E_7 and E_8 ALH* Tesserons as the moduli space of Doubly Periodic Monopoles

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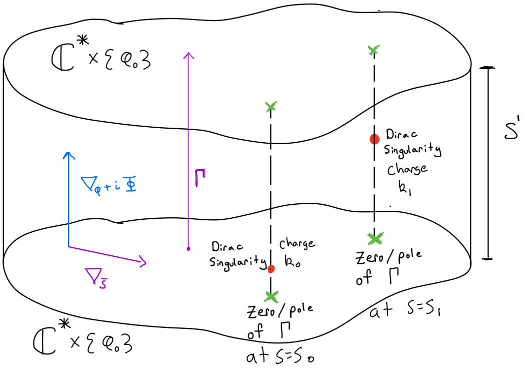
Doubly Periodic Monopoles

- Hermitian Vector Bundle $E \to \mathbb{R} \times S^1 \times S^1$ with coordinates (x, θ, φ)
- Higgs field Φ and connection D_A satisfying Bogomolny equations $*D_A\Phi = -F_A$
- Allow Dirac style singularities
- At most linear growth of Φ as $x \to \pm\infty$.

Reinterpreting the Bogomolny Equations

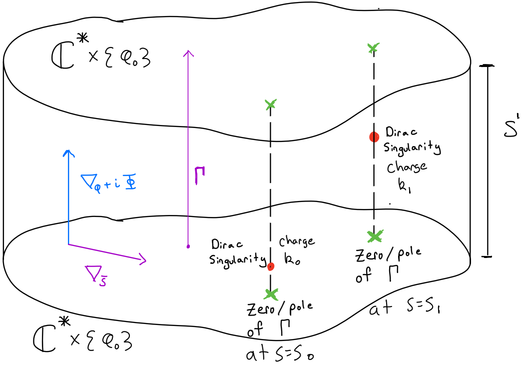


$$\mathbb{R} \times S^1 \times S^1 \simeq \mathbb{C}^* \times S^1 \tag{1}$$

$$(x, \theta, \varphi) \mapsto (s = \exp(x + i\theta), \varphi) \tag{2}$$

$$D_A = \nabla_x dx + \nabla_\theta d\theta + \nabla_\varphi d\varphi \implies \nabla_{\bar{s}} = \frac{1}{2\bar{s}}\left(\nabla_x + i\nabla_\theta\right) \tag{3}$$

Holomorphic view of Bogomolny Equations



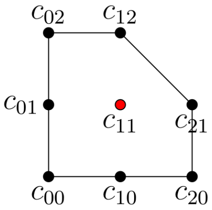
- $\nabla_\varphi + i\Phi$ induces parallel transport in φ direction
- Denote resulting holonomy operator $\Gamma(s) : E_{\varphi_0} \to E_{\varphi_0}$
- $\nabla_{\bar{s}}$ is holomorphic structure on E_{φ_0}.

Spectral Data

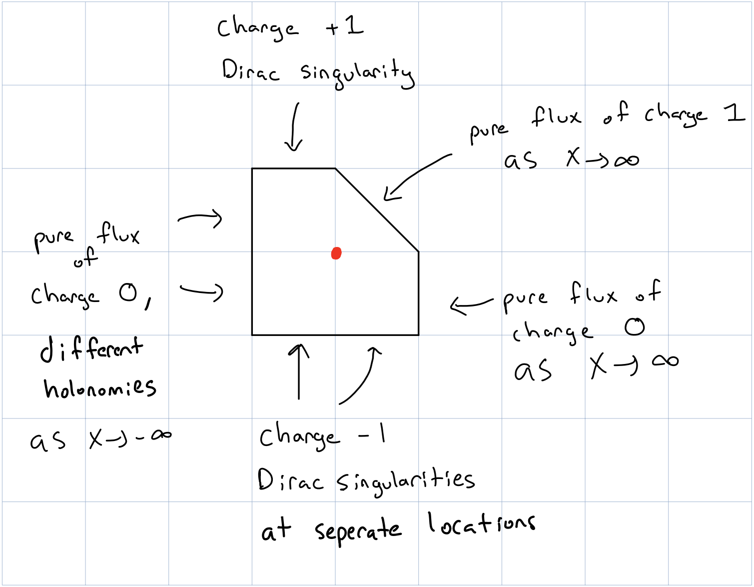
- $\Sigma := \det(\Gamma(s) - t)^{-1}(0)$, curve of eigenvalues of Γ.
- Monopole boundary conditions ensure $\Sigma = p^{-1}(0)$ for polynomial $p(s,t)$.
- From eigenvectors of Γ, get an eigenline bundle $L \to \Sigma$
- Bogomolny equations imply $\nabla_{\bar{s}}\Gamma = 0$, hence $L \to \Sigma$ has a holomorphic structure
- Holomorphic line bundle $L \to \Sigma$ along with parabolic structure at marked points can recover the monopole [**mochizuki20**] [**harris23**]

Newton Polygon

- Consider the polynomial
  $p(s,t) = c_{00} + c_{10}s + c_{20}s^2 + t(c_{10} + c_{11}s + c_{21}s^2) + t^2(c_{02} + sc_{12})$
- We can represent it by a Newton polygon

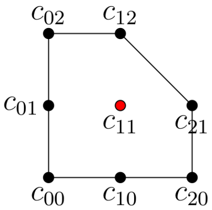


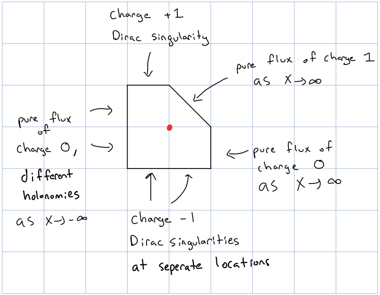
Newton Polygon Interpretation for E_4



Moduli Space, coefficients

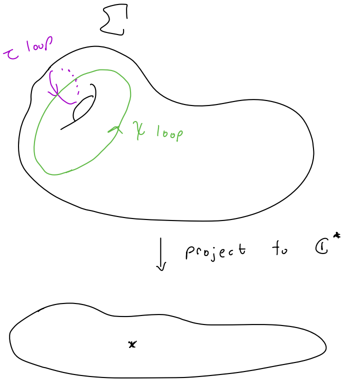
- L^2 deformations of doubly periodic monopoles induce certain deformations in spectral data, meaning L^2 pairs $(\Phi + \phi, D_A + \alpha)$ such that (ϕ, α) satisfy linearized Bogomolny equations
- $p(s,t) = c_{00} + c_{10}s + c_{20}s^2 + t(c_{10} + c_{11}s + c_{21}s^2) + t^2(c_{02} + sc_{12})$
- Boundary terms of p fixed, but c_{11} can change.





Moduli Space, holonomy

- L^2 deformations of doubly periodic monopoles induce certain deformations in spectral data
- Part of the doubly periodic monopole was encoded into a holomorphic structure on a line bundle $L \to \Sigma$
- Spectral curves holonomies also get changed.



Moduli Space, combined

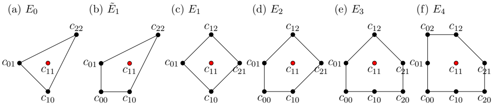
- L^2 doubly periodic monopole deformations correspond to interior coefficient deformations of Newton polygon and holonomy deformations on spectral curve
- Moduli space of doubly periodic monopoles understood by thinking of holomorphic line bundles over spectral curves

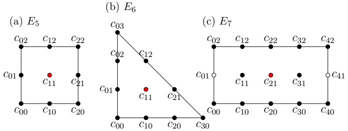
ALH^* Tesserons

- Tesseron is a complete, noncompact hyperkähler manifold of real dimensions 4 with L^2 Riemann curvature tensor
- ALH^* Tesserons have geodesic ball volume growth $\sim r^{4/3}$.
- Classified by intersection forms, types E_{9-b}
- Model ALH^* metric $= \frac{Hb}{2\pi}(dH^2 + d\rho^2 + d\tau^2) + \frac{2\pi}{Hb}(d\chi - \frac{b\rho}{2\pi}d\tau)^2$ **[collins21]** **[hein21]** **[sun21]**

ALH^* Tesserons as moduli space

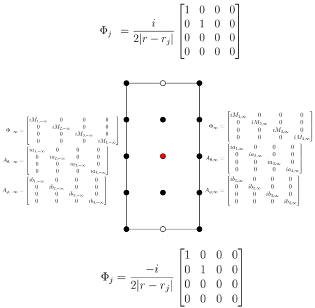
- With regular boundary conditions, interior points represent 4 times the number of L^2 deformations
- Cherkis, Cross, Ward used this to find $E_0, E_1, \dots E_6$ case

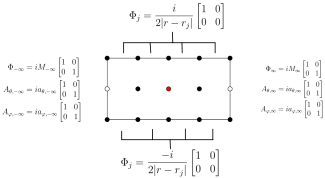




E_7 and E_8

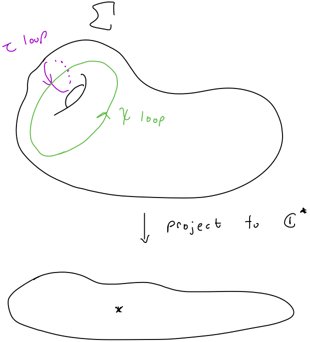
- E_7 and E_8 arise from degenerate boundary conditions, [**kim17**]
- Some interior points of Newton polygon are now fixed

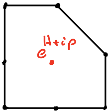




ALH^*, spectral side

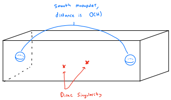
- Interior coefficient is $e^{H+i\rho}$
- Holomorphic line bundle parameterized by holonomies around 1-cycles, coordinates of (τ, χ)
- 4 coordinates of (H, ρ, τ, χ)
- Model ALH^* metric $= \dfrac{Hb}{2\pi}(dH^2 + d\rho^2 + d\tau^2) + \dfrac{2\pi}{Hb}(d\chi - \dfrac{b\rho}{2\pi}d\tau)^2$

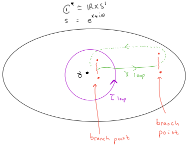




ALH^*, monopole side

- L^2 metric on monopole side should approximate model ALH^* metric
- Φ, D_A split into abelian solutions between monopoles
- Branch points of Σ are like smooth monopoles
- Distance between monopoles is $O(H)$.
- Coordinates (H, ρ, τ, χ)
- H represents size of Higgs field between monopoles
- ρ represents φ holonomy between monopoles
- τ represents θ holonomy between monopoles
- χ represents holonomy by transporting from one monopole to the other, changing sheets, and transporting back.





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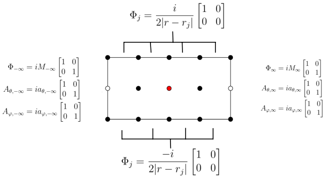
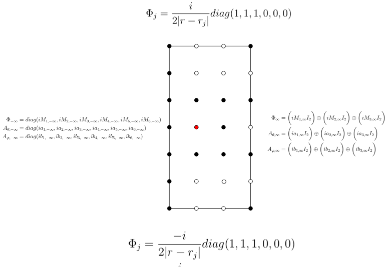
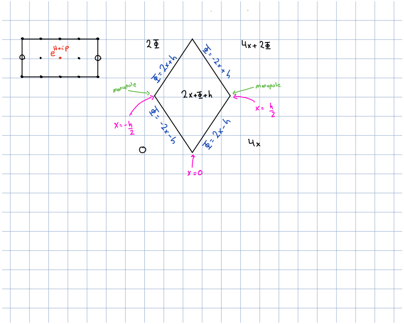
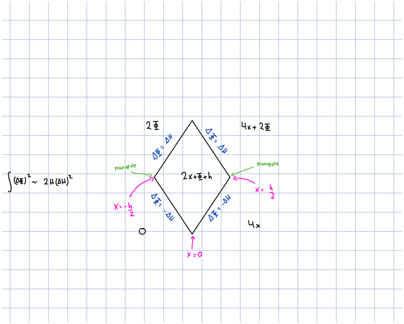




Dual Polygons





Deformations

- For Long E_7, deformations between monopoles is essentially constant
- $\Delta\Phi \sim \Delta H \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $\Delta A_\varphi \sim \Delta\rho \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $\Delta A_\theta \sim \Delta\tau \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $\Delta A_x \sim \frac{\Delta\chi}{H} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Squaring an integrating over a region of length H yields model ALH^* metric
- Model ALH^* metric $= \frac{Hb}{2\pi}(dH^2 + d\rho^2 + d\tau^2) + \frac{2\pi}{Hb}(d\chi - \frac{b\rho}{2\pi}d\tau)^2$

Thank you for your time.

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arXiv:1411.7903v4 [hep-th], 11 Jul 2017.

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Takuro Mochizuki
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Trans. Amer. Math. Soc., 373(1):551–596, 2020

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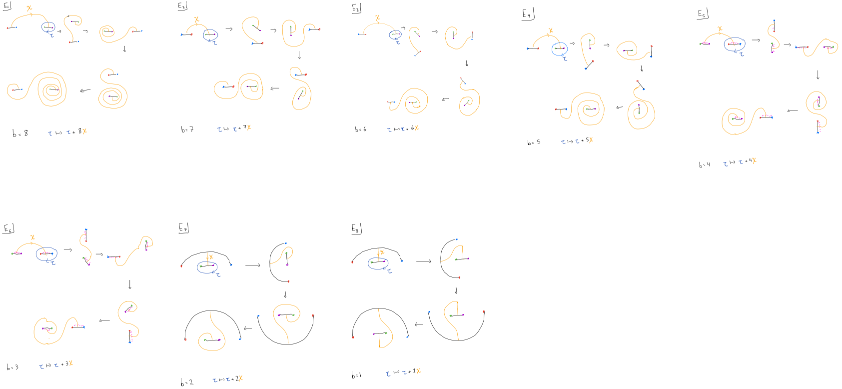
All ALH^* Tesserons from Doubly Periodic Monopoles
In progress

Meaning of b


E1
b = 8   τ ↦ τ + 8χ
E2
b = 7   τ ↦ τ + 7χ
E3
b = 6   τ ↦ τ + 6χ
E4
b = 5   τ ↦ τ + 5χ
E5
b = 4   τ ↦ τ + 4χ
E6
b = 3   τ ↦ τ + 3χ
E7
b = 2   τ ↦ τ + 2χ
E8
b = 1   τ ↦ τ + 1χ