Symmetries of Monopole Spectral Curves

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Outline

Background and Motivation

Background - Euclidean Monopoles

Given a principal $SU(2)$ bundle over $\mathbb{R}^3$ with connection A and section of the adjoint bundle ϕ, the data of a magnetic monopole is a solution of the BPS equation $F = \star D\phi$ satisfying boundary conditions that as $r \to \infty$,

1. $|\phi| = 1 - \frac{k}{2r} + \mathcal{O}(r^{-2})$,
2. $\frac{\partial|\phi|}{\partial\Omega} = \mathcal{O}(r^{-2})$,
3. $|D\phi| = \mathcal{O}(r^{-2})$.

$k \in \mathbb{Z}$ classifies solutions topologically, called the charge.

$4k$-dimensional moduli space of all charge-k monopoles M_k with action of $E(3)$. $(4k-4)$-dimensional submanifold of strongly centred monopoles M_k^0 with action of $O(3)$.

Dynamics of monopoles approximated by geodesic motion in moduli space.

Background - Spectral Curve

Recall minitwistor space $\pi : T\mathbb{P}^1 \cong \mathcal{O}(2) \to \mathbb{P}^1$ with involution $\tau(\zeta,\eta) = (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$ and $L \to T\mathbb{P}^1$ with transition function $\exp(-\eta/\zeta)$.

Given spectral curve $S \subset T\mathbb{P}^1$ a compact algebraic curve in the linear system $|\pi^*\mathcal{O}(2k)|$, the Hitchin conditions on S are

- S has no multiple components,
- S real wrt τ,
- $L^2 \to S$ trivial, $L(k-1) \to S$ real,
- $\forall s \in (0,2)$, $H^0(S, L^s(k-2)) = 0$.

Write S as vanishing of polynomial $P(\eta,\zeta) = \eta^k + \sum_{i=1}^{k} a_i(\zeta)\eta^{k-i}$, $\deg(a_i) = 2i$. $g(S) = (k-1)^2$.

Example: $k = 1$

$$P(\eta,\zeta) = \eta - \left[(x_1 + ix_2) - 2ix_3\zeta + (x_1 - ix_2)\zeta^2\right]$$

Background - Nahm Data

The data $\{ T_i(s) \,|\, T_i \in M_k(\mathbb{C}_\infty),\, s \in [0,2] \}$ is called Nahm data if

- the T_i satisfy Nahm's equation,

$$\frac{dT_i}{ds} = \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} \left[T_j, T_k \right] ,$$

- $\forall s \in (0,2)$, $T_i(s)$ are regular, simple poles at $s = 0, 2$, residues form an irreducible k-dimensional rep of $SU(2)$,
- $T_i(s) = -T_i^\dagger(s),\ T_i(s) = T_i^T(2-s)$.

Nahm's equations have Lax formulation $\dot{L} = [L, M]$,

$$L = (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2, \quad M = -iT_3 + (T_1 - iT_2)\zeta.$$

Example: $k = 1$

$$T_j = ix_j \quad \text{constant.}$$

Background - Circle of Ideas

Theorem (Hitchin, 1983)
TFAE:

1. *the data of a magnetic monopole,*
2. *a spectral curve satisfying the Hitchin conditions,*
3. *Nahm data.*

Spectral curve of the Nahm Lax pair is the associated spectral curve.

Centering monopole corresponds to $a_1 = 0$ in spectral curve, $\text{Tr}(T_i) = 0$ for Nahm data.

Constructing gauge fields from Nahm data involves solving an ODE ("bold adaptation of the ADHM construction of instantons").

Charge-2

Example: Charge-2 Monopoles
dim $M_2^0 = 4$. Quotienting by residual $SO(3)$ action leaves 1-parameter family.

Nahm data $T_j(s) = \frac{1}{2i} f_j(s)\sigma_j$, σ_j Pauli matrices, f_j real functions, gives

$$\dot{f}_1 = f_2 f_3 \quad + \quad \text{cycles},$$

solved in terms of elliptic functions.

Spectral curve

$$\eta^2 + \frac{1}{4}K(k)^2 \left[\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1\right] = 0.$$

Scattering of two 1-monopoles understood in terms of parameter k.

Charge-2 Pictures

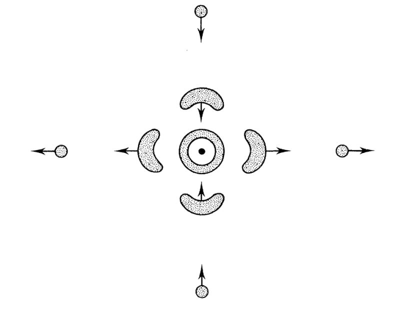


FIGURE 2. Schematic diagram of the direct collision process.

(a) Low-Energy Scattering of Non-Abelian Magnetic Monopoles [AH88]

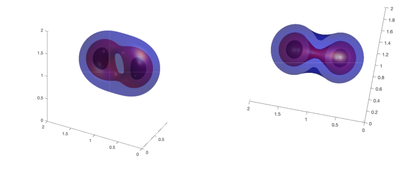


FIGURE 4. Two views of the Energy density $\mathcal{E}(x)$ for $k = 0.8$. Blue corresponds to the isocontour $\mathcal{E}(x) = 0.2$, red to $\mathcal{E}(x) = 0.42$, and dark red to $\mathcal{E}(x) = 0.7$.

(b) The charge 2 monopoles via the ADHMN construction [BE21]

Problem

So what goes wrong? Constructing Nahm data from nothing is hard, better if we have a spectral curve to aim for, but spectral curve conditions are also hard.

Theorem (Braden, 2018)
Spectral curves are transcendental.

Few solutions found this way.

$P(\eta,\zeta)$	G	
η	$SO(3)$	[Hit82]
$\eta\prod_{l=1}^{m}(\eta^2+l^2\pi^2\zeta^2)$	$SO(2)$	[Hit82]
$\prod_{l=0}^{m}(\eta^2+[l+1/2]^2\pi^2\zeta^2)$	$SO(2)$	[Hit82]
$\eta^2+\frac{1}{4}K(k)^2\left[\zeta^4+2(k^2-k'^2)\zeta^2+1\right]$	C_2	[Hur83]
$\eta^3+\alpha\eta\zeta^2+\gamma\zeta^3+\beta(\zeta^6-1)^*$	C_3	[BDE11]

*Coefficients defined implicitly by vanishing of period on genus-2 curve.

Nahm Data from Symmetry Considerations

Representation Theory

Method of Hitchin, Manton, Murray, 1995. $SO(3)$ rep space of (T_i) is

$$\begin{aligned}\mathbb{R}^3 \otimes \mathfrak{sl}_{\mathbb{C}}(k) &= \mathbb{S}^2 \otimes (\mathbb{S}^{2k-2} \oplus \cdots \oplus \mathbb{S}^2), \\ &= (\mathbb{S}^{2k}_{-1} \oplus \mathbb{S}^{2k-2}_0 \oplus \mathbb{S}^{2k-4}_1) \oplus \cdots \oplus (\mathbb{S}^4_{-1} \oplus \mathbb{S}^2_0 \oplus \mathbb{S}^0_1),\end{aligned}$$

$\mathbb{S}^r$ vector space of degree-r homogeneous bivariate polynomials with $PSU(2)$ action. Given $P \in \mathbb{S}^{2r}_i$, $\rho : \mathfrak{so}(3) \to \mathfrak{sl}_{\mathbb{C}}(k)$, realise associated matrix by

$$\mathbb{S}^{2r}_i \to \mathbb{S}^{2r}_{-1} \overset{\text{Pol}}{\to} \mathbb{S}^2 \otimes \mathbb{S}^{2r-2}\Big|_{\mathbb{S}^{2r}_{-1}} \overset{hwv}{\to} \mathbb{S}^2 \otimes \mathbb{S}^{2(r+i)}\Big|_{\mathbb{S}^{2r}_i} \overset{hwv}{\to} \mathbb{R}^3 \otimes \mathfrak{sl}_{\mathbb{C}}(k)\Big|_{\mathbb{S}^{2r}_i}.$$

Mapping respects group action, so given $G \leq SO(3)$ and $P \in (\mathbb{S}^{2r})^G$, get vectors $(S_i) \in (\mathbb{R}^3 \otimes \mathfrak{sl}_{\mathbb{C}}(k))^G$. Note

$$\langle(\rho_i)\rangle = \left(\mathbb{R}^3 \otimes \mathfrak{sl}_{\mathbb{C}}(k)\right)^{SO(3)} \leq (\mathbb{R}^3 \otimes \mathfrak{sl}_{\mathbb{C}}(k))^G \leq \left(\mathbb{R}^3 \otimes \mathfrak{sl}_{\mathbb{C}}(k)\right)^{\{e\}}.$$

Representation Theory

Assuming a spanning set $\left\langle(\rho_i),(S_i^{(j)})\,,\, j=1,\ldots,d\right\rangle = (\mathbb{R}^3\otimes\mathfrak{sl}_{\mathbb{C}}(k))^G$, writing $T_i = x\rho_i + y_j S_i^{(j)}$ (summing over repeated indices) gives equations for G-invariant Nahm data

$$
\begin{aligned}
[\rho_1,\rho_2] &\propto \rho_3,\\
\left[\rho_1, S_2^{(j)}\right] + \left[S_1^{(j)},\rho_2\right] &= \alpha^{(j)}\rho_3 + \beta^{(j,k)}S_3^{(k)},\\
\left[S_1^{(j)}, S_2^{(k)}\right] + \left[S_1^{(k)}, S_2^{(j)}\right] &= (1+1_{j=k})(\gamma^{(j,k)}\rho_3 + \delta^{(j,k,l)}S_3^{(l)}), \quad k\le j,
\end{aligned}
$$

$+$ cycles, which can be solved with Gröbner bases or vectorisation, where the ability to solve imposes constraints on x, y_j. Nahm's equations become

$$
\begin{aligned}
x' &= 2x^2 + \alpha^{(k)}xy_k + \gamma^{(k,l)}y_k y_l,\\
y_j' &= \beta^{(k,j)}xy_k + \delta^{(k,l,j)}y_k y_l.
\end{aligned}
$$

Remains to solve with correct singularities and reality conditions.

Example Calculation

```
load("nahm_data.py")

K.<j> = NumberField(polygen(QQ)^2 + 1)
_.<z,w> = K[]
j = (polygen(K)^2+1).roots(multiplicities=False)[0]

r1 = matrix(K, [[0, 0, 0], [0, 0, -2], [0, 2, 0]])
r2 = matrix(K, [[0, 0, 2], [0, 0, 0], [-2, 0, 0]])
r3 = matrix(K, [[0, -2, 0], [2, 0, 0], [0, 0, 0]])

Q = j*w*z*(w^4 - z^4)

rs, Ss = find_invariant_vectors([r1, r2, r3], Q)
make_hermitian(Ss)

_, constraints = solve_commutation_relations(rs, Ss)
ODEs = ode_system(rs, Ss)
f = spectral_curve(rs, Ss)
Ts = nahm_matrices(rs, Ss)

print("Constraints on coefficients:", constraints)
print("ODEs:")
pretty_print(ODEs)
print("Spectral curve:")
pretty_print(f)
print("Nahm data:")
pretty_print(Ts)
```

```
Succeeded in making all matrices anti-Hermitian:
True
Constraints on coefficients: []
ODEs:
```

$$\left[2x^2 - 50y_0^2, -4xy_0\right]$$

```
Spectral curve:
```

$$\left(480jx^2y_0 - 4000jy_0^3\right)\zeta^5 + \eta^3 + \left(-480jx^2y_0 + 4000jy_0^3\right)\zeta$$

```
Nahm data:
```

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2x+10jy_0 \\ 0 & 2x+10jy_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2x+10jy_0 \\ 0 & 0 & 0 \\ -2x+10jy_0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2x+10jy_0 & 0 \\ 2x+10jy_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

Many New Curves

This approach allowed for the construction of many new curves with distinguished symmetries.

$P(\eta,\zeta)$	G	
$\eta^3 + ia_3\zeta(\zeta^4-1)$	A_4	[HMM95]
$\eta^3 - 6(a^2\pm 4)^{1/3}\kappa^2\eta\zeta^2 + 2i\kappa^3 a\zeta(\zeta^4-1)$	C_4	[HS96b]
$\eta\left\{\eta^2 - K(k)^2\left[k^2(\zeta^4+1)+2(k^2-2)\zeta^2\right]\right\}$	C_2 (inv)	[HS96a]
$\eta^4 + a_4(\zeta^8+14\zeta^4+1)$	S_4	[HMM95]
$\eta^4 + 36ia\kappa^3\eta\zeta(\zeta^4-1) + 3\kappa^4(\zeta^8+14\zeta^4+1)$	A_4	[HS96d]
$\eta\left[\eta^4 + 4a_4(\zeta^8+14\zeta^4+1)\right]$	S_4	[HS96c]
$\eta\left[\eta^6 + a_7\zeta(\zeta^{10}+11\zeta^5-1)\right]$	A_5	[HS96c]

Those with a free parameter (k or κ) give scattering in the moduli space.

Being More Methodical

Q: In charge 3, what remaining potential symmetric monopoles are there? A: Wiman, 1895 (English translation DH, Beckett, Deutsch, 2022) writes down equations for all non-hyperelliptic genus-4 curves and their automorphism groups.

Such curves are $Q \cap C \subset \mathbb{P}^3$, Q either non-singular or a cone. $T\mathbb{P}^1 \hookrightarrow \mathbb{P}^{1,1,2} \cong Q_{\text{cone}}$, and write C in terms of $T\mathbb{P}^1$ coordinates as

$$\eta^3 + f_2(\zeta_0, \zeta_1)\eta^2 + f_4(\zeta_0, \zeta_1)\eta + f_6(\zeta_0, \zeta_1) = 0.$$

Hence all candidate charge-3 monopole spectral curves have been written down, remains to impose reality and other Hitchin conditions.

Full list is too long (21 curves), so need a way to simplify.

Genus-1 Quotients

Idea: Ask for spectral curves S with $G \leq \mathrm{Aut}(S)$ s.t. $g(S/G) = 1$. Can enumerate these based on work of Breuer and others. They can only give G and associated signature of quotient c, but with a bit of work (and Sage's Riemann surfaces functionality) can match these to Wiman.

f	G	c	δ
$\eta^3 + \eta(a\zeta^4 + b\zeta^2 + c) + (d\zeta^6 + e\zeta^4 + f\zeta^2 + g)$	C_2	$[1; 2^6]$	6
$\eta^3 + \eta(a\zeta^3 + b) + (\zeta^6 + c\zeta^3 + d)$	C_3	$[1; 3^3]$	3
$\eta^3 + a\eta\zeta^2 + \zeta(\zeta^4 + 1)$	C_4	$[1; 4^2]$	2
$\eta^3 + \eta[a(\zeta^4 + 1) + b\zeta^2] + \zeta(\zeta^4 - 1)$	C_2^2	$[1; 2^3]$	3
$\eta^3 + a\eta\zeta^2 + (\zeta^6 + 1)$	S_3	$[1; 2^2]$	2
$\eta^3 + a\eta\zeta^2 + (\zeta^6 + 1)$	C_6	$[1; 2^2]$	2
$\eta^3 - \zeta(\zeta^4 + 1)$	A_4	$[1; 2]$	1

A_4 and C_4 curves already found in [HMM95, HS96b]. C_3 family is just A_4 curve. Pullback condition rules out C_2. $\dim_{\mathbb{R}}(M_3^0)^G \leq \delta$, δ determined by c only.

New Spectral Curves!

Theorem (DH, Braden, 2023)
Given $\alpha \in [0,1]$, *define*

$$\tau = \tau(\alpha) = i\,\frac{{}_2F_1(1/6, 5/6, 1; 1-\alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)}.$$

Solving

$$\frac{1}{3}\alpha_2^2 = \frac{1}{4}g_2(1,\tau), \quad \frac{1}{27}\alpha_2^3 - 2\beta^2 = \frac{1}{8}g_3\left(1,\tau\right),$$

with $\mathrm{sgn}(\alpha_2) = \mathrm{sgn}(\alpha)$ *yields a monopole spectral curve with* D_6 *symmetry*

$$\eta^3 + \alpha_2\eta\zeta^2 + \beta(\zeta^6 - 1) = 0.$$

Moreover, the Nahm data is known explicitly in terms of $\wp$*-functions.*

Specialisation of [BDE11] curve made explicit. Can understand scattering in terms of α, and get explicit values for α_2, β when $\alpha = 1/2$.

New Spectral Curves!

Theorem (DH, Braden, 2023)
Given $\alpha \in \mathbb{R}$, $m \in [0,1]$, *and* $\mathrm{sgn} = \pm 1$, *define* g_2, g_3 *by* $g_2 = 12\left(K(m)^2/3\right)^2 q_1(m)$, $g_3 = 4\left(K(m)^2/3\right)^3 (2m-1) q_2(m)$, *where*

$$q_1(m) = \begin{cases} 1 - m + m^2 & sgn = 1 \\ 1 - 16m + 16m^2 & sgn = -1 \end{cases}, \quad q_2(m) = \begin{cases} (m-2)(m+1) \\ 2(32m^2 - 32m - 1) \end{cases}$$

If m *is such that* $g_2 > 0$ *and the polynomial* $(4 - 2\alpha)x^3 - g_2 x - g_3$ *has a real root* x_* *with* $|x_*| < \sqrt{g_2/3}$ *and* $\mathrm{sgn}(x_*) = -\mathrm{sgn}(\alpha)$, *then we may solve*

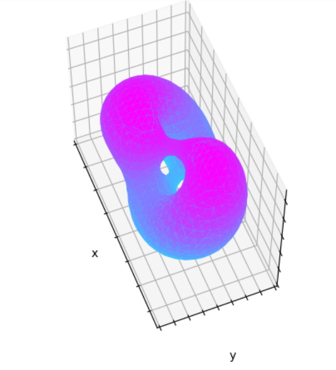
$$a^2 + \frac{b^2}{12} = g_2, \quad \frac{b(b^2 - 36a^2)}{216} + \frac{c^2}{4} = g_3$$

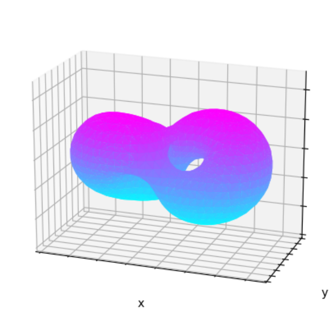
for $a, b, c \in \mathbb{R}$. *Then*

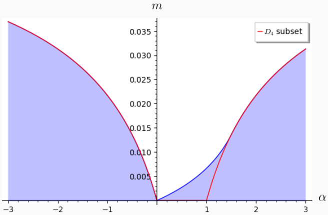
$$\eta^3 + \eta\left[a(\zeta^4 + 1) + b\zeta^2\right] + ic\zeta(\zeta^4 - 1) = 0$$

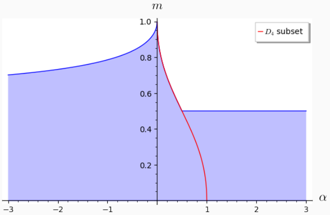
is a monopole spectral curve with V_4 *symmetry. Moreover the Nahm data is given explicitly in terms of elliptic functions.*

New Spectral Curves!

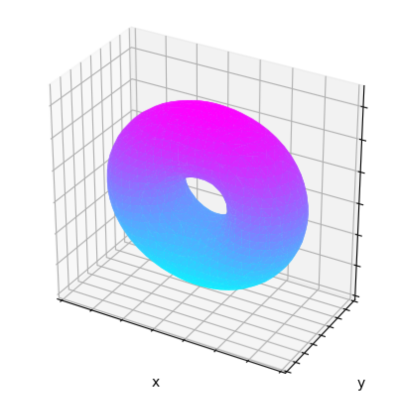




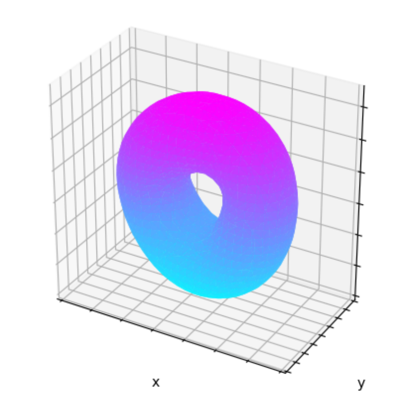




New Spectral Curves

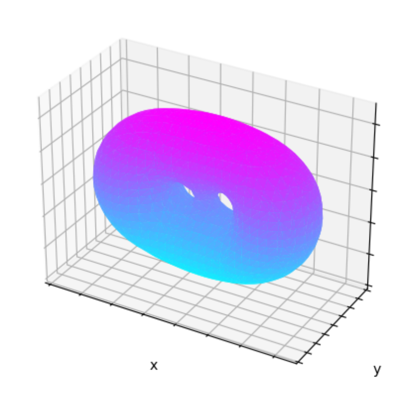


(a) $k = 0.45,\ \alpha = 0.2,\ \Delta > 0,\ b = -3.21$

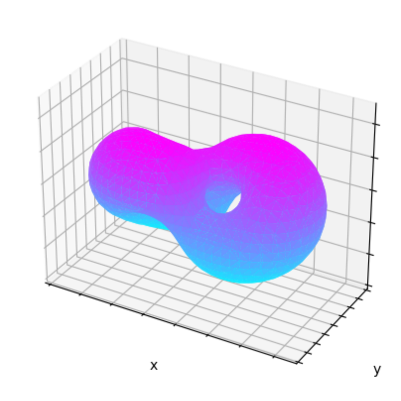


(b) $k = 0.45,\ \alpha = 0.2,\ \Delta > 0,\ b = -7.19$

New Spectral Curves



(a) $k = 0.77,\ \alpha = -2.0,\ \Delta > 0,\ b = 1.42$



(b) $k = 0.77,\ \alpha = -2.0,\ \Delta > 0,\ b = 7.24$

Outlook

Outlook

Questions for charge 3:

- Extend to the other curves on Wiman's list, solving in terms of e.g. hyperelliptic functions?

Further questions:

- Geometry can be extended to higher charges.
- What can we do for hyperbolic monopoles (e.g. [NR07])?
- What happens for higher gauge groups with different symmetry breaking?
- Can we formalise the idea that if $g(S/G) = 1$, we can solve, from a Nahm perspective?
- Interpretation of the representation theory.
- $\dim = \delta - 1$ for elliptic quotients?

Idea of Computation

Using Sutcliffe ansatz, C_3-invariant monopole written in in terms of periodic $A_2^{(1)}$ Toda

$$\dot{a}_i = \frac{1}{2} a_i (b_i - b_{i+1}), \quad \dot{b}_i = a_i^2 - a_{i-1}^2,$$

$i = 1, 2, 3$ taken mod 3. Imposing additional symmetry gives restriction $a_1^2 - a_2^2 = 0 = b_2$. Have constants $\alpha_2 = -b_1^2 + a_0^2 + 2a_1^2$, $\beta = -a_0 a_1^2$, so reduces to elliptic equation. Have to fix the real period of the corresponding lattice, which requires inverting a j-invariant using formula of Ramanujan. Remaining constraints of reality and pole structure fall into place.

Reduction can be seen to come from folding the Dynkin diagram $A_2^{(1)} \to A_2^{(2)}$, and leaves the Bullough-Dodd equation. Alternatively from enforcing the residue to be irreducible.

Idea of Computation

Find matrices

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{f}_1 \\ 0 & f_1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & f_2 \\ 0 & 0 & 0 \\ -\bar{f}_2 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -\bar{f}_3 & 0 \\ f_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and equations reduce to $\dot{f}_1 = \bar{f}_2 \bar{f}_3$ and cycles. Can separate this into equations for $|f_i|$ and $\arg(f_i)$ to solve.

Requiring that the f_i are real is what sets $c = 0$, then these are inversion-symmetric monopoles. Requiring $f_1 = f_2$ sets $a = 0$ and these are monopoles 'with a twist'.

Special Curve

$\tau(1/2) = i$, so this correspond to the square lattice. Coefficients are

$$\alpha_2 = \frac{\sqrt{3}\Gamma(1/4)^4}{8\pi}, \quad \beta = \pm\frac{\Gamma(1/4)^6}{32(\sqrt{3}\pi)^{3/2}}.$$

Related to the "twisted figure-of-eight" monopole of [HS96b], where automorphism group is D_8.

- M. F. Atiyah and N. J. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press, 1988.
- H. W. Braden, A. D'Avanzo, and V. Z. Enolski, *On charge-3 cyclic monopoles*, Nonlinearity **24** (2011), no. 3, 643–675.
- H. W. Braden and V. Z. Enolski, *The charge 2 monopole via the ADHMN construction*, Advances in Theoretical and Mathematical Physics **25** (2021), 791–956, Appendix by David E. Braden, Peter Braden, and H. W. Braden.
- N. J. Hitchin, *Monopoles and geodesics*, Communications in Mathematical Physics **83** (1982), no. 4, 579–602.
- N. J. Hitchin, N. S. Manton, and M. K. Murray, *Symmetric monopoles*, Nonlinearity **8** (1995), no. 5, 661–692.

References ii

- C. J. Houghton and P. M. Sutcliffe, *Inversion symmetric 3-monopoles and the Atiyah - Hitchin manifold*, Nonlinearity **9** (1996), no. 6, 1609.
- ______, *Monopole scattering with a twist*, Nuclear Physics B **464** (1996), no. 1, 59–84.
- ______, *Octahedral and dodecahedral monopoles*, Nonlinearity **9** (1996), no. 2, 385.
- ______, *Tetrahedral and cubic monopoles*, Communications in Mathematical Physics **180** (1996), no. 2, 343–361.
- J. Hurtubise, $\mathrm{SU}(2)$ *monopoles of charge* 2, Communications in Mathematical Physics **92** (1983), no. 2, 195–202.

References iii

P. Norbury and N. M. Romão, *Spectral curves and the mass of hyperbolic monopoles*, Communications in Mathematical Physics **270** (2007), no. 2, 295–333.