A simple monopole model in 3+1D and its three topological quantum numbers

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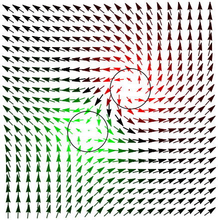
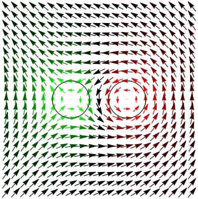




Together with: Joachim Wabnig, Josef Resch, Dominik Theuerkauf, Fabian Anmasser

finite radius of electrons?

Eduard Shpolsky, Atomic physics (Atomnaia fizika), second edition, 1951

“The issue of the radius of the electron is a challenging problem of modern theoretical physics. The admission of the hypothesis of a finite radius of the electron is incompatible to the premises of the theory of relativity. On the other hand, a point-like electron (zero radius) generates serious mathematical difficulties due to the self-energy of the electron tending to infinity.”

Shpolsky’s statement is incorrect,
as I will show

Problem of infinite self-energy of electron

solved by Hendrik Anthony Kramer 1947 :
subtraction of appropriately adjusted infinities from infinities

Martinus Veltman:
talk at the 65th Lindau Nobel Laureate Meeting in Konstanz (2015):

"May be at some future time we know more and we know how to deal with these infinities. May be we find a better theory, where you go to small distances, may be something happens there, but we postpone that problem. All we are going to say is whatever we do, the result for the mass of the electron is what we observe and how that comes about, who cares."

https://www.mediatheque.lindau-nobel.org/videos/34703/martinus-veltman-discovery-higgs-particle

Rotations as Field variables in 3+1D

consider field of rotations of spatial Dreibeins in $\mathbb{M}^4 \ni x$

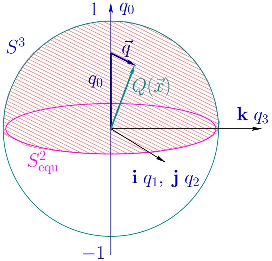
Use rotational group $D(x) \in SO(3)$

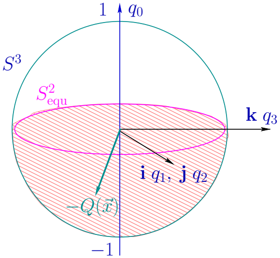
or simpler double covering group of SO(3): SU(2) $\ni Q(x)$

SO(3) versus SU(2)$\simeq \mathbb{S}^3$, $D(x) \leftrightarrow \pm Q(x)$ ... unit Quaternions

$Q(\vec{x}) = q_0(\vec{x}) - \mathrm{i}\vec{\sigma}\vec{q}(\vec{x}) = \cos\alpha(\vec{x}) - \mathrm{i}\vec{\sigma}\vec{n}(\vec{x})\sin\alpha(\vec{x}), \quad q_0^2 + \vec{q}^2 = 1$

Two hemispheres of $\mathbb{S}^3$

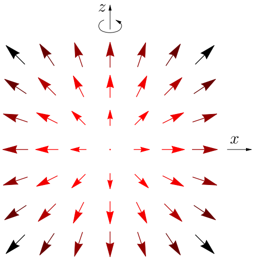




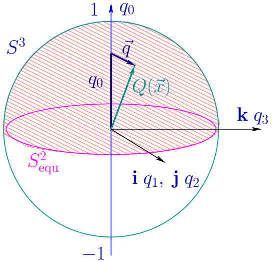
Field configurations $\pm\{Q(x)\}$ are identical

electrons as rotational knots in space

SU(2) is a three dimensional manifold



$R^3 \mapsto \mathrm{SU}(2)$
$x \mapsto Q(x)$



fullfilling Gauß law: $\Pi_2(\mathbb{S}^2) : S^2_\infty \mapsto S^2_{\mathrm{equ}}$
crossing unit charge:
rotate spatial Dreibeins along all axes in space by 2π

remember: Field configurations $\pm\{Q(x)\}$ are identical

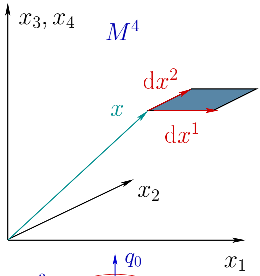
Lagrangian from geometry

$Q(x) \to \vec{\Gamma}_\mu(x) \to \vec{R}_{\mu\nu}(x) \to \mathcal{L}(x)$

Connection one-form = dual photon field

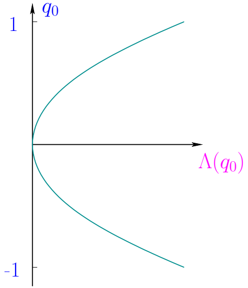
$\partial_\mu Q(x) =: -\mathrm{i}\vec{\Gamma}_\mu(x)\,\vec{\sigma}\,Q(x)$

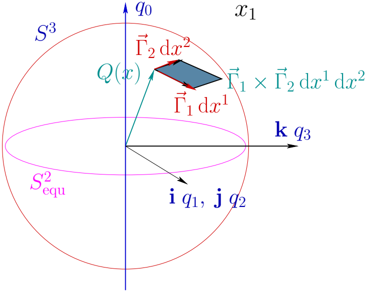
Curvature: $\vec{R}_{\mu\nu}(x) := \vec{\Gamma}_\mu(x) \times \vec{\Gamma}_\nu(x)$



Lagrangian: $\mathcal{L} = -\frac{\alpha_f \hbar c_0}{4\pi}\left(\frac{1}{4}\,\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} + \Lambda\right)$

potential term: $\Lambda(x) = q_0^6/r_0^4$





2D degeneracy of vacuum $\Pi_3(\mathbb{S}^2)$: $\Rightarrow$ two Goldstone bosons = photons

Relation to other models

- a 3D generalisation of the Sine-Gordon model
  1+1D $\to$ 3+1D, 1 dof $\to$ 3 dofs

- a model for soft dual Dirac monopoles
  no Dirac string, no singularity in the origin

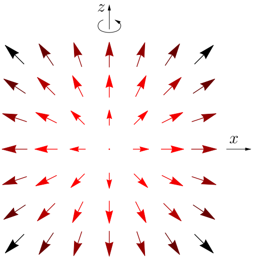
- a modification of the Skyrme model
  short range $\to$ long-range interaction

Relation to nature:

$$
{}^\star\vec{F}_{\mu\nu} := -\frac{e_0}{4\pi\varepsilon_0 c}\vec{R}_{\mu\nu} = \begin{pmatrix} 0 & \vec{B}_1 & \vec{B}_2 & \vec{B}_3 \\ -\vec{B}_1 & 0 & \frac{\vec{E}_3}{c} & -\frac{\vec{E}_2}{c} \\ -\vec{B}_2 & -\frac{\vec{E}_3}{c} & 0 & \frac{\vec{E}_1}{c} \\ -\vec{B}_3 & \frac{\vec{E}_2}{c} & -\frac{\vec{E}_1}{c} & 0 \end{pmatrix}. \tag{1}
$$

Stable minima of energy (topological Solitons)

- hedgehog ansatz: $\vec{n}(x) = \frac{\vec{r}}{r}$,
  $\vec{q}(x) = \vec{n}(x) \sin\alpha(x),\ q_0 = \cos\alpha(x)$,
  $\alpha = \alpha(\rho), \quad \rho = r/r_0$,
  $q_0^2 + \vec{q}^2 = 1$,
  $Q(x) = q_0(x) + \mathrm{i}\vec{\sigma}\vec{q}(x)$,
  soliton covers half of S^3



- minimisation of energy leads to non–linear differential equation

$$\partial_\rho^2 \cos\alpha + \frac{(1-\cos^2\alpha)\cos\alpha}{\rho^2} - m\rho^2 \cos^{2m-1}\alpha = 0$$

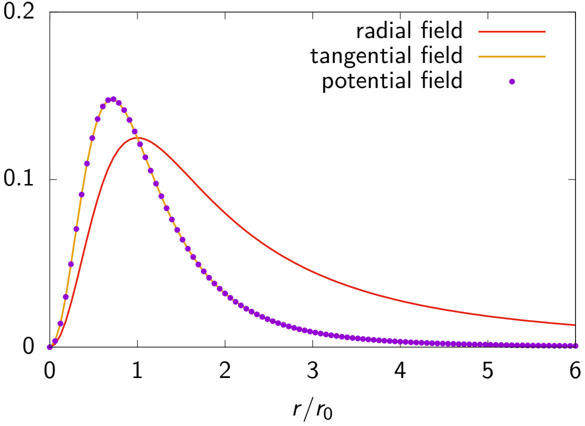
- solution for $m = 3$: $\quad \alpha(\rho) = \arctan(\rho)$.
- energy of soliton $\quad E = \frac{\alpha_f \hbar c}{r_0}\frac{\pi}{4}$
- compare with monopoles? with (non) existing?

$$\alpha_f \hbar c = 1.44\,\mathrm{MeV\,fm},\ m_e c^2 = 0.511\ \mathrm{MeV},\ r_0 = 2.21\ \mathrm{fm}$$

Energy densities **No Divergencies!**

$$\mathcal{L} = -\frac{\alpha_f \hbar c}{4\pi}\left(\frac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} + \frac{q_0^6}{r_0^4}\right), \quad q_0(\rho) = \cos\alpha(\rho) = \frac{1}{\sqrt{1+\rho^2}}$$

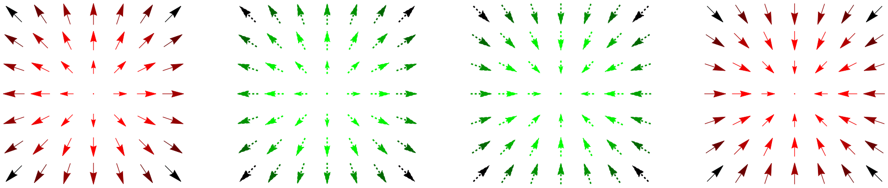
radial energy densities



particle and field are indistinguishable

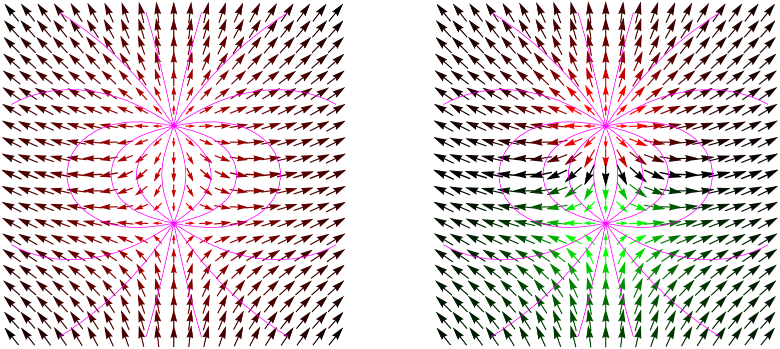
Four classes of solitons

$\mathcal{T} = 1$	$\mathcal{T} = z\Pi$	$\mathcal{T} = z$	$\mathcal{T} = \Pi$
$Z = 1$ $\mathcal{Q} = \frac{1}{2}$	$Z = 1$ $\mathcal{Q} = -\frac{1}{2}$	$Z = -1$ $\mathcal{Q} = \frac{1}{2}$	$Z = -1$ $\mathcal{Q} = -\frac{1}{2}$



Corresponding to Dirac spinor

Field lines of dipole



Left: $S = 0$ configuration, Right: $S = 1$ configuration
field lines = lines of constant $\vec{n}$-field

q_0-distribution of dipole

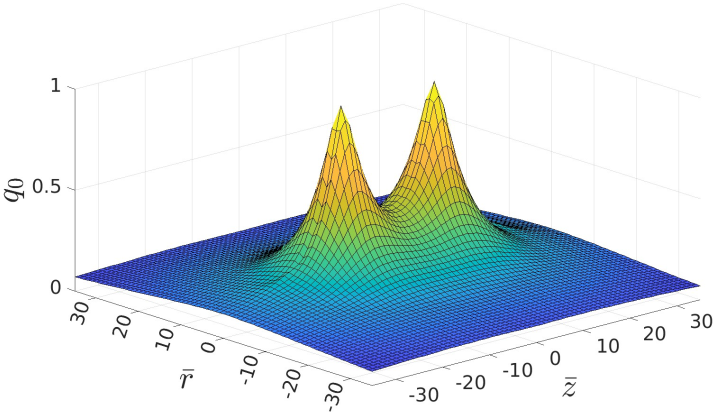


Figure: The energy density for a particle anti-particle solution

Energy density of dipole

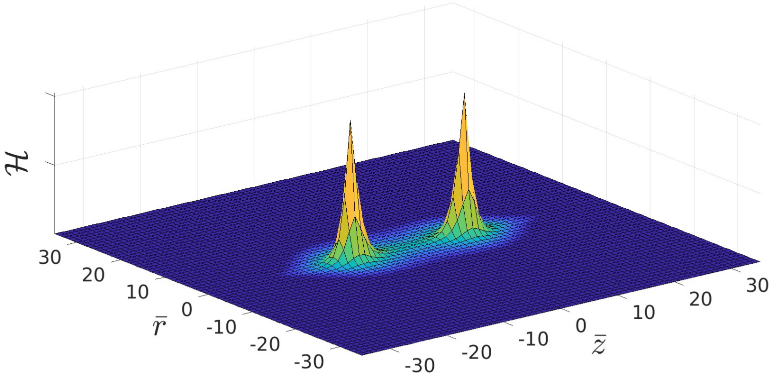
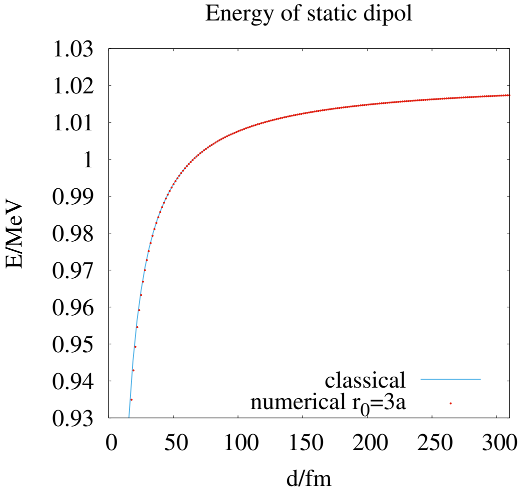


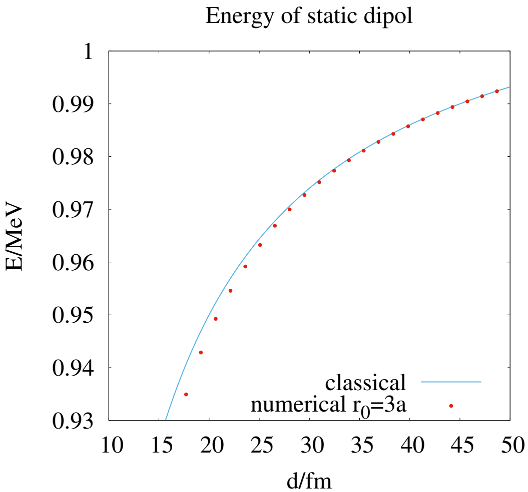
Figure: The energy density for a particle anti-particle solution

Soliton-antisoliton-potential



Total energy of a soliton pair for varying distance r.
together with: Fabian Anmasser and Dominik Theuerkauf

Soliton-antisoliton-potential



Total energy of a soliton pair for varying distance r.
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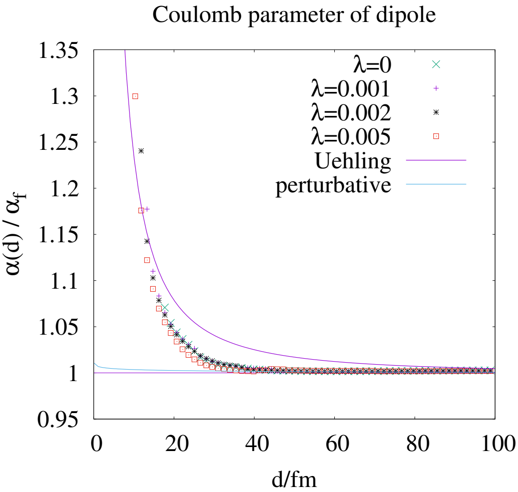
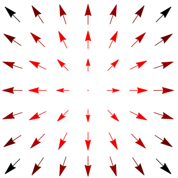
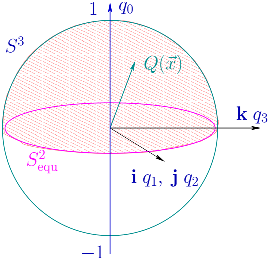
Running of Fine Structure Constant



Spin, a topological quantum number



$R^3 \rightarrow \mathbb{S}^3$



Field configuration $Q(\mathbf{r})$ of unit charge covers hemisphere of $\mathbb{S}^3$, $s = \frac{1}{2}$.

Spin quantum number s

$$s := |\mathcal{Q}| = \left| \frac{1}{V(\mathbb{S}^3)} \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \, \vec{\Gamma}_r (\vec{\Gamma}_\vartheta \times \vec{\Gamma}_\varphi) \right|$$

Magnetic quantum numbers $m_s = \pm 1/2$: Upper and lower hemisphere

$$\mathcal{Q} = \chi \cdot S$$

T.D.Lee: Why does the mass violate chiral symmetry?

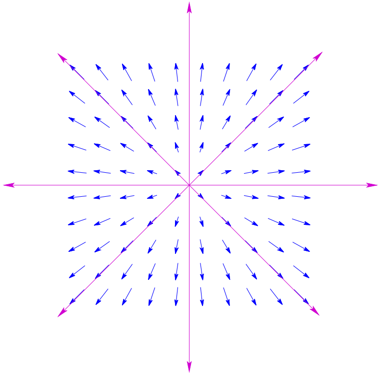
Monopole is wired to surrounding space

flux lines $\equiv$ lines of constant $\vec{n}$-field

flux lines $\equiv$ strings

they connect the soliton with the surrounding,

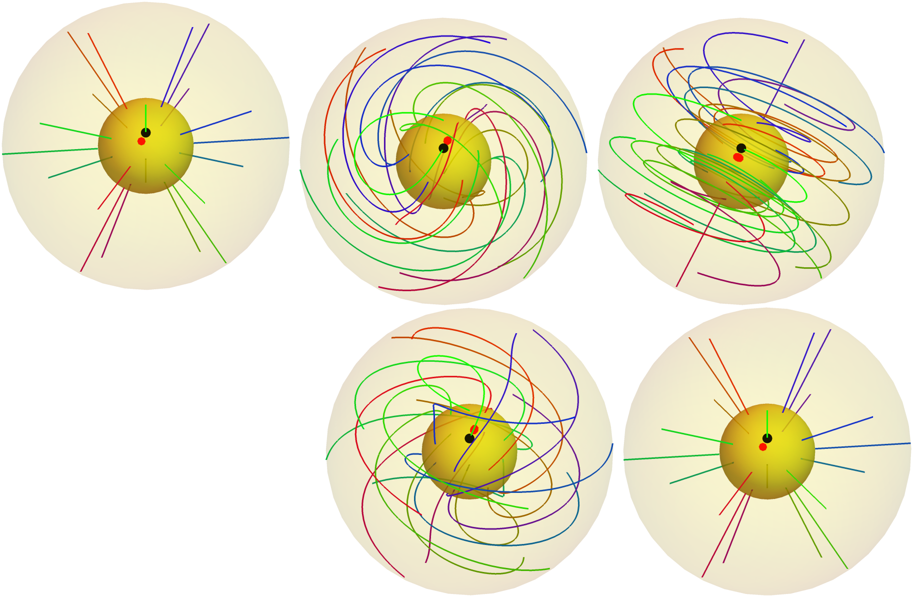
with other charges



after 4π-rotation
soliton configuration is restored

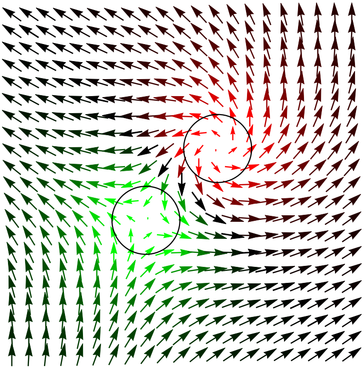
a property of spin-$1/2$-particles

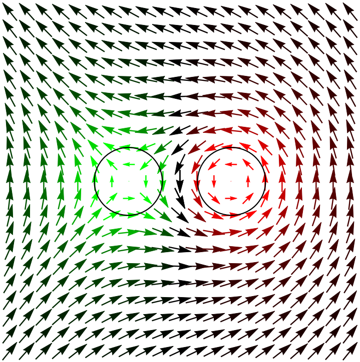
4π-rotations. . . wired to surroundings



Spin, an angular momentum

Symmetry broken vacuum, $Q(\infty) = -\mathrm{i}\sigma_3$, Field at infinity is constant
No rigid rotation possible





$S = 1$, Charge Zero

Imagine we have only Space-Time

What can we explain?

- Non-trivial metric: $g_{\mu\nu} \quad \rightarrow \quad$ Gravitation

- Rotating frames in $\mathbb{R}^3$: $D(x) \in SO(3)$

  Topological excitations $\leftrightarrow$ Topological quantum numbers

  $\Pi_3(\mathbb{S}^3) = \mathbb{Z} \quad \leftrightarrow$ spin

  $\Pi_2(\mathbb{S}^2) = \mathbb{Z} \quad \leftrightarrow$ charge

  $\Pi_3(\mathbb{S}^2) = \mathbb{Z} \quad \leftrightarrow$ photon number

  Non-topological excitations:

  dark matter?

  dark energy?

Comparison to Maxwell's electrodynamics

1. The Lagrangian is Lorentz covariant, thus the laws of special relativity are respected.
2. Charges have Coulombic fields fulfilling Gaußes law.
3. Charges interact via $\frac{1}{r^2}$ electric fields, they feel Coulomb and Lorentz forces.
4. A local U(1) gauge invariance is respected.
5. There are two dofs of massless excitations for photons.

In distinction to Maxwell's electrodynamics

1. Electric charges are quantised, like the magnetic charges of Dirac monopoles. Charge is a topological quantum number.
2. By topological construction, mirror properties of particles and antiparticles.
3. The mass of solitons is completely due to field energy and finite.
4. The self-energy of charges is finite and does not need regularisation and renormalisation.
5. Charges and their fields are described by the same SO(3) dofs.
6. SO(3) dofs interpreted as orientations of spatial Dreibeins.
7. Gauge symmetry a geometrical phenomenon,basis changes on S^3.
8. Spin has usual quantisation properties and combination rules.
9. 4 basic configurations of solitons, quantum numbers of Dirac spinors.

In distinction to Maxwell's electrodynamics

10. Solitons and antisolitons have opposite internal parity.
11. Solitons are characterised by a chirality quantum number which can be related to the sign of the magnetic quantum number.
12. Spin contributes to angular momentum due to internal rotations.
13. The canonical energy-momentum tensor is automatically symmetric.
14. Static charges are described by the spatial components of vector fields. Moving charges need time-dependent fields.
15. r-dependence of charge by finite size of solitons $\rightarrow$ running coupling.
16. Local U(1) gauge invariance explained, bases choice on $\mathbb{S}^2$.
17. Photon number $\rightarrow$ Gaußian linking number of fibres on $\mathbb{S}^2$.
18. Photon number changes by interaction with charges.

Rather unexpected

1. Spin and magnetic moment are dynamical properties only.
2. Electric and magnetic field vectors are perpendicular to each other
3. Existence of unquantised magnetic currents is allowed.
4. α-waves in $q_0 = \cos\alpha$ contribute to (dark) matter density.
5. α-waves lead to additional forces on particles and are a possible origin of quantum fluctuations.
6. Potential term allows mechanism of cosmic inflation
7. Potential term contributes to dark energy.

Aftermath

Physics is measurements of distances of objects and times of events.

This may indicate, that

Physics is geometry and not algebra.

Finally, one should use the algebra to describe the geometry.

General Relativity:

Wheeler: “Spacetime tells matter how to move;

matter tells spacetime how to curve.

My addition for Electrodynamics:

$\cdots$ Charges and electromagnetic fields tell space how to rotate.

Everything on earth is finite, besides $\cdots$

Thanks

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