Classical Solutions of Spherically Symmetric Gravitating Skyrmion

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The Einstein-Skyrme Model

The Einstein-Skyrme model is a gravity model generated by the Skyrme Field as the matter field. The back-reactions between both gravitational fields and the Skyrme field are allowed. In other words, the dynamics of the Skyrme field respond to the gravitational field generated by itself. Sometimes, this model is also called the gravitating Skyrmion model.

Because the gravitational fields obey Einstein's general relativity, the coupling between gravity and the Skyrmion is minimal, that is, the interaction is introduced via metric tensor determinant and the covariant derivatives on curved spacetime without any additional terms in the Lagrangian.

Einstein's GR

Let $\mathcal{M}^{1,d}$ be a $1+d$ dimensional spacetime manifold with negative signature, $\gamma_{\mu\nu}$s are the components of metric tensor on $\mathcal{M}^{1,d}$, and $\gamma = \det[\gamma_{\mu\nu}]$. Einstein's GR is a system of equations that is equivalent to the Euler-Lagrange equation (or, least-action principle) of the Einstein-Hilbert action

$$S_{\text{Einstein}} = \int \sqrt{-\gamma}\frac{R}{16\pi} d^{d+1}x + S_{\text{Matter}}, \tag{1}$$

where $R = \gamma^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar and $R_{\mu\nu}$s are the components of Ricci tensor.

Einstein's GR

Let ϕ^a be the matter field. The Euler-Lagrange equations from (1) by variation of $\gamma_{\mu\nu}$ gives the Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}R\gamma_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2}$$

and the Euler-Lagrange equations by variation of ϕ^a are equivalent to the conservation of matter energy-momentum tensor

$$\nabla^\mu T_{\mu\nu} = 0, \tag{3}$$

where the energy-momentum tensor is defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{\text{Matter}}}{\delta\gamma^{\mu\nu}} \tag{4}$$

The system of equations contains up to second-order derivatives of every field.

Coupling to Skyrme Field

A convenient way to construct the Skyrme model for arbitrary spacetime dimensions is known as the Skyrme-Sigma model proposed by [Arthur et al., 1996]. Let ϕ^a be the components of the Skyrme field, ϕ. The field $\phi : \mathcal{M}^{1,d} \to S^d$ have a d dimensional sphere target space, such that, $\phi^a\phi^a = 1$ with $a = 0, 1, 2, \dots, d$.

For the usual spacetime with $d = 3$, the constraint $\phi^a\phi^a = 1$ is equivalent to the unitarity of $SU(2)$ valued field, $U^\dagger U = \mathbf{1}_{2\times 2}$, and U can be written in ϕ^a as

$$U = \phi^0 \mathbf{1}_{2\times 2} + i(\phi^1\sigma^1 + \phi^2\sigma^2 + \phi^3\sigma^3), \tag{5}$$

Coupling to Skyrme Field

The action of the Skyrme-Sigma model in arbitrary spacetime dimension is given by

$$\mathcal{S}_{\text{Matter}} = -\int \sqrt{-\gamma} \left[C_0 V + \sum_{n=1}^{d} \frac{C_n}{(n!)^2} \phi^{a_1}_{[\mu_1} \dots \phi^{a_n}_{\mu_n]} \phi^{a_1}_{[\nu_1} \dots \phi^{a_n}_{\nu_n]} \gamma^{\mu_1 \nu_1} \dots \gamma^{\mu_n \nu_n} \right] d^{d+1}x, \tag{6}$$

with C_ns are the coupling constants of each term and $\phi^a_\mu = \nabla_\mu \phi^a$. An explicit expression for $T_{\mu\nu}$ for this action is too complicated for arbitrary d. Thus, it is necessary to introduce an ansatz for ϕ which acts as a coordinate system on S^d, ensuring that $\phi^a \phi^a = 1$ is always satisfied.

Hedgehog Ansatz

For a spherically symmetric spacetime, the metric tensor is given by

$$ds^2 = -N(t,r)dt^2 + \frac{1}{f(t,r)}dr^2 + r^2 d\Omega_{d-1}^2, \tag{7}$$

with $d\Omega_{d-1}^2 = d\theta_1^2 + \sum_{p=2}^{d-1}\prod_{q=1}^{p-1}\sin^2\theta_q \; d\theta_p^2$ is the metric tensor on the spherical submanifold, S^{d-1}, and θ_ps are the angular coordinates.

A good ansatz for this spherically symmetric system can be defined using the angular coordinates above, namely

$$\begin{aligned} \phi^0 &= \cos\xi(t,r),\; \phi^1 = \sin\xi(t,r)\cos\theta_1, \\ &\ldots \\ \phi^{d-1} &= \sin\xi(t,r)\sin\theta_1\ldots\sin\theta_{d-2}\cos\theta_{d-1}, \\ \phi^d &= \sin\xi(t,r)\sin\theta_1\ldots\sin\theta_{d-2}\sin\theta_{d-1}. \end{aligned} \tag{8}$$

Hedgehog Ansatz

To simplify the problem, the topological charge is assumed to be constant, hence the boundary conditions are

$$\xi(t,0) = \pi, \quad \lim_{r\to\infty} \xi(t,r) = 0, \tag{9}$$

for all setups with no singularity at the origin, $r = 0$.

An exception has to be made for the Skyrme black hole where we can only integrate the topological current at regions beyond the horizon. The standard approach is to introduce a shooting parameter ξ_h, such that

$$\lim_{r\to r_h} \xi(t,r) = \xi_h, \quad \lim_{r\to\infty} \xi(t,r) = 0. \tag{10}$$

Clasiccal Solutions With Spherical Symmetry

In this work, a classical solution is defined to be a C^2-real function of all coordinates of spacetime, $h(t, r, \dots)$, that satisfies the dynamical equations of fields.

A spherically symmetric solutions are defined to be functions of only time and radial coordinates and their corresponding dynamical equations should not depend on any angular coordinates or functions of angular coordinates.

As such, a spherically symmetric action (in the sense that it is invariant under $d-1$ dimensional rotation) implies spherically symmetric solutions.

Existence of Unique Solutions

The problem of spherically symmetric gravitating Skyrmion is then reduced to solving coupled differential equations of three functions

$$N(t,r),\ f(t,r),\ \xi(t,r), \tag{11}$$

that is effectively a two-dimensional hyperbolic PDE problem.

In order to show that unique solutions to such a problem exist, we can employ the Banach fixed-point theorem

Banach Fixed-Point Theorem

Let d be a metric on complete space X. If there exist $\lambda \in [0,1)$ for every $x_1, x_2 \in X$ such that $T : X \to X$ satisfies $d(T(x_2), T(x_1)) \leq \lambda d(x_2, x_1)$, then there exist a unique fixed point, $x^* \in X$ of T.

Existence of Unique Solutions

General strategy:

1. Find a good function space, X, containing C^2 functions, that is large enough to contain the possible solutions, usually equipped with sup-norm.
2. Define a characteristic, usually the null curve in the radial direction, and then define a map using the integral of dynamical equations along the characteristic.
3. Show that the map is a ball-to-ball map inside X.
4. Show that the map satisfies the Lipschitz condition.

Stability of Solutions

There are lots of definitions for stable solutions and stable systems of equations. Some of them can be applied to a single model and provide more insights into how the system is protected from singularity formations or decays.

- Derick's stability: There exists a finite non-zero spatial scaling that minimizes the energy of the system.
- Topologically supported stability: The BPS bound indicates that the vacuum is not accessible for solitons with non-zero topological degrees.
- Linear dynamical stability: steady-state solutions are stable under time-dependent perturbation.

Some Important Works

- Unique spherically symmetric global solutions of the dynamical Einstein-Scalar system in $1+3$ spacetime exist. The solutions can be obtained through the characteristic method [Christodolou, 1986]
- The standard Einstein-Skyrme model admits asymptotically flat star and hairy black hole solutions in the static $1+3$ spacetime case. There are families of solutions that are stable under linear harmonic perturbation [(Luckock and Moss, 1986),(Heusler et al., 1991),(Droz et al., 1991), (Bizon and Chmaj, 1992)]. The case of asymptotically AdS and dS spacetime has also been studied [(Shiiki and Sawado, 2005),(Brihaye and Delsate, 2006)]
- The quartic (Skyrme) term is necessary for the existence of generalized Skyrme hairy black hole solutions in $1+3$ spacetime [(Gudnason et al., 2016),(Adam et al., 2016),(Perapechka and Shnir, 2017)]
- The sextic term is necessary for the existence of both Skyrme star and hairy black hole solutions in $1+4$ spacetime [Brihaye et al. 2017]
- In $1+d$ spacetime, Skyrme hairy black hole solutions exist for general coupling in Skyrme Lagrangian. Stable solutions under harmonic perturbation exist [Gunara et al., 2021]

Dynamical Gravitating Skyrmions in 3+1 Bondi spacetime

The metric (7) actually poses a problem when the horizon is formed, implying that the metric does not admit classical solutions due to coordinate singularity. We can avoid this problem by introducing a new coordinate system

$$ds^2 = -e^{2\rho} du^2 - 2e^{\rho+\sigma} du dr + r^2 d^2\Omega, \qquad (12)$$

where u is the new time-like coordinate, satisfying $\sqrt{N} dt - \frac{dr}{\sqrt{f}} = e^{\rho} du$.

In this coordinate system, traversing the horizon only changes the sign of metric components and we can always apply the usual boundary condition for Skyrmions.

Dynamical Gravitating Skyrmions in 3+1 Bondi spacetime

In this work (arXiv:2210.15895), we consider the standard Skyrme model without potential and sextic terms,

$$\mathcal{S}_{\text{Skyrme}} = -\int \sqrt{-\gamma} \left[C_1 \phi^a_\kappa \phi^a_\tau \gamma^{\kappa\tau} + C_2 \phi^a_{[\kappa} \phi^b_{\alpha]} \phi^a_{[\tau} \phi^b_{\beta]} \gamma^{\kappa\tau} \gamma^{\alpha\beta} \right] d^4x \; . \quad (13)$$

Since the Skyrme model can be considered mass-less because there is no potential term, then the null-curve, χ, satisfying

$$\frac{dr}{du} = -\frac{e^{\rho-\sigma}}{2}, \quad (14)$$

is a suitable characteristic.

Let $h(u,r) = \partial_r(r\xi(u,r))$, and $h_0(r)$ is the initial data of $h(u,r)$. We can use two function spaces below

$$\tilde{X} = \{h(.,.) \in C^1([0,u_0] \times [0,\infty)) \mid \|h\|_{\tilde{X}} < \infty\}, \quad (15)$$
$$\tilde{Y} = \{h(.,.) \in C^1([0,u_0] \times [0,\infty)) \mid \|h\|_{\tilde{Y}} < \infty\}, \quad (16)$$

with the norm $\|h\|_{\tilde{X}}$ and $\|h\|_{\tilde{Y}}$ given by

$$\|h\|_{\tilde{X}} := \sup_{u\in[0,u_0]} \sup_{r\geq 0} \left\{(1+r)^2|h(u,r)| + (1+r)^3 \left|\frac{\partial h}{\partial r}(u,r)\right|\right\}, \quad (17)$$
$$\|h\|_{\tilde{Y}} := \sup_{u\in[0,u_0]} \sup_{r\geq 0} \left\{(1+r)^2|h(u,r)|\right\}. \quad (18)$$

Let $g \equiv e^{\rho+\sigma}$, $\tilde{g} \equiv e^{\rho-\sigma}$, $\alpha \equiv 8\pi C_1$ and $\bar{h} = \frac{1}{r}\int_0^r h(u,s)ds$.

Dynamical Gravitating Skyrmions in 3+1 Bondi spacetime

The dynamical equations of (ξ, ρ, σ) can be recast into a single integro-differential equation on the characteristic curve, χ, namely

$$\frac{dh}{du'} = Ph + Q, \tag{19}$$

with

$$P \equiv \left[\frac{g-\tilde{g}}{2r} - \frac{\alpha}{2} gr \left(2\frac{\sin^2 \bar{h}}{r^2} + \frac{\sin^4 \bar{h}}{r^4}\right)\right]_\chi - \frac{1}{2}\frac{d}{du'} \ln\left(1 + 2\frac{\sin^2 \bar{h}}{r^2}\right)_\chi \tag{20}$$

$$\begin{aligned} Q \equiv & -\left[\bar{h}\frac{g-\tilde{g}}{2r} - \frac{\alpha}{2}\bar{h}gr\left(2\frac{\sin^2 \bar{h}}{r^2} + \frac{\sin^4 \bar{h}}{r^4}\right)\right]_\chi + \left(\frac{2\frac{\sin^2 \bar{h}}{r^2}}{1+2\frac{\sin^2 \bar{h}}{r^2}}\right)_\chi \frac{d}{du'}\bar{h}\Big|_\chi \\ & - g\cos\bar{h}\frac{\sin\bar{h}}{r}\Big|_\chi \left(\frac{1+\frac{\sin^2 \bar{h}}{r^2}}{1+2\frac{\sin^2 \bar{h}}{r^2}}\right)_\chi + \frac{\bar{h}}{2}\frac{d}{du'}\ln\left(1+2\frac{\sin^2 \bar{h}}{r^2}\right)_\chi . \end{aligned} \tag{21}$$

Dynamical Gravitating Skyrmions in 3+1 Bondi spacetime

Main Result

For every smooth initial data $h_0(r) \in \tilde{X}$, there exists $u_0 > 0$ such that unique smooth classical solution of spherically symmetric Einstein-Skyrme system, $h(u,r)$, exist on the interval $[0, u_0]$, taking $h_0(r) = h(0,r)$ as its initial data. This solution behaves like $h(u,r) = O(r^{-2})$ and $\frac{\partial h(u,r)}{\partial r} = O(r^{-3})$ as r goes to infinity for every u in $[0, u_0]$.

The existence cannot be extended to a global solution

Remark

Banach fixed-point theorem requires that $u_0 < \min[\eta, \nu]$ with,

$$\eta < \frac{1}{4\alpha\pi^4}\mathcal{W}_0\left(\frac{1}{60\alpha\pi^4}\right), \quad (22)$$

$$\nu < \frac{1}{\alpha\pi}\mathcal{W}_0\left(2\sqrt{3}\alpha^{5/2}\right), \quad (23)$$

where $\mathcal{W}_0(x)$ is the principal Lambert's W-Function.

Conclusions and Outlooks

1. The generalized Einstein-Skyrme system can be reduced into an analysis of PDE when an ansatz with certain symmetry is assumed.
2. There exists a family of classical solutions for dynamical standard Einstein-Skyrme system in $1+3$ spacetime with finite time interval. The time interval cannot be extended to infinity for non-zero topological charges.
3. We cannot be sure of the fate of gravitating Skyrmions beyond the time interval shown above. Since the topological charge is kept constant, then the only possible fate is either producing singularities or producing steady-state configurations that have been known in previous studies.
4. The more general case where the boundary conditions are not assumed to be constant is still an open problem for gravitating Skyrmions. The study is important in order to understand the connection between the charge of black holes and star configurations.

Thank You