1, x_2, x_3]$ unique up to equivalence, such that, for $i \leq 0$,

$$\zeta_{p^i,\mathcal{L}}(s) = \zeta_{\mathbb{Z}_p^3}(s) - Z_f(s-2)\zeta_p(2s-2)p^{(2-s)(i+1)}(1-p^{-1})^{-1},$$

where $Z_f(s)$ is Igusa's local zeta function associated to f.

Zeta function of fermion $\mathfrak{spin}_{\mathbb{Z}_p}\left(\frac{1}{2}\right)$ Lie algebra

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$$\mathfrak{spin}_{\mathbb{Z}_p}\left(\frac{1}{2}\right) = \mathbb{Z}_p S_- \oplus \mathbb{Z}_p S_+ \oplus \mathbb{Z}_p S_z, \qquad (4.1)$$

where

$$[S_z, S_+] = 2S_+, \quad [S_z, S_-] = -2S_-, \quad [S_+, S_-] = S_z.$$

We obtain

$$R(x) = \begin{pmatrix} & x_3 & -2x_1 \\ -x_3 & & 2x_2 \\ 2x_1 & -2x_2 & \end{pmatrix}.$$

The ternary quadratic form is given by: $f(x) = x_3^2 + 4x_1x_2$. Specifically for a fermion when $p = 2$ the function f defines a smooth conic in a projective 2-space which has a good reduction

Igusa Local Zeta function

Using the Igusa local zeta function we observe that;

$$Z_f(s-2) = \frac{(1-p^{-1})(1-p^{-1}t)}{(1-pt^2)(1-pt)}.$$

From Theorem 1 we obtain the formula

$$\begin{aligned} \zeta_{\mathfrak{spin}_{\mathbb{Z}_p}(\frac{1}{2})}(s) &= \zeta_{\mathbb{Z}_p^3} - \frac{(1-p^{-1}t)p^2t}{(1-pt)(1-p^2t)(1-pt^2)(1-p^2t^2)} \quad (4.2) \\ &= \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-1)\zeta_p(2s-2)\zeta_p(3s-1)^{-1}. \end{aligned}$$

Ghost polynomials of particles

Theorem 2 M. Du Sautoy et al.

The only ghost polynomials associated with the fermionic $\mathrm{Spin}_{\mathbb{Z}_p}(\frac{1}{2})$ Lie groups are the GO_{2l+1}, GSp_{2l} or GO^{+}_{2l} of type $\Pi(B_l)$, $\Pi(C_l)$ and $\Pi(D_l)$ respectively.

Theorem 3

Every fermionic ghost polynomial, GO_{2l+1}, GSp_{2l} and GO_{2l} of type $\Pi(B_l)$, $\Pi(C_l)$ and $\Pi(D_l)$ is a friendly ghosts.

Hounkonnou *et al.* $\mathcal{R}(p,q)-$deformed quantum algebras 18

$\mathcal{R}(p,q)-$deformed quantum algebras

Let p and q be two positive real numbers such that $0 < q < p \leq 1$. We consider a meromorphic function $\mathcal{R}$ defined on $\mathbb{C}\times\mathbb{C}$ by:

$$\mathcal{R}(u,v) = \sum_{s,t=-l}^{\infty} r_{st} u^s v^t, \tag{4.3}$$

with an eventual isolated singularity at the zero, where r_{st} are complex numbers, $l \in \mathbb{N}\cup\{0\}$, $\mathcal{R}(p^n,q^n) > 0, \forall n \in \mathbb{N}$, and $\mathcal{R}(1,1) = 0$ by definition. We denote by $\mathbb{D}_R$ the bi-disk

$$\begin{aligned} \mathbb{D}_R &:= \prod_{j=1}^{2} \mathbb{D}_{R_j} \\ &= \left\{ w = (w_1,w_2) \in \mathbb{C}^2 : |w_j| < R_j \right\}, \end{aligned}$$

Hounkonnou and Bukweli $\mathcal{R}(p,q)-$Derivative

defined by Hadamard formula as follows:

$$\lim \sup_{s+t\longrightarrow\infty} \sqrt[s+t]{|r_{st}|R_1^s\, R_2^t} = 1.$$

We define the following linear operators defined on $\mathcal{O}(\mathbb{D}_R)$

$$\begin{aligned} Q:\Psi \longmapsto Q\Psi(z): &= \Psi(qz),\\ P:\Psi \longmapsto P\Psi(z): &= \Psi(pz),\\ \partial_{p,q}:\Psi \longmapsto \partial_{p,q}\Psi(z): &= \frac{\Psi(pz)-\Psi(qz)}{(p-q)z} \end{aligned}$$

$\mathcal{R}(p,q)-$ derivative given by:

$$\partial_{\mathcal{R}(p,q)} := \partial_{p,q}\frac{p-q}{P-Q}\mathcal{R}(P,Q) = \frac{p-q}{p^P-q^Q}\mathcal{R}(p^P,q^Q)\partial_{p,q}(5.1)$$

Theorem

Theorem 4

For an arbitrary function f, and a real number a, we set

$$\int_0^a f(z)d_{\mathcal{R}(p,q)}z = \frac{q^Q - p^P}{q^n - p^n}\frac{(q-p)a}{\mathcal{R}(q^Q,p^P)}\sum_{r=0}^{n} p^r/q^{r+1}f(p^r/q^{(r+1)}a) \quad (5.2)$$

if $\left|\frac{p}{q}\right| < 1 \leq \phi_1.$

$$\int_0^a f(z)d_{\mathcal{R}(p,q)}z = \frac{p^P - q^Q}{p^n - q^n}\frac{(p-q)a}{\mathcal{R}(p^P,q^Q)}\sum_{r=0}^{n} q^r/p^{r+1}f(q^r/p^{(r+1)}a) \quad (5.3)$$

if $\left|\frac{q}{p}\right| < 1 \leq \phi_2.$

$(p,q)-$ Jagannathan-Srinivasa derivative

Particular Cases

The following conditions can help retrieve some relevant $q-$analogues and $(p,q)-$analogues:

(a) $q-$Heine derivative, $q-$ Quesne derivative, $q-$ Biedenharn-Macfarlane derivative, $(p^{-1},q)-$ Chakrabarty - Jagannathan derivative, Hounkonnou-Ngompe generalization of $q-$ Quesne derivative and

(b) Setting $\mathcal{R}(p,q)=1$, we have the $(p,q)-$ Jagannathan-Srinivasa derivative:

$$\partial_{p,q}\Psi(z)=\frac{\Psi(pz)-\Psi(qz)}{z(p-q)}.$$

$\mathcal{R}(p,q)-$Gamma function

Definition 4 [Sadjang]

Let z be a complex number, we define the $\mathcal{R}(p,q)-$Gamma function as

$$\Gamma_{\mathcal{R}(p,q)}(z) = \frac{(\xi_1 \ominus \xi_2)^{\infty}_{\mathcal{R}(p,q)}}{(\xi_1^z \ominus \xi_2^z)^{\infty}_{\mathcal{R}(p,q)}}(\xi_1 - \xi_2)^{1-z} \tag{5.4}$$

for $0 < \xi_2 < \xi_1$, with $\xi_1, \xi_2 \in \xi_i, \quad i = 1, 2$. Also, if $\xi_1 = p \quad$ and $\quad \xi_2 = q$ with $\mathcal{R}(p,q) = 1$ and $0 < q < p$, one obtains the $\Gamma_{p,q}(z)$ function. This further reduces to $\Gamma_q(z)$ function, if we set $p = 1$.

The $\mathcal{R}(p,q)-$power basis and the $\mathcal{R}(p,q)-$ factorial

$$[n]_{\mathcal{R}(p,q)}! = \frac{(\xi_1 \ominus \xi_2)^{n}_{\mathcal{R}(p,q)}}{(\xi_1 - \xi_2)^n}.$$

$\mathcal{R}(p,q)-$Power Basis

Definition 5 [Sadjang]

Let n be a non-negative integer. The $\mathcal{R}(p,q)-$analogue of subtraction are defined by

(i) $(x\ominus a)^n_{\mathcal{R}(p,q)} = (x-a)(x\xi_1 - a\xi_2)(x\xi_1^2 - a\xi_2^2)\cdots(x\xi_1^{n-1} - a\xi_2^{n-1})$

(ii) $(x\ominus a)^{-n}_{\mathcal{R}(p,q)} = \dfrac{1}{(x\xi_1^{-n}\ominus \xi_2^{-n}a)^n_{\mathcal{R}(p,q)}}$

(iii) $(x\ominus a)^{-n}_{\mathcal{R}(p,q)} = (x\ominus a)^m_{\mathcal{R}(p,q)}(x\xi_1^m\ominus \xi_2^m a)^n_{\mathcal{R}(p,q)}$

(iv) $(x\oplus a)^n_{\mathcal{R}(p,q)} = (x+a)(x\xi_1 + \xi_2 a)(x\xi_1^2 + a\xi_2^2)\cdots(x\xi_1^{n-1} + a\xi_2^{n-1})$

1. if $n\geq 1$ respectively. We can further extend the results as follows:

(v) $(x\oplus y)^\infty_{\mathcal{R}(p,q)} = \prod_{i=0}^\infty (x\xi_1^i + y\xi_2^i)$

(vi) $(x\ominus y)^\infty_{\mathcal{R}(p,q)} = \prod_{i=0}^\infty (x\xi_1^i - y\xi_2^i)$

Properties of $\mathcal{R}(p,q)-$Gamma Function

From the definition 4 we have the following results:

(i) $\Gamma_{\mathcal{R}(p,q)}(z+1) = [z]_{\mathcal{R}(p,q)}\Gamma_{\mathcal{R}(p,q)}(z)$

(ii) $\Gamma_{\mathcal{R}(p,q)}(n+1) = [n]_{\mathcal{R}(p,q)}!$ for a non-negative integer n

(iii) $\Gamma_{\mathcal{R}(p,q)}(2z)\Gamma_{\mathcal{R}(p^2,q^2)}\left(\frac{1}{2}\right) = (\xi_1+\xi_2)^{2z-1}\Gamma_{\mathcal{R}(p^2,q^2)}(z)\Gamma_{\mathcal{R}(p^2,q^2)}(z+\frac{1}{2})$

(iv) $\frac{1}{\Gamma_{\mathcal{R}(p,q)}(z)} = \prod_{n=0}^{\infty} \frac{(\xi_1 \ominus \xi_2^{n+z})^{\infty}_{\mathcal{R}(p,q)}}{(\xi_1 \ominus \xi_2^{n+1})^{\infty}_{\mathcal{R}(p,q)}}(\xi_1-\xi_2)^{z-1}$

$\mathcal{R}(p,q)-$Beta function

From the definition of the $\mathcal{R}(p,q)-$gamma function we can have the following:

$$\beta_{\mathcal{R}(p,q)}(x,y) = \frac{\Gamma_{\mathcal{R}(p,q)}(x)\Gamma_{\mathcal{R}(p,q)}(y)}{\Gamma_{\mathcal{R}(p,q)}(x+y)}, \qquad (5.5)$$

$$= \frac{\frac{(\xi_1\ominus\xi_2)^{\infty}_{\mathcal{R}(p,q)}}{(\xi_1^x\ominus\xi_2^x)^{\infty}_{\mathcal{R}(p,q)}}(\xi_1-\xi_2)^{1-x}\frac{(\xi_1\ominus\xi_2)^{\infty}_{\mathcal{R}(p,q)}}{(\xi_1^y\ominus\xi_2^y)^{\infty}_{\mathcal{R}(p,q)}}(\xi_1-\xi_2)^{1-y}}{\frac{(\xi_1\ominus\xi_2)^{\infty}_{\mathcal{R}(p,q)}}{(\xi_1^{x+y}\ominus\xi_2^{x+y})^{\infty}_{\mathcal{R}(p,q)}}(\xi_1-\xi_2)^{x-y}},$$

where $x, y \in \mathbb{Z}_p$.

Properties of the $\mathcal{R}(p,q)-$Beta Function

(i) $\beta_{\mathcal{R}(p,q)}(x,y+1)=\dfrac{[y]_{\mathcal{R}(p,q)}}{[x+y]_{\mathcal{R}(p,q)}}\beta_{\mathcal{R}(p,q)}(x,y)$

(ii) $\beta_{\mathcal{R}(p,q)}(x+1,y)=\dfrac{[x]_{\mathcal{R}(p,q)}}{[x+y]_{\mathcal{R}(p,q)}}\beta_{\mathcal{R}(p,q)}(x,y)$

(iii) $\beta_{\mathcal{R}(p,q)}(x+n,y)=\dfrac{(\xi_1^x\ominus\xi_2^x)^n_{\mathcal{R}(p,q)}}{(\xi_1^{x+y}\oplus\xi_2^{x+y})^n_{\mathcal{R}(p,q)}}\beta_{\mathcal{R}(p,q)}(x,y)$

(iv) $\beta_{\mathcal{R}(p,q)}(x+1,y)+\beta_{\mathcal{R}(p,q)}(x,y+1)=$
$\dfrac{[x]_{\mathcal{R}(p,q)}+[y]_{\mathcal{R}(p,q)}}{[x+y]_{\mathcal{R}(p,q)}}\beta_{\mathcal{R}(p,q)}(x,y)$

(v) $\beta_{\mathcal{R}(p,q)}(x+1,y+1)=\dfrac{[x]_{\mathcal{R}(p,q)}+[y]_{\mathcal{R}(p,q)}}{[x+y+1]_{\mathcal{R}(p,q)}[x+y]_{\mathcal{R}(p,q)}}\beta_{\mathcal{R}(p,q)}(x,y)$

Fermionic $p-$adic q integral

T. Kim and Koblitz

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}$, which we put as $f \in UD(\mathbb{Z}_p)$, if the quotient

$$F_{f(x,y)} = \frac{f(x) - f(y)}{x - y}$$

have a limit as $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, fermionic p-adic q integral on $\mathbb{Z}_p$ by T. Kim is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \qquad (6.1)$$

For $f : \mathbb{Z}_p \longrightarrow \mathbb{C}_p$, we can also have

$p-$adic q integral

$$I_q(f) = \lim_{N\to\infty} \frac{1}{[P^N]_q} \sum_{0\leq x\leq p^N} q^x f(x) \tag{6.2}$$

$$= \lim_{N\to\infty} \sum_{0\leq x\leq p^N} f(x)\mu_q(x+p^N\mathbb{Z}_p) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x).$$

When $q=-1$ we have the following:

$$I_{-1}(f) = \lim_{q\to -1} I_q(f) = \lim_{q\to -1} \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) \tag{6.3}$$

Now given, $f_n(x)=f(x+n)$ we can obtain $f_1(x)=f(x+1)$ then

$$I_{-1}(f_1) = -\lim_{N} \sum^{P^N-1} f(x)(-1)^x + 2f(0)$$

$\mathcal{R}(\rho, q)$-Volkenborn integral

Theorem 5 Araci et al.

For $f \in UD(\mathbb{Z}_p)$, we have

$$q^n I_{\mathcal{R}(\rho,q)}(f_n) - \rho^n I_{\mathcal{R}(\rho,q)} = \rho^n \frac{(\rho - q)}{\rho^n - q^n} \frac{(\rho^\rho - q^Q)}{\mathcal{R}(\rho^\rho, q^Q)} \times \sum_{a=0}^{n-1} \left(\frac{q}{\rho}\right)^a \left(\frac{f'(a)}{\ln q - \ln \rho} + f(a)\right) \tag{6.4}$$

where $f_n(x) = f(x+n)$ and for $n = 1$ we obtain;

$$q I_{\mathcal{R}(\rho,q)}(f_1) - \rho I_{\mathcal{R}(\rho,q)} = \frac{\rho(\rho^\rho - q^Q)}{\mathcal{R}(\rho^\rho, q^Q)} \left(\frac{f'(0)}{\ln q - \ln \rho} + f(0)\right) \tag{6.5}$$

where $f_1(x) = f(x+1)$.

$p-$adic $\mathcal{R}(\rho, q)-$Gamma Functions

We shall be denote without loss of generality by:

$$(n!)^p = \prod_{\substack{j<n \\ (p,j)=1}} j$$

and

$$\Gamma^p(x) = \lim_{n\to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j.$$

for easy notation in the subsequent development. Now for $n \in \mathbb{N}$, the $p-$adic $\mathcal{R}(\rho, q)-$factorial function can be written as $(n!)^p_{\mathcal{R}(\rho,q)} = \prod_{\substack{j<n \\ (p,j)=1}} [j]_{\mathcal{R}(\rho,q)}.$

$p-$adic $\mathcal{R}(\rho, q)-$Gamma Functions

Definition 6 [Duran et al.]

Let ρ and $q \in \mathbb{C}_p$ with

$$|\rho - 1|_p < 1$$

and

$$|q - 1|_p < 1,$$

$\rho \neq 1, \quad q \neq 1$. We introduce the p-adic $\mathcal{R}(\rho, q)-$factorial function $(x!)^p_{\mathcal{R}(\rho,q)}$ as follows

$$\Gamma^p_{\mathcal{R}(\rho,q)}(x) = \lim_{n \to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} [j]_{\mathcal{R}(\rho,q)}. \tag{7.1}$$

$p-$adic $\mathcal{R}(\rho, q)-$Gamma Functions

Lemma 1 [W. Schikhof, Menken et al., Duran et al.]

For all $x \in \mathbb{Z}_p$ the following results hold;

$$\Gamma^p_{\mathcal{R}(\rho,q)}(0) = 1, \quad \Gamma^p_{\mathcal{R}(\rho,q)}(1) = -1, \quad \text{and} \quad |\Gamma^p_{\mathcal{R}(\rho,q)}(x)|_P = 1. \tag{7.2}$$

Theorem 6 [W. Schikhof, Menken et al., Duran et al.]

The following recurrence formula holds true for all $z \in \mathbb{Z}_p$:

$$\Gamma^p_{\mathcal{R}(\rho,q)}(z+1) = \delta^p_{\mathcal{R}(\rho,q)}[z]\Gamma^p_{\mathcal{R}(\rho,q)}(z) \tag{7.3}$$

where

$$\delta^p_{\mathcal{R}(\rho,q)}[z] = \begin{cases} -[z]_{\mathcal{R}(\rho,q)} & \text{if} \quad |z|_p = 0, \\ -1 & \text{if} \quad |z|_p < 1. \end{cases}$$

$p-$adic $\mathcal{R}(\rho, q)-$Gamma Functions

Remark

From the $\mathcal{R}(\rho, q)-$numbers, we have the product rule:

$$[kp]_{\mathcal{R}(\rho,q)} = [k]_{\mathcal{R}(\rho^p,q^p)}[p]_{\mathcal{R}(\rho^p,q^p)}.$$

Theorem 7 [Duran et al 2019, Menken et al 2013.]

The following recurrence formula holds true for all $n \in \mathbb{N}$:

$$\Gamma^p_{\mathcal{R}(\rho,q)}(n+1) = (-1)^{n+1}\frac{[n]_{\mathcal{R}(\rho,q)}!}{[p]^{\left\lfloor \frac{n}{p}\right\rfloor}_{\mathcal{R}(\rho,q)}\left[\left\lfloor \frac{n}{p}\right\rfloor\right]_{\mathcal{R}(\rho^p,q^p)}!}, \qquad (7.4)$$

$p-$adic $\mathcal{R}(\rho, q)-$Gamma Functions

Lemma 2

Let m_n be the sum of digits of $n = \sum_{j=0}^{m} a_j p^j$ $(a_m \neq 0)$ in base p. Then

$$\left[\left\lfloor \frac{n}{p} \right\rfloor\right]_{\mathcal{R}(\rho^p, q^p)}! = (-1)^{n+1-m} \left(-[p]_{\mathcal{R}(\rho^p, q^p)}\right)^{\frac{(n-m_n)}{(p-\mathbf{1})}} \times \prod_{j=0}^{m-1} \frac{\left[\left\lfloor \frac{n}{p^{j+1}} \right\rfloor\right]_{\mathcal{R}(\rho^p, q^p)}!}{\left[\left\lfloor \frac{n}{p^{j}} \right\rfloor\right]_{\mathcal{R}(\rho^p, q^p)}!} \prod_{i=0}^{m} \Gamma^p_{\mathcal{R}(\rho, q)}\left(\left[\left\lfloor \frac{n}{p} \right\rfloor\right] + 1\right) \tag{7.5}$$

$p-$adic $\mathcal{R}(\rho, q)-$Gamma Functions

Lemma 2 Cont'd

$$[n]_{\mathcal{R}(\rho^p,q^p)}! = (-1)^{n+1-m}\left(-[p]_{\mathcal{R}(\rho^p,q^p)}\right)^{\frac{(n-m_n)}{(p-1)}}\left[\left\lfloor\frac{n}{p}\right\rfloor\right]_{\mathcal{R}(\rho^p,q^p)}! \qquad (7.6)$$

$$\times\prod_{j=0}^{m-1}\frac{\left[\left\lfloor\frac{n}{p^{j+1}}\right\rfloor\right]_{\mathcal{R}(\rho^p,q^p)}!}{\left[\left\lfloor\frac{n}{p^{j}}\right\rfloor\right]_{\mathcal{R}(\rho^p,q^p)}!}\prod_{i=0}^{m}\Gamma^p_{\mathcal{R}(\rho,q)}\left(\left[\left\lfloor\frac{n}{p^{i}}\right\rfloor\right]+1\right)$$

$p-$adic $\mathcal{R}(\rho, q)-$Gamma Functions

Theorem 8

For a prime number p and m_n be the sum of digits of $n = \sum_{j=o}^{m} a_j p^j$ in base p where $n \in \mathbb{N}$. For $0 \leq k \leq m$ and $j = 0, 1, \cdots, m$ then the following identity holds:

$$\frac{\left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\mathcal{R}(\rho,q)}!}{[p]_{\mathcal{R}(\rho,q)}^{\left\lfloor \frac{n}{p^j} \right\rfloor} \left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\mathcal{R}(\rho^p,q^p)}!} = \prod_{k=1}^{\left\lfloor \frac{n}{p^j} \right\rfloor} \frac{\rho^k - q^k}{\rho^{kp} - q^{kp}}. \tag{7.7}$$

Consequently we obtain;

$p-$adic $\mathcal{R}(\rho,q)-$Gamma Functions

Theorem 8 Cont'd

$$[n]_{\mathcal{R}(\rho^p,q^p)}! = (-1)^{\frac{n-m_n}{(p-1)+n+1-m}} \prod_{k=1}^{\left\lfloor \frac{n}{p^1} \right\rfloor} \frac{\rho^k - q^k}{\rho^{kp} - q^{kp}} \cdots \prod_{k=1}^{\left\lfloor \frac{n}{p^m} \right\rfloor} \frac{\rho^k - q^k}{\rho^{kp} - q^{kp}} \quad (7.8)$$

$$\times \prod_{i=0}^{m} \Gamma^p_{\mathcal{R}(\rho,q)}\left(\left\lfloor \frac{n}{p^j} \right\rfloor + 1 \right).$$

$p-$adic $\mathcal{R}(\rho,q)-$Beta functions

For any $x, y \in \mathbb{Z}_p$ the $p-$adic beta function $\beta_p : \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow \mathbb{Q}_p$ is given by:

$$\beta_p(x+y) = \frac{\Gamma_p(x)\Gamma_p(y)}{\ldots} \quad (7.9)$$

$p-$adic $\mathcal{R}(\rho, q)-$Beta functions

Definition 7

For the purpose of notation we shall denote the $p-$adic beta function as follows:

$$\beta^p(x+y) = \frac{\Gamma^p(x)\Gamma^p(y)}{\Gamma^p(x+y)}. \tag{7.10}$$

Let ρ and $q \in \mathbb{C}_p$ with $|\rho - 1|_p < 1$ and $|q-1|_p < 1$, $\rho \neq 1, \quad q \neq 1$. We define the $p-$adic $\mathcal{R}(\rho, q)-$beta function via the $p-$adic $\mathcal{R}(\rho, q)-$gamma functions as follows:

$$\beta^p_{\mathcal{R}(\rho,q)}(x, y) = \frac{\Gamma^p_{\mathcal{R}(\rho,q)}(x)\Gamma^p_{\mathcal{R}(\rho,q)}(y)}{\Gamma^p_{\mathcal{R}(\rho,q)}(x+y)} \tag{7.11}$$

for $x, y \in \mathbb{Z}_p$.

$p-$adic $\mathcal{R}(\rho, q)-$Beta functions

Theorem 9

The $\mathcal{R}(\rho, q)-$Beta Functions have the following properties:

(i) $$\beta^p_{\mathcal{R}(\rho,q)}(x, y+1) = \frac{\delta^p_{\mathcal{R}(\rho,q)}(y)}{\delta^p_{\mathcal{R}(\rho,q)}(x+y)}\beta^p_{\mathcal{R}(\rho,q)}(x, y)$$

(ii) $$\beta^p_{\mathcal{R}(\rho,q)}(x+1, y) = \frac{\delta^p_{\mathcal{R}(\rho,q)}(x)}{\delta^p_{\mathcal{R}(\rho,q)}(x+y)}\beta^p_{\mathcal{R}(\rho,q)}(x, y)$$

(iii) $$\beta^p_{\mathcal{R}(\rho,q)}(x+1, y) = \frac{\delta^p_{\mathcal{R}(\rho,q)}(x)}{\delta^p_{\mathcal{R}(\rho,q)}(y)}\beta^p_{\mathcal{R}(\rho,q)}(x, y+1)$$

(v) $$\beta^p_{\mathcal{R}(\rho,q)}(x+1, y) + \beta^p_{\mathcal{R}(\rho,q)}(x, y+1) = \frac{\delta^p_{\mathcal{R}(\rho,q)}(x) + \delta^p_{\mathcal{R}(\rho,q)}(y)}{\delta^p_{\mathcal{R}(\rho,q)}(x+y)}\beta^p_{\mathcal{R}(\rho,q)}(x, y)$$

$p-$adic $\mathcal{R}(\rho, q)-$Beta functions

Theorem 9 Cont'd

(vi) $\beta^p_{\mathcal{R}(\rho,q)}(x+1, y+1) = \dfrac{\delta^p_{\mathcal{R}(\rho,q)}(x) + \delta^p_{\mathcal{R}(\rho,q)}(y)}{\delta^p_{\mathcal{R}(\rho,q)}(x+y+1)\delta^p_{\mathcal{R}(\rho,q)}(x+y)}$
$\times \beta^p_{\mathcal{R}(p,q)}(x, y)$

(vii) $\beta^p_{\mathcal{R}(\rho,q)}(x, y) + \beta^p_{\mathcal{R}(\rho,q)}(x+y, z) + \beta^p_{\mathcal{R}(\rho,q)}(x+y+z, w)$
$= \dfrac{\Gamma^p_{\mathcal{R}(\rho,q)}(x)\Gamma^p_{\mathcal{R}(\rho,q)}(y)\Gamma^p_{\mathcal{R}(\rho,q)}(z)\Gamma^p_{\mathcal{R}(\rho,q)}(w)}{\Gamma^p_{\mathcal{R}(\rho,q)}(x+y+z+w)}$

(viii) $\beta^p_{\mathcal{R}(\rho,q)}(x, 1-x) = -\Gamma^p_{\mathcal{R}(\rho,q)}(x)\Gamma^p_{\mathcal{R}(\rho,q)}(1-x).$

Veneziano Amplitude

Definition 8

Veneziano model is

$$A_0^{\infty}(s,t,u)=\int_0^1 dx|x|^{-\alpha(s)-1}|1-x|^{-\alpha(t)-1}=\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

where $\alpha(s)=1+\frac{s}{2}$, $s+t+u=-8$ and therefore $\alpha(s)+\alpha(t)+\alpha(u)=-1$. Adding the terms related by crossing symmetry

$$A^{\infty}=A_0^{\infty}(s,t,u)+A_0^{\infty}(t,u,s)+A_0^{\infty}(u,s,t) \tag{8.1}$$

and define β^{∞} by

$$A^{\infty}(s,t,u)=g_{\infty}^2\beta^{\infty}(-\alpha(s),-\alpha(t))$$

$p-$adic beta amplitude

with

$$\beta^{\infty}(-\alpha(s),-\alpha(t))=\int_0^1 dx|x|^{-\alpha(s)-1}|1-x|^{-\alpha(t)-1} \tag{8.2}$$

in which the integration range is the real field $\mathbb{R}=\mathbb{Q}_\infty$ rather than just $(0,1)$. The $p-$adic string amplitude is defined by replacing $\mathbb{R}$ with the $p-$adic number field $\mathbb{Q}_p$.

$$\beta^{p}(-\alpha(s),-\alpha(t))=\int_{\mathbb{Q}_p} dx|x|_p^{-\alpha(s)-1}|1-x|_p^{-\alpha(t)-1} \tag{8.3}$$

$$=\prod_{x=s,t,u}\left(\frac{1-p^{-\alpha(x)-1}}{1-p^{\alpha(x)}}\right) \tag{8.4}$$

Amplitude and Riemann zeta function

E. Witten and O. Freund

Recall the definition of the Riemann zeta function:

$$\zeta(z)=\prod_p\left(\frac{1}{1-p^{-z}}\right)\equiv\sum_{r=1}^{\infty}\left(\frac{1}{r^z}\right) \tag{8.5}$$

in order to rewrite

$$\prod_p\beta^p(-\alpha(s),-\alpha(t))=\prod_{x=s,t,u}\frac{\zeta(-\alpha(x))}{\zeta(1+\alpha(x))}$$

.

$p-$adic Gravity

Aref'eva et al.

Aref'eva, Dragovich, Frampton, and Volovich considered extending the wave function of the universe by considering the p-adic gravity, specifically they gave the p-adic analogue of the Einstein equation as follows

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} - \Lambda g_{\mu\nu} \tag{8.6}$$

$g_{\mu\nu}$ is $p-$adic valued gravitational field.

Concluding remarks

Currently working on these(Open to collaborations and joint work)

1. Some applications of the $p-$adic $\mathcal{R}(\rho,q)-$Gamma and Beta function to the string amplitude.
2. Connections with solitons, D-branes, taychons and some applications to Feymann amplitude
3. Understanding the $p-$adic Matter and its impact in Cosmology.
4. Some applications to Macdonald polynomials, Coulombs branching, Hilbert schemes and quiver varieties.
5. Some experimental observations of the $p-$adic matter and see how it links to Holony graphs(Bruhat Tits tree) as well as superstring symmetry.
6. Look at the Rieman zeta functions again and see some connections with all the above in $p-$adic field theory.

QUESTIONS?

THANKS FOR YOUR ATTENTION!