Tesserons from Monopoles

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Tesserons

We require all Riemannian four-manifolds below to be complete with L^2 Riemann curvature:

$$\int_{M^4} |Rm|^2 dVol < \infty$$

- **Gravitational Instanton** (defined by Hawking `77) is any solution of the vacuum Einstein equations:

$$Rm = \Lambda g.$$

- **Self-dual Gravitational Instanton** has self-dual Riemann curvature tensor

$$Rm = * Rm.$$

Locally, these are hyperkähler!

- Main interest: **Hyperkähler gravitational instanton**:

“Complete hyperkähler manifold of real dimension 4 with L^2 Riemann curvature tensor”

Q: What should we call it?

Based on the underlying quaternionic structure:

Quaternion? Quatron? Twiston? Hamilton?

- A **Tesseron** is a “Complete noncompact hyperkähler manifold of real dimension 4 with L^2 Riemann curvature tensor.”

Classification of Tesserons was recently completed:

- ALE: Kronheimer ‘89
- ALF: Minerbe ’07,’08
- ALG & ALH: G. Chen and X.-X. Chen ‘15; G.Chen and Viaclovsky ‘21
- ALG*: G. Chen and Viaclovsky ‘21
- ALH*: Hein, Sun, Viaclovsky, Zhang ’21; Collins, Jacob, Lin ’21; Lee, Lin ‘22

Tesserons are distinguished by their asymptotic **Volume Growth:**

A ball of geodesic ball of radius R

Space	Vol (B_R)
ALE	R^4
ALF	R^3
ALG	R^2
ALH	R, $R^{4/3}$

Asymptotic Model

As it happens, all tesseron metrics locally have asymptotic triholomorphic isometry.

According to Gibbons-Hawking, a metric with such an isometry locally has the form

$$g = V\vec{x}^2 + \frac{(d\tau + \omega)^2}{V}, \text{ where } *_3 dV = d\omega,$$

Tesserons' model ends have (locally):

- ALE $V = \frac{N}{2|\vec{x}|}$
- ALF $V = \ell + \frac{N}{2|\vec{x}|}$
- ALG $V = C + \frac{N}{2}\ln(x_1^2 + x_2^2)$
- ALH $V = C + Nx_1$

Current literature distinguishes:

ALG* and ALH* are spaces with $N \neq 0$, and

ALG and ALH are with $N = 0$ (locally constant fiber).

Prototypical example:

Asymptotic metric:

- ALE $V = \frac{N}{2|\vec{x}|}$

$\mathbb{R}^4$ metric in 'radial coordinates'

$$g = \frac{1}{2x} d\vec{x}^2 + 2x(d\theta + \omega)^2$$

Circle fibration (growing circle):
Quotient: $\mathbb{R}^4/\Gamma$,
$\Gamma \subset SU(2)$.

- ALF $V = \ell + \frac{N}{2|\vec{x}|}$

The **Taub-NUT**:

$$g = (\ell + \frac{1}{2x}) d\vec{x}^2 + \frac{(d\theta + \omega)^2}{\ell + \frac{1}{2x}}$$

Circle fibration (with stabilizing circle):

- ALG $V = C + \frac{N}{2} \ln(x_1^2 + x_2^2)$

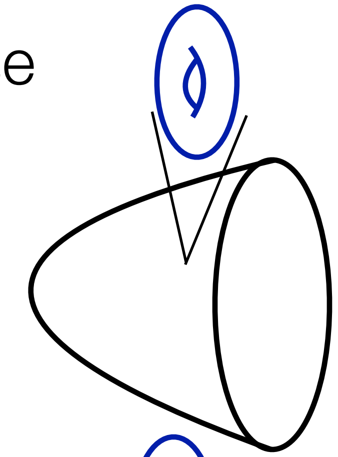
(ALG* if $N \neq 0$)

Elliptic Fibrations:

$$g = \tau_2 dz d\bar{z} + \frac{|d\theta_a + \tau d\theta_b|^2}{\tau_2},$$

$$\tau = \tau_1 + i\tau_2 = C + N \frac{i}{2\pi} \ln z$$

Over cone base

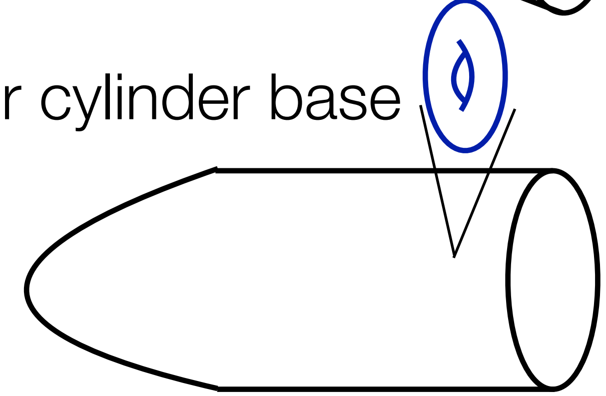


- ALH $V = C + Nx_1$

(ALH* if $N \neq 0$)

$$\tau = \tau_1 + i\tau_2 = C + iNz$$

Over cylinder base



Classification

ALE: $\mathbb{R}^4$, $A_{k\geq 1}$, $D_{k\geq 1}$, and E_6, E_7, E_8 $A_0 = \mathbb{R}^4, A_1 =$ Eguchi-Hanson

ALF: $\mathbb{R}^3 \times S^1$, $A_{k\geq 0}$ and $D_{k\geq 0}$

$A_0 =$ Taub-NUT

$A_k = (k+1)$-centered multi-Taub-NUT

$D_0 =$ Atiyah-Hitchin,

$D_1 =$ deformation of double cover of D_0

$\mathbf{D_2} =$ deformation of $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$

ALG:

& ALG*

$\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$

$\mathbb{R}^2 \times T^2$, D_0, D_1, D_2, D_3, $\mathbf{D}_4$,

Ш₂ , Ш₁ , Ш₀

ALH:

& ALH*

$\mathbb{R} \times T^3$, E_0, E_1, $\tilde{E}_1$, E_2, E_3, E_4, E_5, E_6, E_7, E_8, $\mathbf{\frac{1}{2}K3}$

Naive Parameter Count

m_n denotes n real parameters specifying the form of infinity and
m “interior” parameters.

ALE: $\mathbb{R}^4$, $A_{k\geq 1}$, $D_{k\geq 1}$, and $E_{k=6,7,8}$.

$(0)_0$ $(3k)_0$ $(3k)_0$ $(3k)_0$

Note: additional isometries
reduce this Naive count,
e.g. A_1 ALE — $(1)_0$
and A_1 ALF — $(1)_1$

ALF: $\mathbb{R}^3 \times S^1$, $A_{k\geq 0}$ and $D_{k\geq 0}$

$(0)_1$ $(3k)_1$ $(3k)_1$

ALG:

& ALG*

$\mathbb{R}^2 \times T^2$, $(0)_3$

I^*_{4-k} $D_{k=0,1,2,3}$ $(3k)_3$

I^*_0 $\mathbf{D}_4$, $(3k)_3$

$\mathbf{E}_{\mathbf{k=6,7,8}}$ $(3k)_1$

$\mathbf{Ш}_{\mathbf{k=0,1,2}}$ $(2+3k)_1$ (?)

i.e.

IV^* III^* II^*

$\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$

IV III II

$\mathbf{Ш}_2$, $\mathbf{Ш}_1$, $\mathbf{Ш}_0$

ALH:

& ALH*

$\mathbb{R} \times T^3$, $(0)_7$

E_0, E_1,

$\tilde{E}_1$,

E_2, E_3, E_4, E_5, E_6, E_7, E_8, $(3k)_3$

$\frac{\mathbf{1}}{\mathbf{2}}\mathbf{K3}$ $(3x8)_7$

Discrete Painleve Eqs.

Noumi et al 1998
Hidetaka Sakai 2001
(Table 2)

The list of ALG$^{(*)}$ and ALH$^{(*)}$ tesserons coincides with Discrete Painlevé Equations (and associated rational surfaces)!

Note: As a result all ALG and ALG* tesserons are moduli spaces of Hitchin systems!

ALG & ALG* **Rational** Painlevé:

$$\begin{tikzcd}
E_8^{(1)} & E_7^{(1)} & E_6^{(1)} & D_4^{(1)} & A_3^{(1)} & (A_1+A_1)^{(1)} & A_1'^{(1)} & A_0^{(1)} \\
E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] \arrow[dl] & D_3 \arrow[r] & D_2 \arrow[r] & D_1 \arrow[r] & D_0 \\
Ш_0 & Ш_1 \arrow[l] & Ш_2 \arrow[l] & & & & & \\
A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & & & & &
\end{tikzcd}$$

ALH **Trigonometric** Painlevé:

$$\begin{tikzcd}
 & & & & & & & \tilde{E}_1^{(1)} & \\
 & & & D_5^{(1)} & & (A_2+A_1)^{(1)} & & & \\
E_8^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_6^{(1)} \arrow[r] & E_5^{(1)} \arrow[r] & E_4^{(1)} \arrow[r] & E_3^{(1)} \arrow[r] & E_2^{(1)} \arrow[r] \arrow[uur] & E_1^{(1)} \arrow[r] & E_0^{(1)} \\
 & & & & A_4^{(1)} & & (A_1+A_1')^{(1)} & A_1'^{(1)} & A_0^{(1)}
\end{tikzcd}$$

ALH* **Elliptic** Painlevé:

$$E_8^{(1)} = \frac{1}{2}K3$$

Tesserons as Moduli Spaces of Monopoles

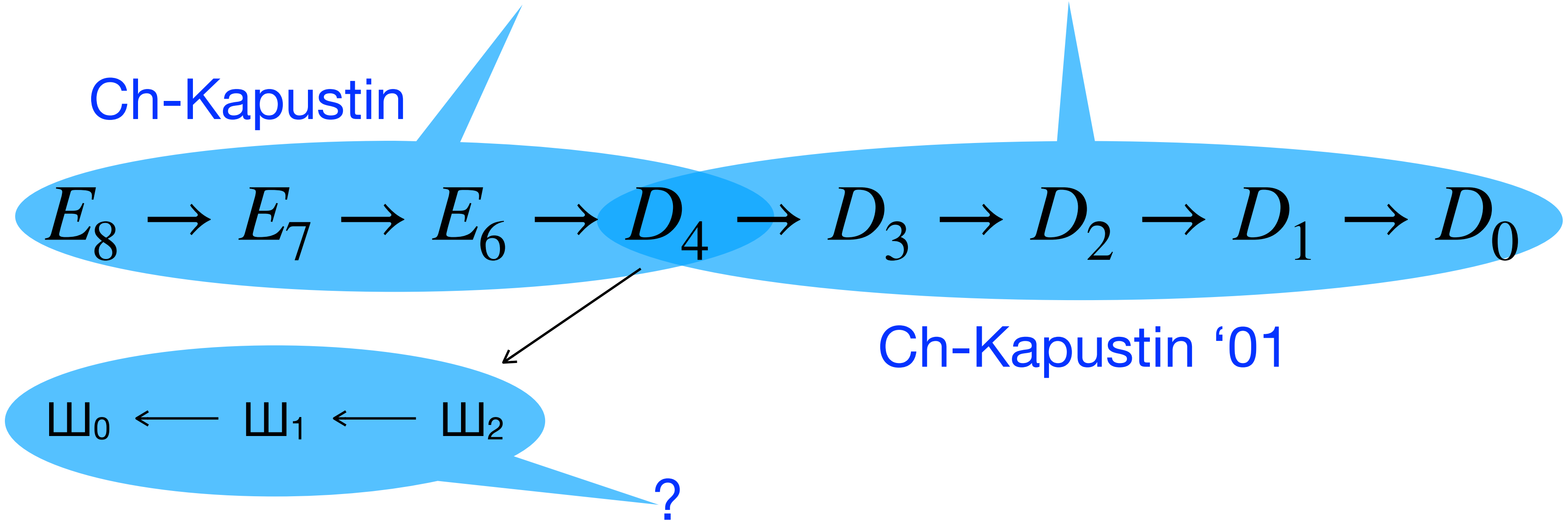
ALE A? D? E?

ALF One monopole Centered 2 monopoles Centered 2 monopoles with k Dirac sing. One monopole with k Dirac sing.

$\mathbb{R}^3 \times S^1,$ $D_0, D_1,$ $D_k,$ A_k

Atiyah-Hitchin ‘88 Ch-Kapustin ‘97

ALG & ALG* Equivariant instantons on $(\mathbb{R}^2 \times T^2)/\mathbb{Z}_n$ Periodic singular monopoles



Ch-Kapustin

$$E_8 \to E_7 \to E_6 \to D_4 \to D_3 \to D_2 \to D_1 \to D_0$$

Ch-Kapustin ‘01

$$\text{Ш}_0 \longleftarrow \text{Ш}_1 \longleftarrow \text{Ш}_2$$

?

ALH Doubly periodic singular monopoles

$$\begin{tikzcd}
 & & & & & & & \tilde{E}_1 & \\
E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & E_5 \arrow[r] & E_4 \arrow[r] & E_3 \arrow[r] & E_2 \arrow[r] \arrow[ur] & E_1 \arrow[r] & E_0
\end{tikzcd}$$

Thomas Harris `23 Rebekah Cross `15,`19; Ch-Ward `12; Ch `14

ALH* Two triply periodic monopoles with 2+2 Dirac singularities

$$\frac{1}{2}K3$$

Cherkis-Jardim Charbonneau - Hurtubise ‘10

Monopole

A monopole on a three-dimensional Riemannian manifold (X, g) is

1) a Hermitian vector bundle $E \to X$ and

2) a pair (A, Φ) of a connection A on E and

an skew-hermitian endomorphism Φ of E,

satisfying the Bogomolny Equation

$$F_A = * \, d_A \Phi,$$

and appropriate asymptotic conditions.

Monopole on		**Moduli Space type:**
$X = \mathbb{R}^3$	Monopole	ALF
$X = \mathbb{R}^2 \times S^1_R$	Periodic Monopole	ALG
$X = \mathbb{R}^1 \times T^2_{A,\tau}$	Doubly Periodic Monopole = Monopole Wall = Monowall	ALH

Singular Monopoles

Simple Dirac singularities at marked points $p_1^-, \ldots, p_{\nu_-}^-$ and $p_1^+, \ldots, p_{\nu_+}^+$: $\Phi = i\begin{pmatrix} \pm\frac{1}{2|\vec{x}-\vec{p}_\sigma^{\pm}|} & 0 \\ 0 & 0_{n-1,n-1} \end{pmatrix} + O(1)$

Note: More generally the charge is any cocharacter of the gauge group. (Talk by Thomas Harris on Friday.)

Boundary conditions:

$\mathbb{R}^3$

$$\Phi^g(\vec{x}) = \frac{i}{2\pi}\begin{pmatrix} \lambda - \frac{Q}{2|\vec{x}|} & 0 \\ 0 & -\lambda + \frac{Q}{2|\vec{x}|} \end{pmatrix} + O(r^{-2})$$

Finite energy =>
center of mass is a “modulus”

$\mathbb{R}^2 \times S^1$

Coordinates
$z = x + iy,\ \varphi$

$$\Phi^g = \frac{i}{2\pi}\mathrm{diag}(v_j + q_j \log|z| + \mathrm{Re}\frac{\mu_j}{z}) + O(1/|z|^2)$$

$$A^g = \frac{1}{2\pi}\mathrm{diag}((q_j \arg z + [b_j + \mathrm{Im}\frac{\mu_j}{z}])d\theta + \alpha_j d\arg z) + O(|z|^{-2})$$

Infinite energy =>
center of mass is fixed

$\mathbb{R} \times T^2$

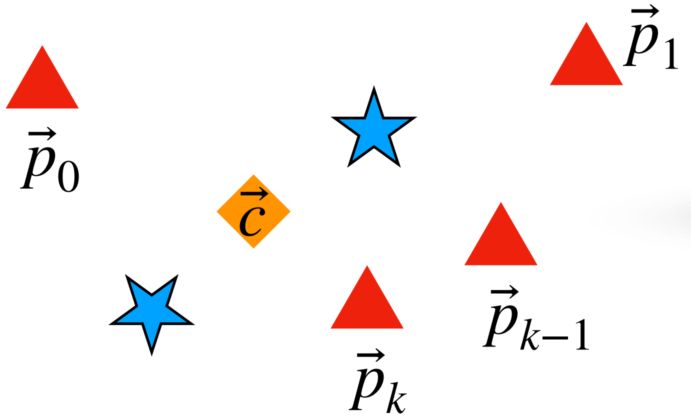
Coordinates
x, θ, φ

$$\Phi^g = \frac{i}{2\pi}\mathrm{diag}(Q_j^{\pm} x + M_j^{\pm}) + O(1/x)$$

$$A^g = -\frac{i}{2\pi}\mathrm{diag}(Q_j^{\pm}\theta d\varphi + \chi_{j,\theta}^{\pm} d\theta + \chi_{j,\varphi}^{\pm} d\varphi) + O(1/x)$$

Infinite energy =>
center of mass is fixed

Monopoles in $\mathbb{R}^3$



Moduli Space:

- A single monopole moduli: position in $\mathbb{R}^3$ and phase in S^1

$$\Phi^g(\vec{x}) = \frac{i}{2\pi}\begin{pmatrix} \lambda - \frac{1}{2|\vec{x}|} & 0 \\ 0 & -\lambda + \frac{1}{2|\vec{x}|} \end{pmatrix} + O(r^{-2})$$

$\mathbb{R}^3 \times S^1_{\frac{1}{\sqrt{\lambda}}}$

- A single monopole with $k+1$ simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_k \in \mathbb{R}^3$

A$_k$ ALF = multi-Taub-NUT with NUTs at $\vec{p}_0, \ldots, \vec{p}_k$

- Two monopoles in $\mathbb{R}^3$: two positions and two phases
  => 8 dim moduli space with triholomorphic isometry

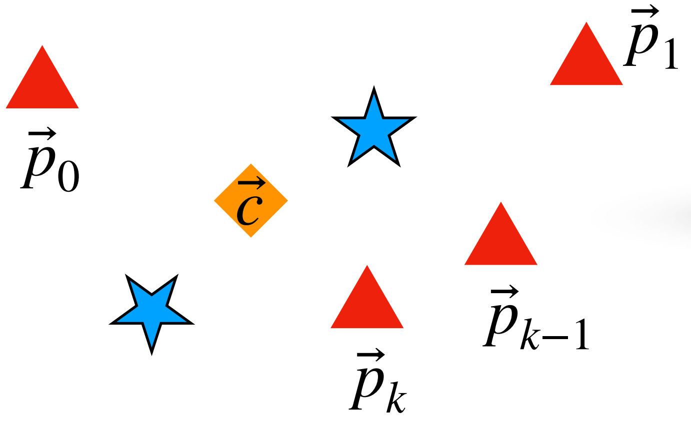
Two centered monopoles in $\mathbb{R}^3$ with center at $\vec{c} \in \mathbb{R}^3$

D$_0$ ALF = Atiyah-Hitchin

Two centered monopoles with k simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_{k-1} \in \mathbb{R}^3$

D$_k$ ALF

This picture leads to direct relations between these spaces!



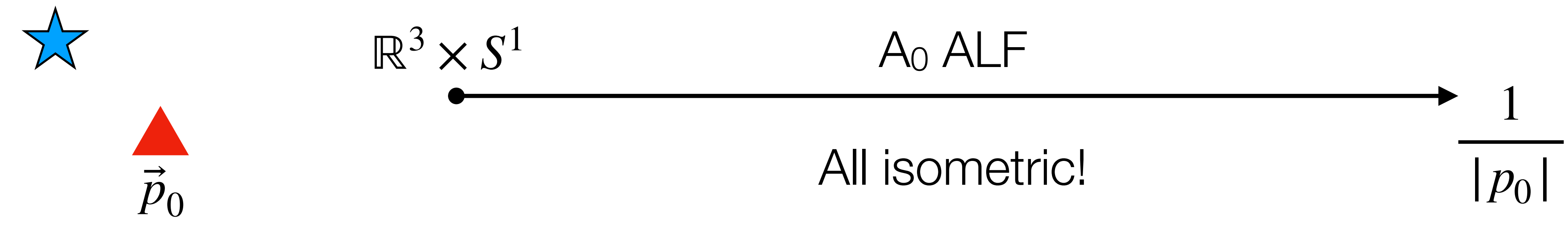
Relations between ALF Spaces

A single with $k+1$ simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_k \in \mathbb{R}^3$ — A_k ALF

Two centered monopoles with k simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_{k-1} \in \mathbb{R}^3$ — D_k ALF

$$\begin{tikzcd}
{} \arrow[r] & A_k \arrow[r, "p_k \to \infty"] & A_{k-1} \arrow[r, "p_{k-1} \to \infty"] & \ldots \arrow[r] & A_1 \arrow[r, "p_1 \to \infty"] & A_0 \arrow[r, "p_0 \to \infty"] & \mathbb{R}^3 \times S^1 \\
{} \arrow[r] & D_{k+1} \arrow[u, "c \to \infty"'] \arrow[r, "p_k \to \infty"] & D_k \arrow[u, "c \to \infty"'] \arrow[r, "p_{k-1} \to \infty"] & \ldots \arrow[r] & D_2 \arrow[u, "c \to \infty"'] \arrow[r, "p_1 \to \infty"] & D_1 \arrow[u, "c \to \infty"'] \arrow[r, "p_0 \to \infty"] & D_0 \arrow[u, "c \to \infty"']
\end{tikzcd}$$

Very schematically:



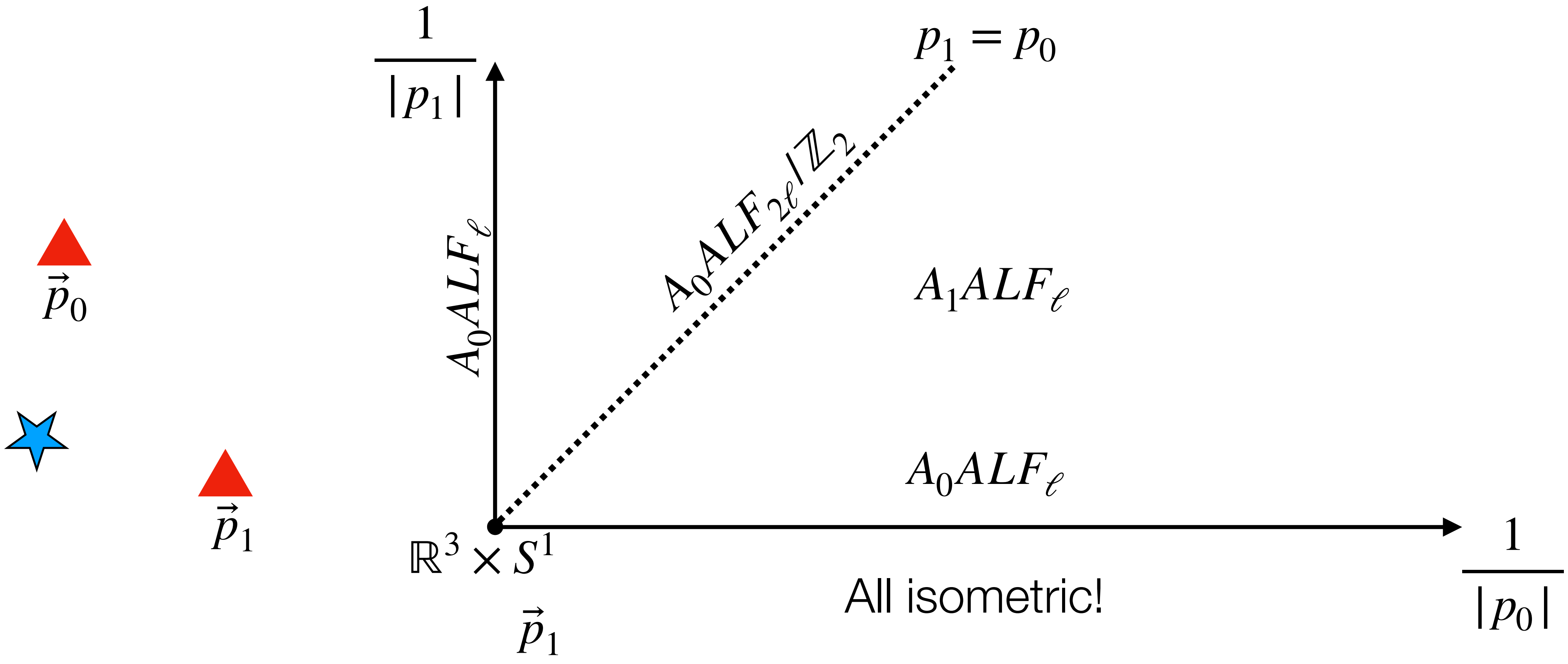
$\mathbb{R}^3 \times S^1$

A_0 ALF

$\frac{1}{|p_0|}$

All isometric!

$\vec{p}_0$



$\frac{1}{|p_1|}$

$p_1 = p_0$

$A_0 ALF_\ell$

$A_0 ALF_{2\ell}/\mathbb{Z}_2$

$A_1 ALF_\ell$

$A_0 ALF_\ell$

$\mathbb{R}^3 \times S^1$

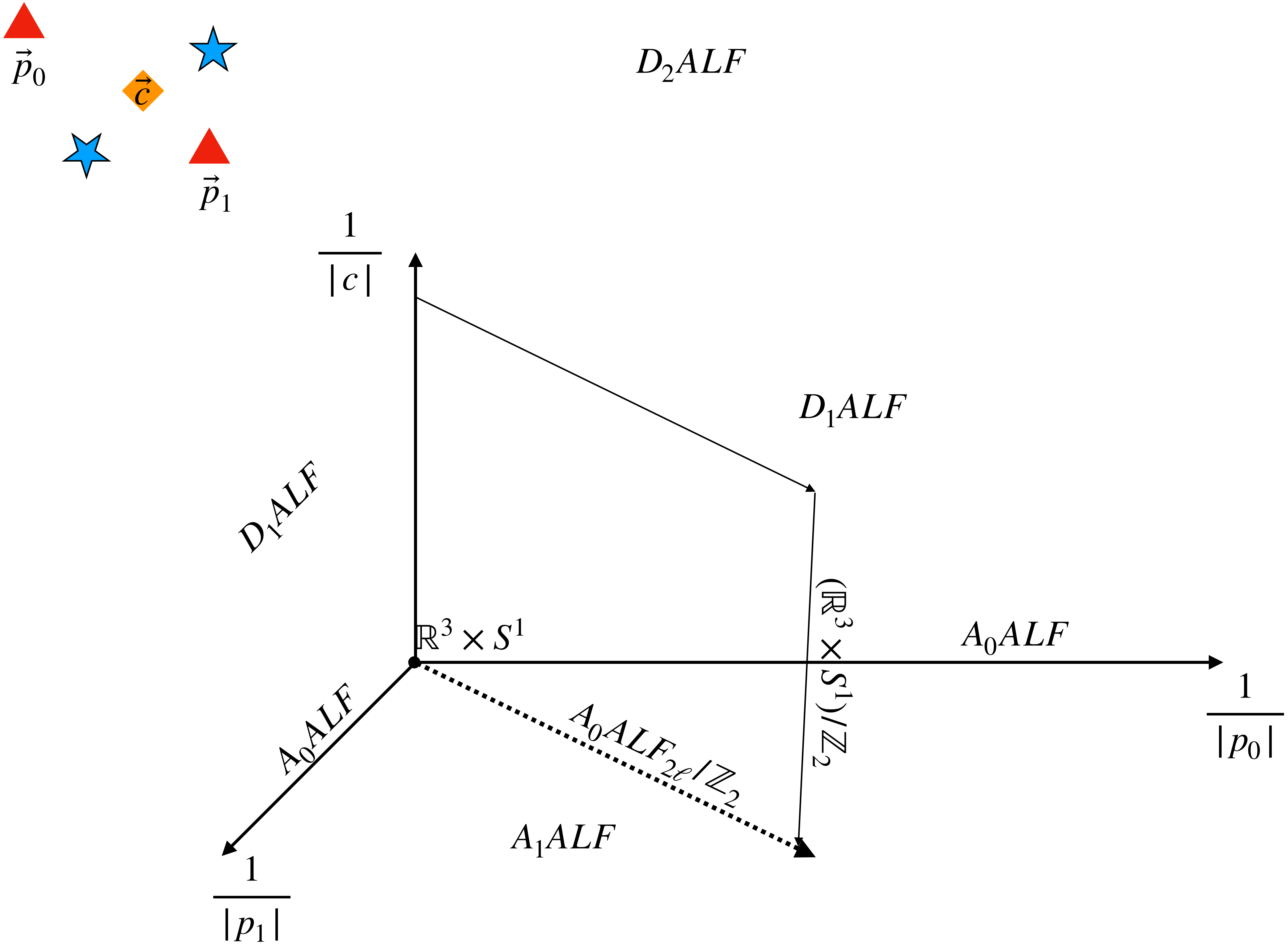
$\vec{p}_1$

$\frac{1}{|p_0|}$

All isometric!

$\vec{p}_0$

$\vec{p}_1$


$\vec{p}_0$
$\vec{c}$
$\vec{p}_1$
D_2ALF
$\frac{1}{|c|}$
D_1ALF
D_1ALF
$\mathbb{R}^3 \times S^1$
$(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$
A_0ALF
$\frac{1}{|p_0|}$
A_0ALF
$A_0ALF_{2\ell}/\mathbb{Z}_2$
A_1ALF
$\frac{1}{|p_1|}$

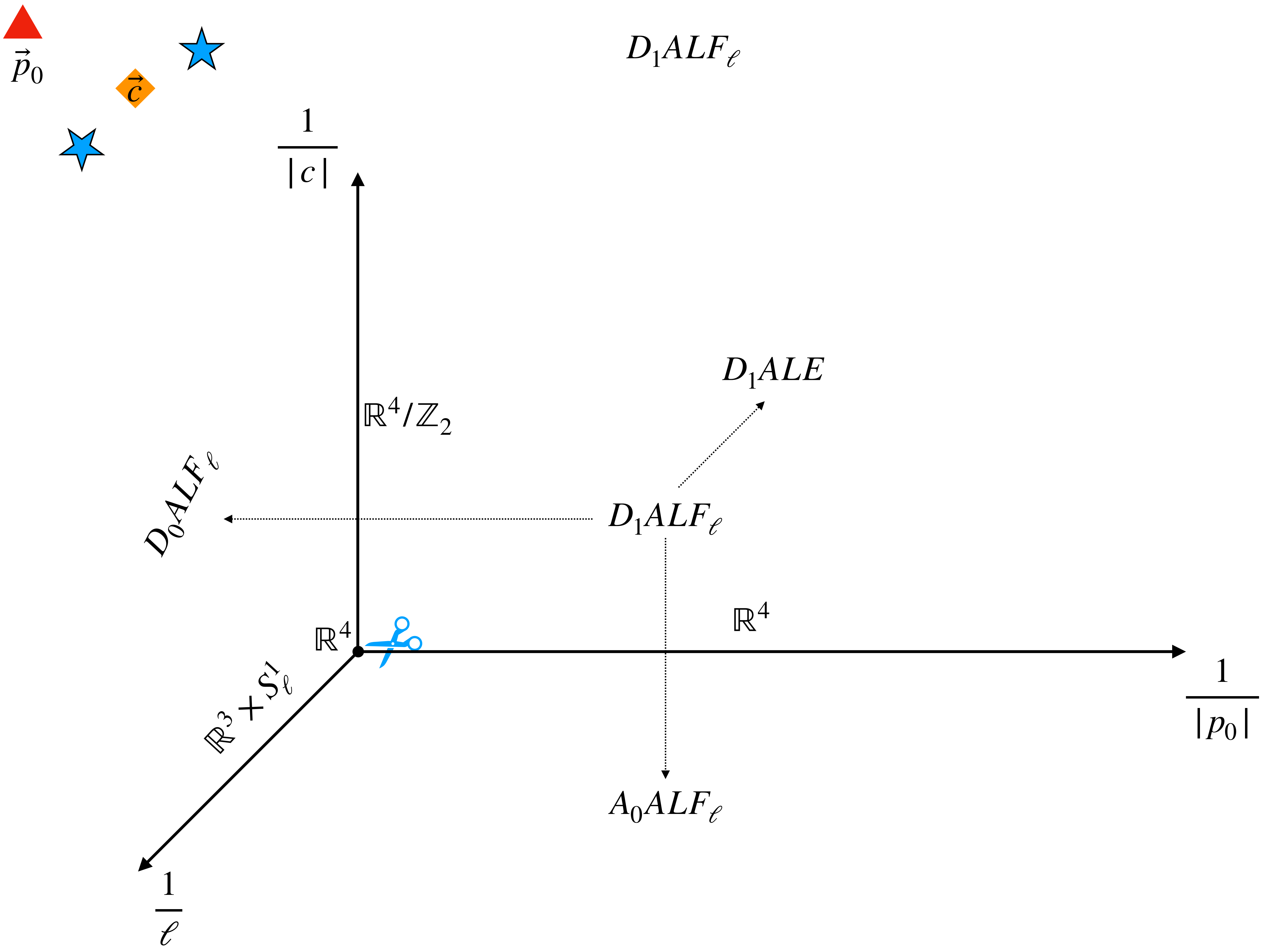
Relation to ALE

ALE

ALF

$$\begin{tikzcd}
 & & & {} \arrow[r] & A_3 \arrow[rr, "p_3\to\infty"] & & A_2 \arrow[rr, "p_2\to\infty"] & & A_1 \arrow[rr, "p_1\to\infty"] & & \mathbb{R}^4 & & \\
 & & {} \arrow[r] & D_4 \arrow[rr, "p_3\to\infty"] \arrow[ur] & & D_3 \arrow[rr, "p_2\to\infty"] \arrow[ur] & & D_2 \arrow[rr, "p_1\to\infty"] \arrow[ur] & & D_1 \arrow[rr, "p_0\to\infty"] \arrow[ur] & & \mathbb{R}^4/\mathbb{Z}_2 & \\
E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[ur] & & & & & & & & & & \\
 & & & {} \arrow[r] & A_3 \arrow[rr, "p_3\to\infty"] \arrow[uuu] & & A_2 \arrow[rr, "p_2\to\infty"] \arrow[uuu] & & A_1 \arrow[rr, "p_1\to\infty"] \arrow[uuu] & & A_0 \arrow[rr, "p_0\to\infty"] \arrow[uuu] & & \mathbb{R}^3\times S^1_{\frac{1}{\sqrt{\lambda}}} \arrow[uuull, dashed, "\lambda\to0"'] \\
 & & {} \arrow[r] & D_4 \arrow[rr, "p_3\to\infty"] \arrow[ur, "c\to\infty"] \arrow[uuu, "\lambda\to0"] & & D_3 \arrow[rr, "p_2\to\infty"] \arrow[ur] \arrow[uuu] & & D_2 \arrow[rr, "p_1\to\infty"] \arrow[ur] \arrow[uuu] & & D_1 \arrow[rr, "p_0\to\infty"] \arrow[ur] \arrow[uuu] & & D_0 \arrow[ur] \arrow[uuu] &
\end{tikzcd}$$

Moral: A- and D-ALE spaces are moduli spaces of monopoles of holomorphic charge 1 or centered monopoles of holomorphic charge 2.


$\vec{p}_0$
$\vec{c}$
D_1ALF_ℓ
$\frac{1}{|c|}$
D_1ALE
$\mathbb{R}^4/\mathbb{Z}_2$
D_0ALF_ℓ
D_1ALF_ℓ
$\mathbb{R}^4$
$\mathbb{R}^4$
$\mathbb{R}^3 \times S^1_\ell$
$\frac{1}{|p_0|}$
A_0ALF_ℓ
$\frac{1}{\ell}$

$\mathbb{R}^2 \times S^1_R$

Periodic Monopoles

$$\Phi^g = \frac{i}{2\pi}\mathrm{diag}(v_j + q_j \log|z| + \mathrm{Re}\frac{\mu_j}{z}) + O(1/|z|^2)$$
$$A^g = \frac{1}{2\pi}\mathrm{diag}((q_j \arg z + [b_j + \mathrm{Im}\frac{\mu_j}{z}])d\theta + \alpha_j d\arg z) + O(|z|^{-2})$$

Of the six parameters $\mu_j \in \mathbb{C}, v_j, b_j, q_j, \alpha_j \in \mathbb{R}$ three are true parameters of the moduli space and three determine the center of mass of the periodic monopole.

Two periodic monopoles can have up to 4 singularities.
Moduli space of two periodic monopoles with k singularities $= D_k$ ALG space.

This realization makes it clear that as $R \to \infty$, a periodic monopole tends to a monopole in $\mathbb{R}^3$ and D_k ALG space approaches D_k ALF space.

Question: What is the place of E- and Ш-ALG spaces?

I do not have a realization of these spaces as moduli spaces of monopoles,
only as limits of such spaces;
instead, let us realize them as moduli spaces of instantons on $\mathbb{R}^2 \times T^2$
equivariant under the action of the cyclic group $\mathbb{Z}_n, n = 3,4,6$.

Equivariant DP Instanton

Consider YM Instantons on $\mathbb{R}^2 \times T^2$, where $T^2 = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$

This admits cyclic action $(z, v) \mapsto (\omega z, \omega^{\pm 1} v)$, with $\omega = e^{\frac{2\pi}{n} i} \in \mathbb{Z}_n$

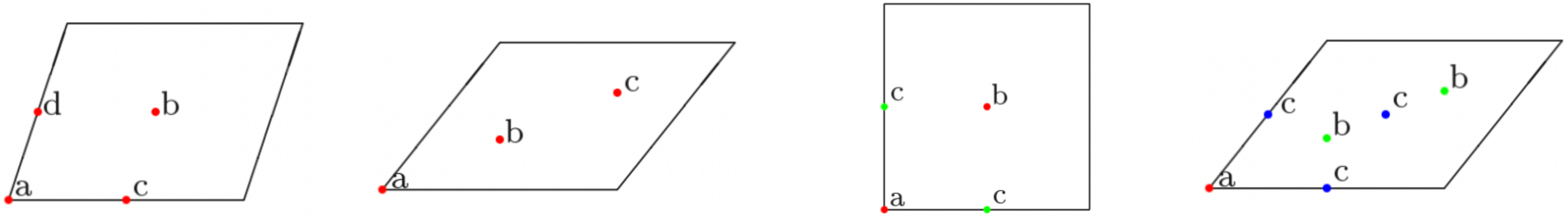
$$A(\omega z, \omega v) = V^{-1} A(z, v) V$$

3 real parameters: size and shape of T^2

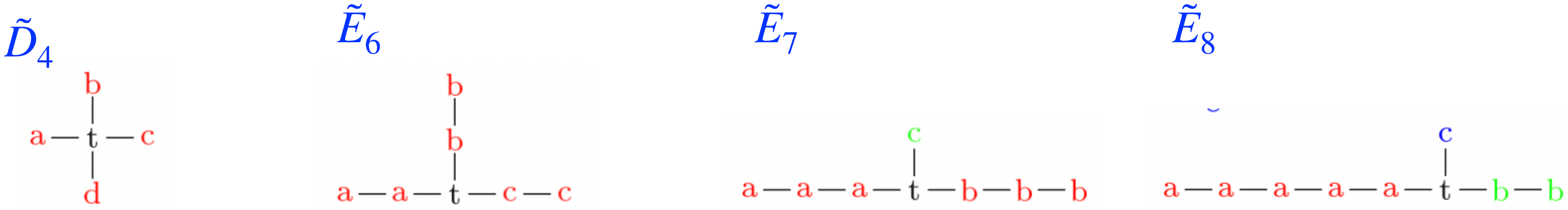
Only 1 real parameter: fixed shape of T^2

$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$
any τ	$\tau = \rho := e^{\frac{2\pi}{6} i}$	$\tau = i$	$\tau = \rho := e^{\frac{2\pi}{6} i}$

Fixed points —> Asymptotic of spectral curve:



Intersection diagram of the resolution:



Instanton gauge groups:

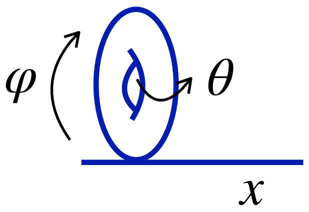
U(4) U(3) U(4) U(6)

Including ALG

$$\begin{tikzcd}
 & & & & & \cdots \arrow[r] & A_3 \arrow[r, "p_3\to\infty"] & A_2 \arrow[r, "p_2\to\infty"] & A_1 \arrow[r, "p_1\to\infty"] & \mathbb{R}^4 & \\
\text{ALE} & & & & \cdots \arrow[r] & D_4 \arrow[r, "p_3\to\infty"] \arrow[ur] & D_3 \arrow[r, "p_2\to\infty"] \arrow[ur] & D_2 \arrow[r, "p_1\to\infty"] \arrow[ur] & D_1 \arrow[r, "p_0\to\infty"] \arrow[ur] & \mathbb{R}^4/\mathbb{Z}_2 & \\
 & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[urr] & & & & & & & \\
 & & & & & \cdots \arrow[r] & A_3 \arrow[r, "p_3\to\infty"] \arrow[uuu] & A_2 \arrow[r, "p_2\to\infty"] \arrow[uuu] & A_1 \arrow[r, "p_1\to\infty"] \arrow[uuu] & A_0 \arrow[r, "p_0\to\infty"] \arrow[uuu] & \mathbb{R}^3\times S^1_{\frac{1}{\sqrt{\lambda}}} \arrow[uuul, dashed, "\lambda\to0"'] \\
\text{ALF} & & & & \cdots \arrow[r] & D_4 \arrow[r, "p_3\to\infty"] \arrow[ur] \arrow[uuu] & D_3 \arrow[r, "p_2\to\infty"] \arrow[ur] \arrow[uuu] & D_2 \arrow[r, "p_1\to\infty"] \arrow[ur] \arrow[uuu] & D_1 \arrow[r, "p_0\to\infty"] \arrow[ur] \arrow[uuu] & D_0 \arrow[ur] \arrow[uuu] & \\
 & & & & & & \text{Ш}_2 \arrow[r] & \text{Ш}_1 \arrow[r] & \text{Ш}_0 & & \\
\text{ALG} & & & & & D_4 \arrow[r, "p_3\to\infty"] \arrow[ur] \arrow[uu] & D_3 \arrow[r, "p_2\to\infty"] \arrow[uu] & D_2 \arrow[r, "p_1\to\infty"] \arrow[uu] & D_1 \arrow[r, "p_0\to\infty"] \arrow[uu] & D_0 \arrow[uu, "R_2\to\infty"'] & \\
 & E_8 \arrow[r] \arrow[uuuuu] & E_7 \arrow[r] \arrow[uuuuu] & E_6 \arrow[urr] \arrow[uuuuu] & & & & & & &
\end{tikzcd}$$

Doubly Periodic Monopoles

Ki-Myeong Lee ’98
Fujimori, Nitta, Ohta, Sakai
and Yamazaki ’08



Monowall = Monopole Wall = Monopole on $\mathbb{R}_x \mathrm{x} S^1_\theta \mathrm{x} S^1_\varphi$

- Abelian monowall - da=*dϕ => d*dϕ=0

thus ϕ is harmonic on $\mathbb{R}$xS^1xS^1, (excluding exponential growth) ϕ=Qx+M with $Q \in \mathbb{Z}$, M∊$\mathbb{R}$.

- In general - $F_A = {}^*d_A \Phi$

a) with boundary conditions $\Phi = \frac{i}{2\pi}\mathrm{diag}(Q_j^\pm x + M_j^\pm) + O(1/x)$

$$A = -\frac{i}{2\pi}\mathrm{diag}(Q_j^\pm \theta d\varphi + \chi^\pm_{j,\theta} d\theta + \chi^\pm_{j,\varphi} d\varphi) + O(1/x)$$

and

b) simple Dirac singularities at marked points $p_1^-, \ldots, p_{\nu_-}^-$ and $p_1^+, \ldots, p_{\nu_+}^+$.

$$\Phi = i\begin{pmatrix} \pm\frac{1}{2|\vec{x} - \vec{p}^\pm_\sigma|} & 0 \\ 0 & 0_{n-1,n-1} \end{pmatrix} + O(1)$$

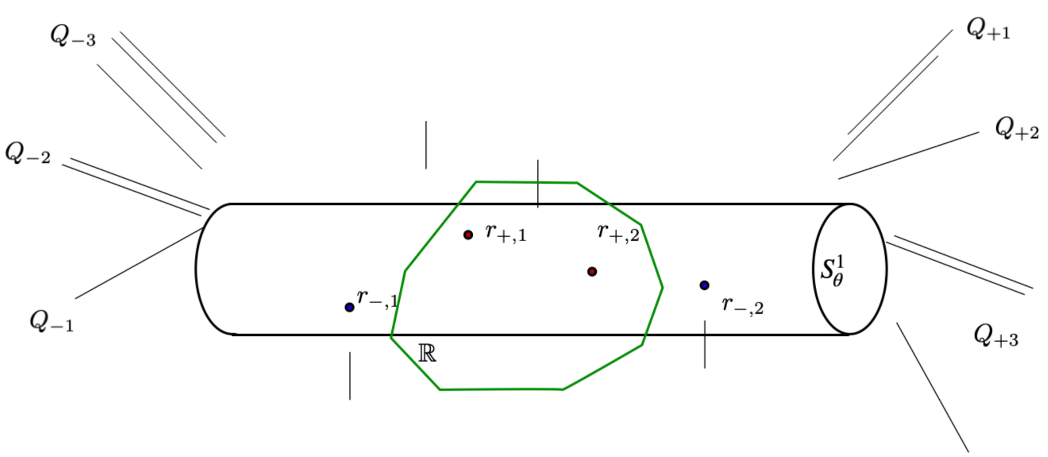
The charges $Q_j^\pm$ are rational, with the denominator equal to the multiplicity of $(Q_j^\pm, M_j^\pm)$.

Ch-Ward ’12
Ch ’14
Ch-Cross ‘19

Monopole charges + singularities —> Newton polygon N

Number of moduli = 4 x Internal integer points of N

Number of parameters = 3 x [(Perimeter integer points of N) -3]

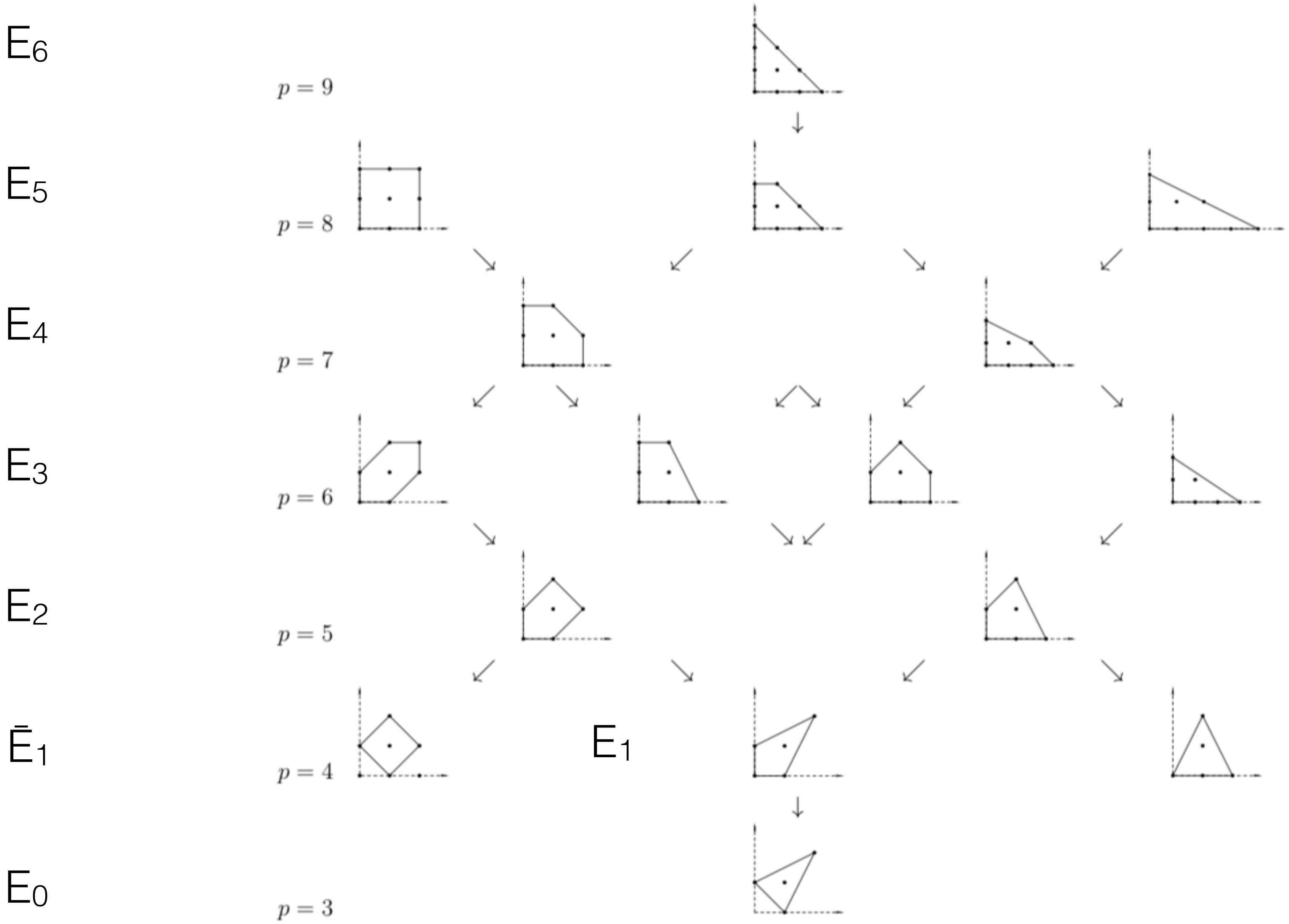


SL(2,ℤ) moduli space isometry generated by

S = Nahm transform and

T= Adding constant magnetic field $(A,\Phi)\mapsto(A-\theta d\varphi,\Phi+x1)$.

All integer Newton polygons with a single internal point up to SL(2,ℤ):

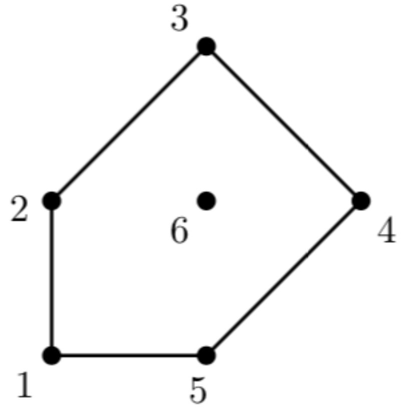


Secondary Polyhedron

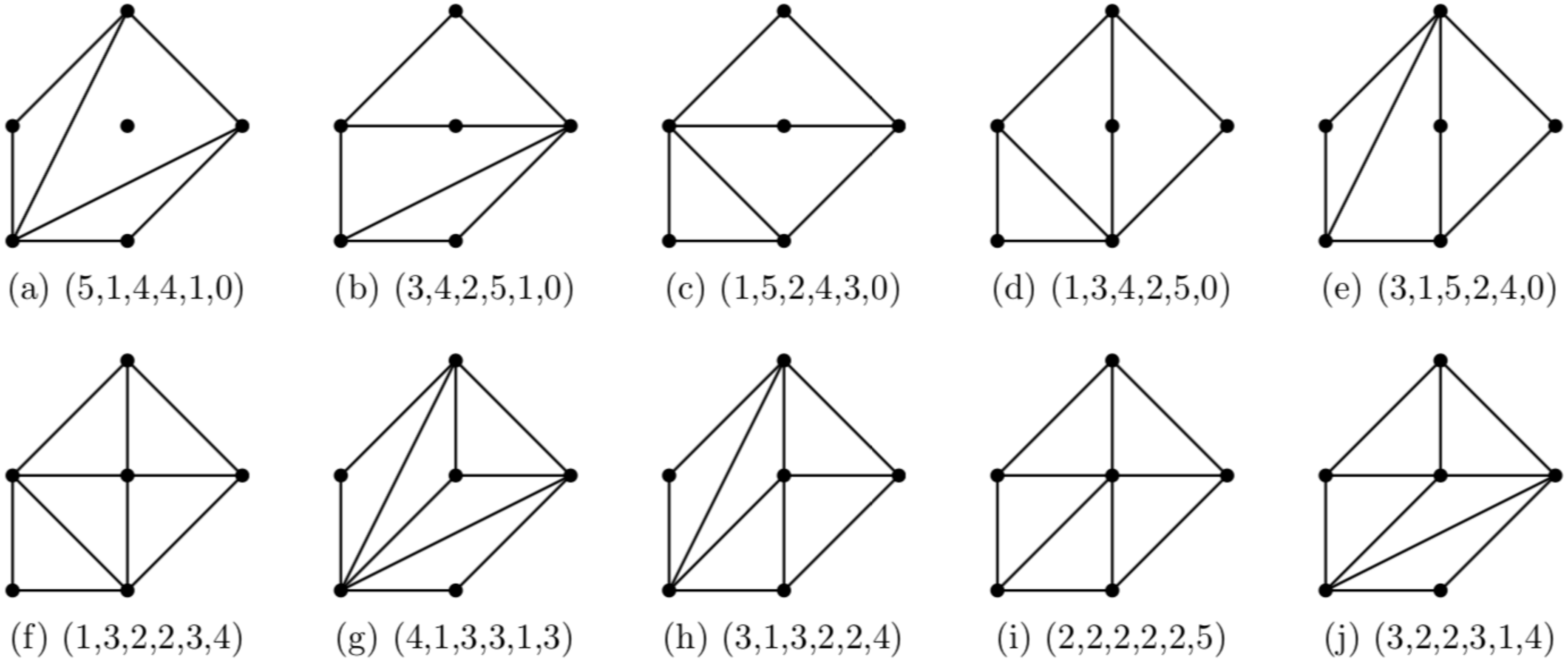
Gelfand, Kapranov,
Zelevinsky circa ’90

Organizes the phase space into sectors.

Given a Newton polygon, label its vertices:



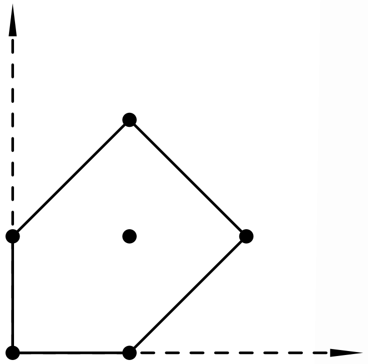
Each regular subdivision gives a |N|-dimensional vector:

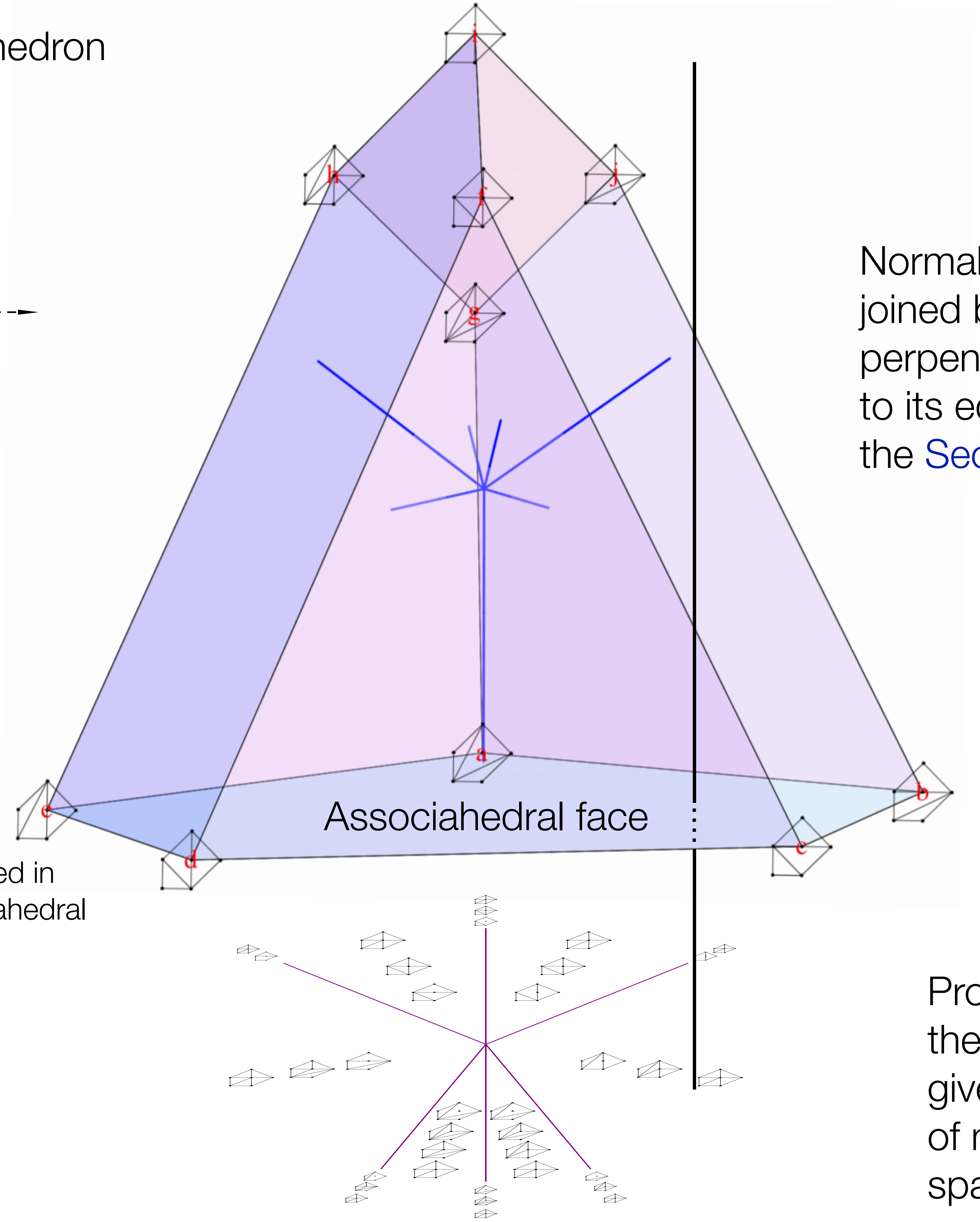


(a) (5,1,4,4,1,0)
(b) (3,4,2,5,1,0)
(c) (1,5,2,4,3,0)
(d) (1,3,4,2,5,0)
(e) (3,1,5,2,4,0)
(f) (1,3,2,2,3,4)
(g) (4,1,3,3,1,3)
(h) (3,1,3,2,2,4)
(i) (2,2,2,2,2,5)
(j) (3,2,2,3,1,4)

Vector from triangulations are the vertices of the minimal convex hull of all these vectors.

Secondary polyhedron of





Normals to its faces,
joined by wedges
perpendicular
to its edges form
the Secondary Fan.

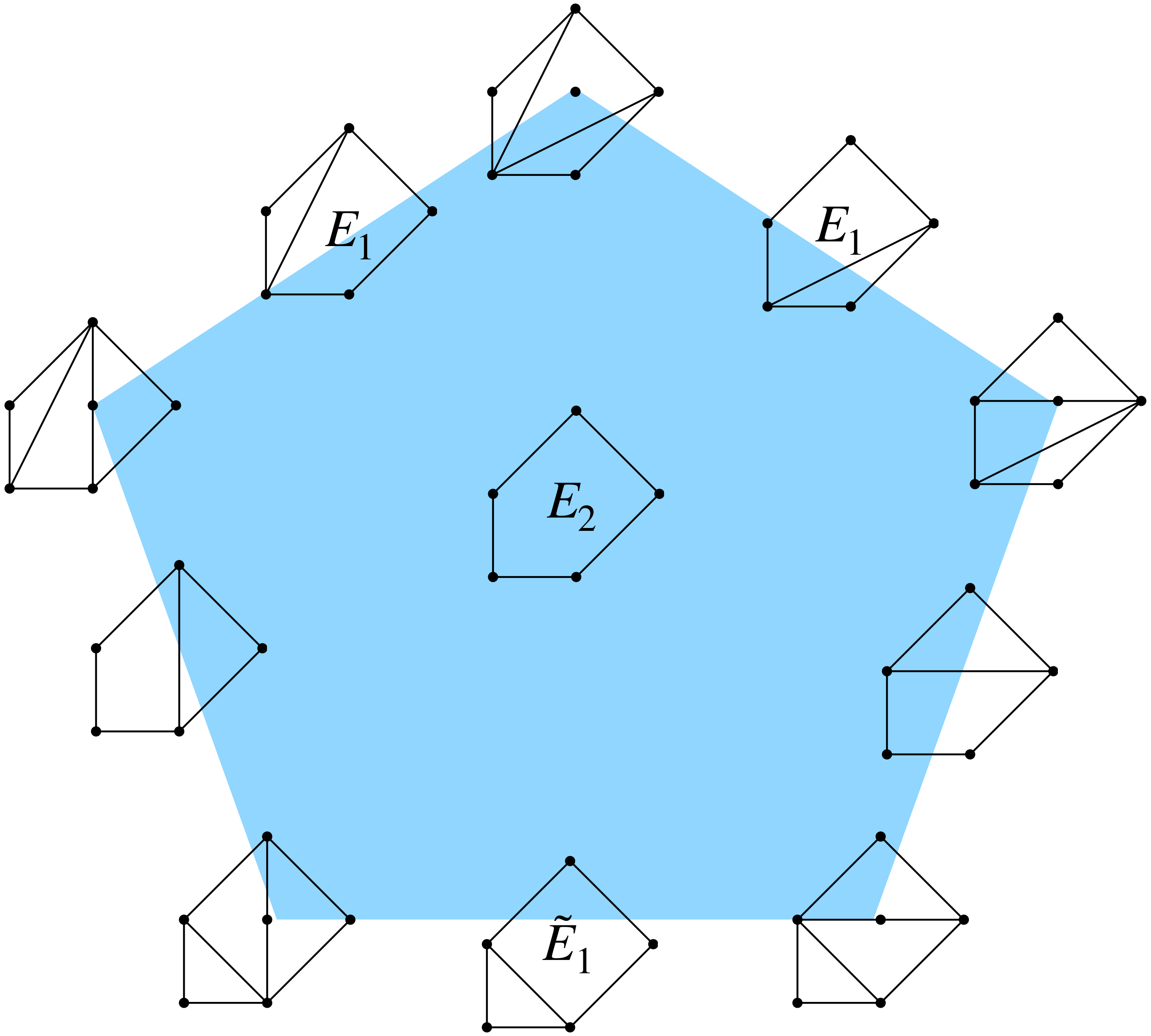
No internal points involved in subdivisions on the associahedral face

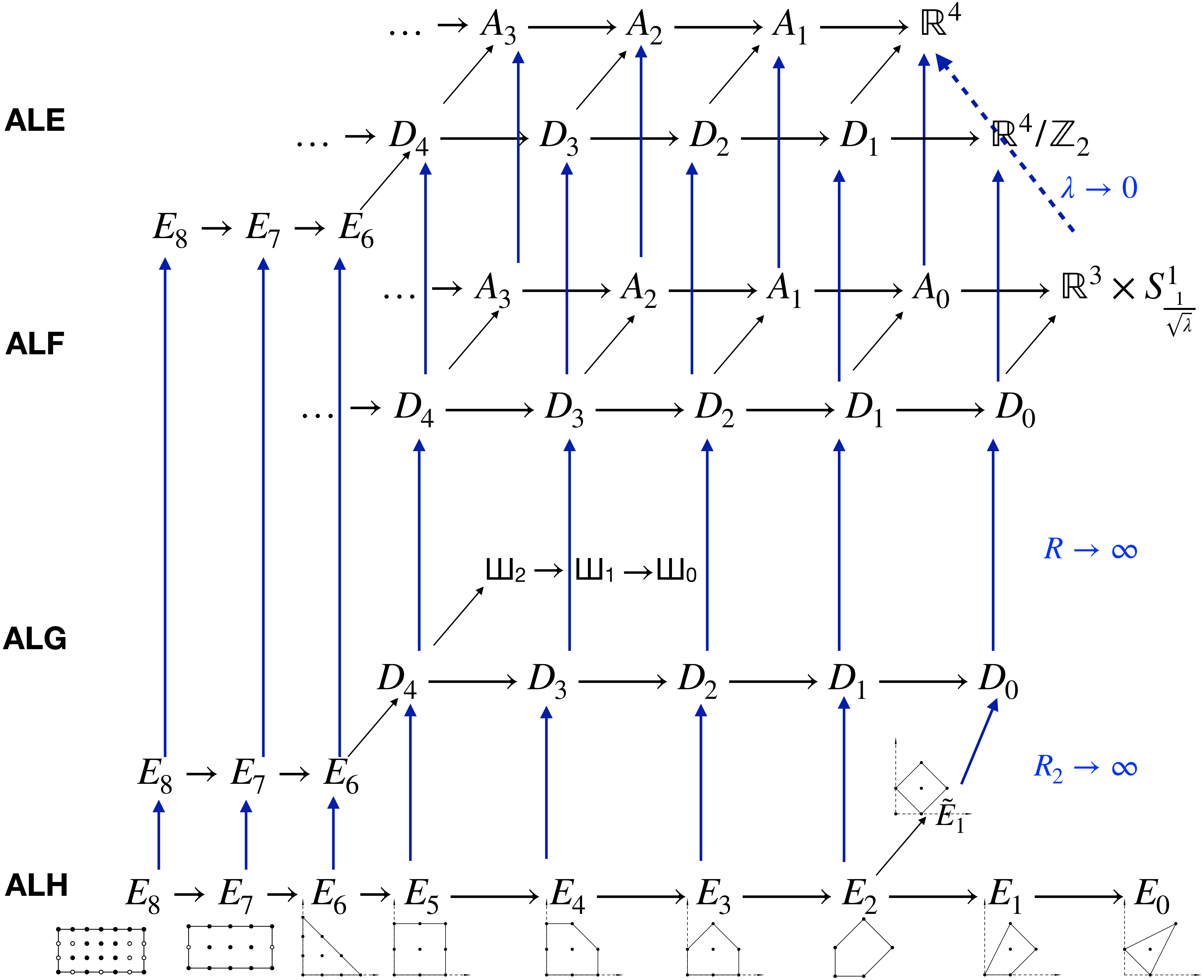
Projection of
the Secondary Fan
gives the phase space
of monowall moduli
spaces.

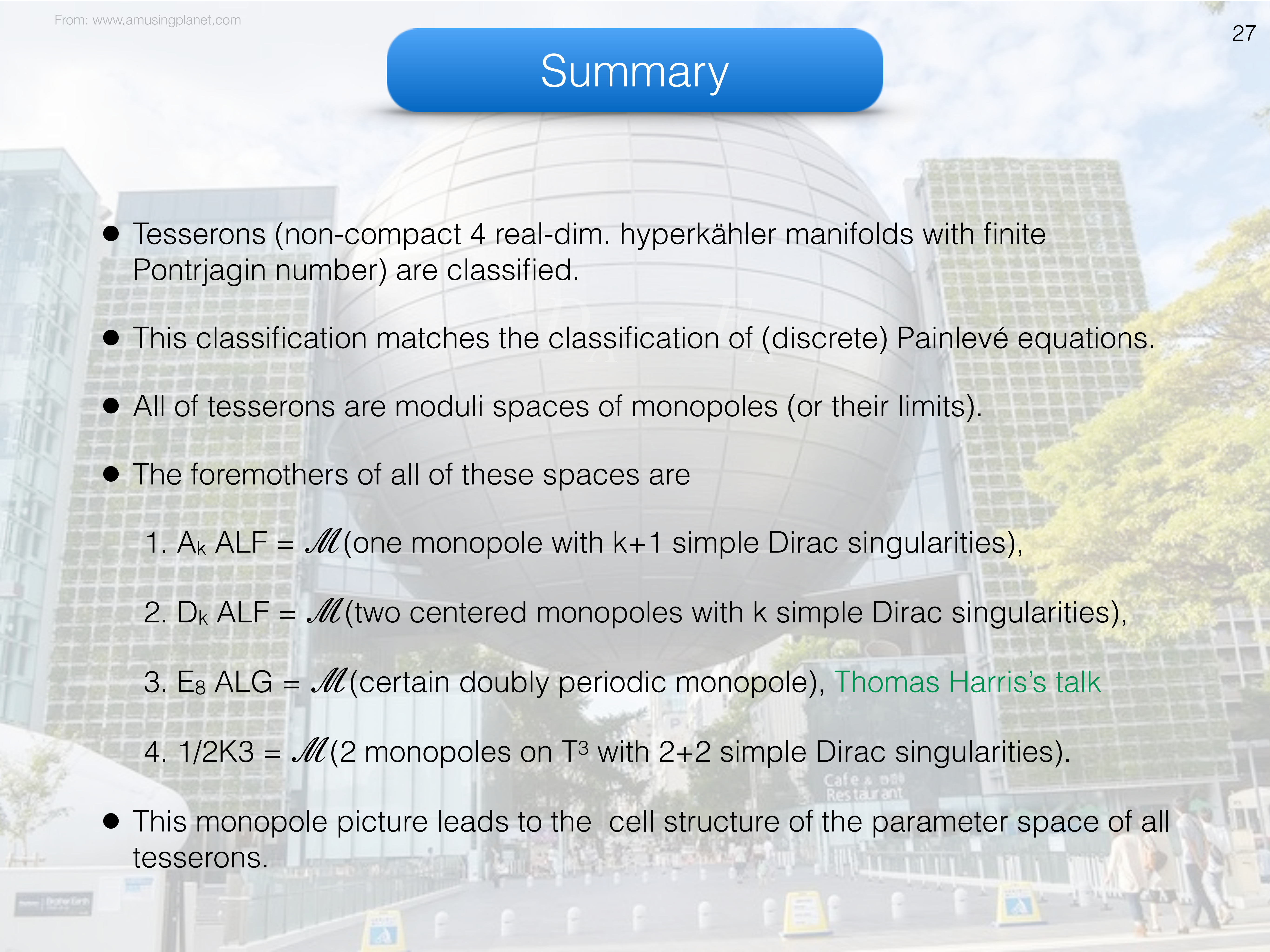
Space of all ALH metrics

The parameter space of ALH metrics is fibered over the "universal ALH associahedron".

For example:




ALE
ALF
ALG
ALH
$\dots \to A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow \mathbb{R}^4$
$\dots \to D_4 \longrightarrow D_3 \longrightarrow D_2 \longrightarrow D_1 \longrightarrow \mathbb{R}^4/\mathbb{Z}_2$
$\lambda \to 0$
$E_8 \to E_7 \to E_6$
$\dots \to A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow \mathbb{R}^3 \times S^1_{\frac{1}{\sqrt{\lambda}}}$
$\dots \to D_4 \longrightarrow D_3 \longrightarrow D_2 \longrightarrow D_1 \longrightarrow D_0$
$R \to \infty$
Ш$_2$ → Ш$_1$ →Ш$_0$
$D_4 \longrightarrow D_3 \longrightarrow D_2 \longrightarrow D_1 \longrightarrow D_0$
$E_8 \to E_7 \to E_6$
$R_2 \to \infty$
$\tilde{E}_1$
$E_8 \to E_7 \to E_6 \to E_5 \longrightarrow E_4 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$



From: www.amusingplanet.com

Summary

- Tesserons (non-compact 4 real-dim. hyperkähler manifolds with finite Pontrjagin number) are classified.
- This classification matches the classification of (discrete) Painlevé equations.
- All of tesserons are moduli spaces of monopoles (or their limits).
- The foremothers of all of these spaces are
  1. A_k ALF = $\mathscr{M}$(one monopole with k+1 simple Dirac singularities),
  2. D_k ALF = $\mathscr{M}$(two centered monopoles with k simple Dirac singularities),
  3. E_8 ALG = $\mathscr{M}$(certain doubly periodic monopole), Thomas Harris's talk
  4. 1/2K3 = $\mathscr{M}$(2 monopoles on T^3 with 2+2 simple Dirac singularities).
- This monopole picture leads to the cell structure of the parameter space of all tesserons.