On construction and integrability of the noncommutative extended KP equation and the noncommutative extended mKP equation

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- Generalization of soliton theory to noncommutative(nc) counterparts is an interesting topic. Some nc versions of integrable systems and their integrability have been invstigated (Paniak, Hamanaka & Toda, Wang & Wadati, Nimmo & Gilson, Li etc.). Besides, connections of nc OPs, nc Painléve equations, quasi-Schur functions and so on to nc integrable systems have been attracting much attention.
- Nc integrable systems: super, matrix, quaternion integrable systems and nc integrable systems caused by replacing the normal product by Moyal star product. They have dynamical applications in string theory, quantum Hall effect, quantum information and computing. Since we only assume that functions are noncommutative in general, corresponding results are applicable to the above-mentioned cases. Nc KdV equation: $U_t - 3UU_x - 3U_xU + U_{xxx} = 0.$
- In commutative setting, some results of nc integrable systems can be transformed to the ones of the corresponding commutative integrable systems. Super integrable systems can be treated as nc integrable systems too. This reveals the advantage of investigating nc integrable systems instead of commutative ones.

- Part I: The twisted derivation and its Gauge transformation proposed by us, from which Darboux transformations can be constructed for some well-known (nc) integrable systems which lead to determinant/quasideterminant solutions to (nc) integrable systems.

- Part II: KP equation is the most fundamental among many soliton equations. Its extensions are extensively studied. There are different methods to construct (squared eigenfunction symmetry constraint, variation of constants) and solve soliton equations with self-consistent sources (IST, D-bar, DT, Bäcklund transformation, Hirota's method).

Outline

- A brief review of quasideterminants
- The twisted derivation and Gauge transformation
  - Darboux transformations for several types of nc integrable systems and quasideterminant solutions
- The nc extended KP equation
  - Lax pair
  - Quasi-Wronskian solutions by variation of constants
  - Quasi-Grammian solutions by variation of constants
- The nc extended modified KP equation
  - Lax pair
  - Quasi-Wronskian solutions by variation of constants
  - Quasi-Grammian solutions by variation of constants
- Miura transformation
- Future work

Quasideterminants - Definition

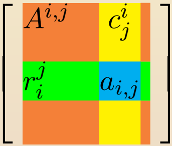
Developed since early 1990s by Gelfand and Retakh; recent review article Gelfand et al (2005) *Advances in Mathematics*, **193**, 56-141.

Definition

An $n \times n$ matrix $A = (a_{i,j})$ over a ring (non-commutative, in general) has n^2 *quasideterminants* written as $|A|_{i,j}$. Defined recursively by

$$|A|_{i,j} = a_{i,j} - r_i^j (A^{i,j})^{-1} c_j^i, \quad A^{-1} = (|A|_{j,i}^{-1})_{i,j=1,\dots,n}.$$

Notation: $A =$



It is obvious that $|A|_{i,j} = (-1)^{i+j} \dfrac{\det(A)}{\det(A^{i,j})}$ in commutative case.

Quasideterminants - Simplest cases

Case $n = 1$. Let $A = (a)$, then $|A|_{1,1} = a$.

Case $n = 2$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$|A|_{1,1} = \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c, \quad |A|_{1,2} = \begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d,$$

$$|A|_{2,1} = \begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a, \quad |A|_{2,2} = \begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b.$$

From this we can obtain the matrix inverse

$$A^{-1} = \begin{pmatrix} |A|_{1,1}^{-1} & |A|_{2,1}^{-1} \\ |A|_{1,2}^{-1} & |A|_{2,2}^{-1} \end{pmatrix} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix}.$$

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Quasideterminants - Continued

For a simpler notation we often use a prototype square partitioned matrix $\begin{pmatrix} A & B \\ C & d \end{pmatrix}$, where d is a single entry.

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B.$$

d also could be a matrix:

$$\begin{vmatrix} A_{n\times n} & B_{n\times l} \\ C_{m\times n} & \boxed{D_{m\times l}} \end{vmatrix} = D - CA^{-1}B,$$

which leads to the essential difference of quasideterminants from determinants!

Quasideterminants - Invariance

The following formula can be used to understand the effect on a quasideterminant of certain elementary row operations involving addition and multiplication on the left

$$\left| \begin{pmatrix} E & 0 \\ F & g \end{pmatrix} \begin{pmatrix} A & B \\ C & d \end{pmatrix} \right|_{n,n} = \begin{vmatrix} EA & EB \\ FA + gC & \boxed{FB + gd} \end{vmatrix} = g \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}$$

There is an analogous invariance under column operations involving addition and multiplication on the right.

Quasideterminants - Noncommutative Jacobi Identity

Noncommutative Jacobi identity

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}$$

Jacobi identity (compared to nc Jacobi identity)

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & C \\ E & i \end{vmatrix} \begin{vmatrix} A & B \\ D & f \end{vmatrix} - \begin{vmatrix} A & B \\ E & h \end{vmatrix} \begin{vmatrix} A & C \\ D & g \end{vmatrix}$$

Quasideterminants - Quasi-Plücker coordinates

Given an $(n+k)\times n$ matrix A, we denote A^i the ith row of A and A^I the submatrix with rows in a subset I of $\{1,2,\cdots,n+k\}$.

Given $i,j\in\{1,2,\ldots,n+k\}$ and I such that $\#I=n-1$ and $j\notin I$, the *(right) quasi-Plücker coordinates* are

$$r_{ij}^I = r_{ij}^I(A) := \begin{vmatrix} A^I \\ A^i \end{vmatrix}_{ns} \begin{vmatrix} A^I \\ A^j \end{vmatrix}_{ns}^{-1} = -\begin{vmatrix} A^I & 0 \\ A^i & \boxed{0} \\ A^j & 1 \end{vmatrix}$$

for any column index $s\in\{1,\ldots,n\}$.

A useful consequence is the identity

$$\begin{vmatrix} A^I & 0 \\ A^i & \boxed{0} \\ A^j & 1 \end{vmatrix}^{-1} = \begin{vmatrix} A^I & 0 \\ A^i & 1 \\ A^j & \boxed{0} \end{vmatrix}.$$

Plücker Relations

A simple case:

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$$

Quasideterminants - Homological relations

Quasideterminants are not independent. They are related by quasi-Plücker coordinates, named homological relations

$$\begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & B & 0 \\ D & f & \boxed{0} \\ E & h & 1 \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix}$$

Derivatives of general quasideterminants

A_k: k-th column of A. A^k: k-th row of A. $I = \sum_{k=1}^{n} e_k e_k^t$.
e_k: column n-vector with 1 in the i-th row and 0 elsewhere.

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B - CA^{-1}B' + CA^{-1}A'A^{-1}B$$

Derivatives of Quasi-Wronskian:

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & e_k \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ (A^k)' & \boxed{(B^k)'} \end{vmatrix}.$$

Derivatives of Quasi-Grammian with $A' = \sum_{i=1}^{k} E_i F_i$:

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' + \begin{vmatrix} A & B \\ C' & \boxed{0} \end{vmatrix} + \begin{vmatrix} A & B' \\ C & \boxed{0} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & E_i \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ F_i & \boxed{0} \end{vmatrix}.$$

The twisted derivation and Gauge transformation

Consider an associative, unital algebra $\mathcal{A}$ (not necessarily graded) over ring K. Suppose that there is a homomorphism $\sigma\colon \mathcal{A} \to \mathcal{A}$ (i.e. for all $\alpha \in K$, $a, b \in \mathcal{A}$, $\sigma(\alpha a) = \alpha\sigma(a)$, $\sigma(a+b) = \sigma(a) + \sigma(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$) and a *twisted derivation* $D\colon \mathcal{A} \to \mathcal{A}$ satisfying $D(K) = 0$ and $D(ab) = D(a)b + \sigma(a)D(b)$.

Gauge transformation: $G_\theta = \sigma(\theta)D\theta^{-1} = D - D(\theta)\theta^{-1}$, $\theta \in \mathcal{A}$.

Theorem

Assume that $\theta_0, \theta_1, \theta_2, \ldots$ *be a sequence in* $\mathcal{A}$. *Let* $\phi[0] = \phi$ *and for* $n \in \mathbb{N}$

$$\phi[n] = D(\phi[n-1]) - D(\theta[n-1])\theta[n-1]^{-1}\phi[n-1],$$

where $\theta[n] = \phi[n]|_{\phi \to \theta_n}$. *Then, for* $n \in \mathbb{N}$,

$$\phi[n] = \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & \phi \\ D(\theta_0) & \cdots & D(\theta_{n-1}) & D(\phi) \\ \vdots & & \vdots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) & D^{n-1}(\phi) \\ D^n(\theta_0) & \cdots & D^n(\theta_{n-1}) & \boxed{D^n(\phi)} \end{vmatrix}. \qquad (1)$$

- C.X. Li & J.J.C. Nimmo, *Proc. R. Soc. A*, **466**(2010): 2471-2493

Derivative Here $D = \partial_x$ satisfies $D(ab) = D(a)b + aD(b)$ and σ is the identity mapping.
(*eg. (nc) KP equation, (nc) Toda lattice equation!*)

$$\widetilde{\phi} = (\partial_x - \partial_x(\theta)\theta^{-1})\phi = \phi_x - \theta_x\theta^{-1}\phi = \begin{vmatrix} \theta & \phi \\ \theta_x & \boxed{\phi_x} \end{vmatrix},$$

$$\widetilde{\phi}_n = (\partial_x - \partial_x(\theta_n)\theta_n^{-1})\phi_n = \phi_n - \theta_n\theta_{n+1}^{-1}\phi_{n+1}.$$

Forward difference The homomorphism is the shift operator $T(a(x)) = a(x+h)$ and the twisted derivation is

$$\Delta(a(x)) = \frac{a(x+h) - a(x)}{h},$$

satisfying $\Delta(ab) = \Delta(a)b + T(a)\Delta(b)$.
(*eg. (nc) Hirota-Miwa equation!*)

$$\widetilde{\phi}_n = (\Delta_i - \Delta_i(\theta_n)\theta_n^{-1})\phi_n = a_i^{-1}(\phi_{n+1} - \theta_{n+1}\theta_n^{-1}\phi_n),$$

where $\Delta_i = a_i^{-1}(T_i - 1)$ and T_i is the shift operator.

Superderivative For $a, b \in \mathcal{A}$, a superalgebra, the superderivative $D = \partial_\theta + \theta\partial_x$ satisfies $D(ab) = D(a)b + \hat{a}D(b)$, where $\hat{}$ is the grade involution, x is an even variable, θ is an odd Grassmann variable.
(*eg. Super KdV equation!*)

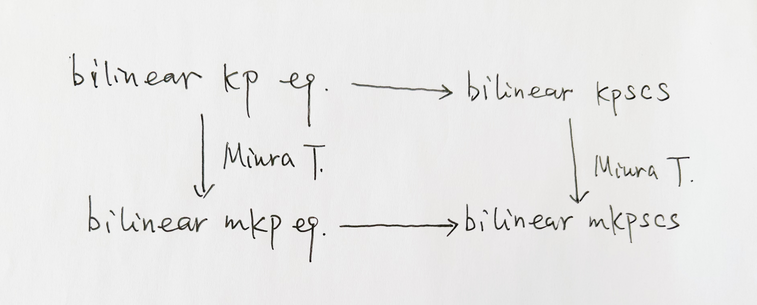
$$\widetilde{\phi} = D(\phi) - D(\theta)\theta^{-1}\phi.$$

Jackson derivative The homomorphism is a q-shift operator defined by $S_q(a(x)) = a(qx)$ and the twisted derivation is

$$D_q(a(x)) = \frac{a(qx) - a(x)}{(q-1)x}.$$

satisfying $D_q(ab) = D_q(a)b + S_q(a)D_q(b)$.
(*works for (nc) q-discrete $2D$ Toda lattice equation too!*)

Nc extended integrable systems



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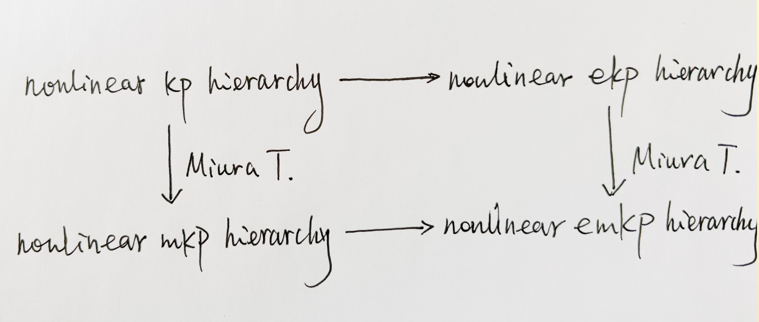

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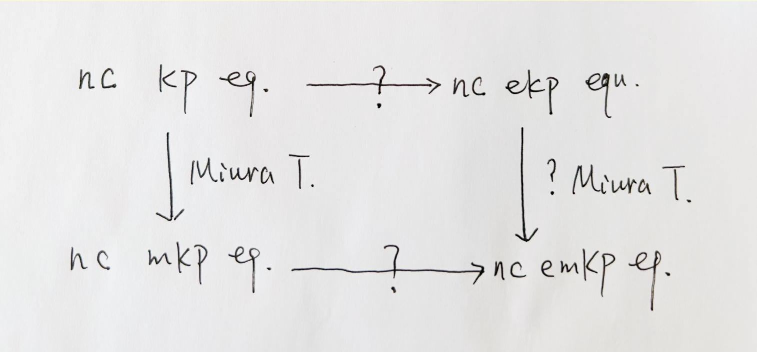

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The nc extended KP equation

The nc KP reads as

$$(v_t + v_{xxx} + v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = 0. \qquad (2)$$

It has the quasi-Wronskian solution

$$v = -2Q(0,0) = -2 \begin{vmatrix} \hat{\Theta} & e_N \\ \Theta^{(N)} & \boxed{0} \end{vmatrix} \qquad (3)$$

where $\hat{\Theta}$ is the $N \times N$ Wronskian matrix of $\theta_1, \cdots, \theta_N$ w.r.t. x, e_N is the column vector with 1 in the N-th row and 0 elsewhere, $\Theta = (\theta_1, \cdots, \theta_N)$, $\Theta^{(N)}$ is the N-th derivative of Θ w.r.t. x and $\theta_i,\ i = 1, \cdots, N$ satisfy the dispersion relations

$$\partial_y \theta_i = \partial_x^2 \theta_i, \quad \partial_t \theta_i = -4\partial_x^3 \theta_i. \qquad (4)$$

C. Gilson & J.J.C. Nimmo. On a direct approach to quasideterminant solutions of a noncommutative KP equation, *J. Phys. A: Math. Theor.* **40** (2007) 3839–3850.

Lax pair

The nc extended KP equation:

$$(v_t + v_{xxx} + v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = -2\left(\sum_{i=1}^{M} q_i r_i\right)_x,$$

$$q_{i,y} = q_{i,xx} + v_x q_i,$$

$$r_{i,y} = -r_{i,xx} - r_i v_x.$$

Lax pair:

$$\begin{cases} \phi_y = (\partial^2 + v_x)(\phi), \\ \phi_t = -\left[4\partial^3 + 6v_x\partial + 3v_{xx} + 3v_y - \sum_{i=1}^{M} q_i \partial^{-1} r_i\right](\phi) \end{cases}$$

H.X. Wu, J.X. Liu and C.X. Li. Quasideterminant solutions of the extended noncommutative Kadomtsev–Petviashvili hierarchy, *Theor. Math. Phys.* **192**(2017): 982–999. Darboux transformation!

Quasi-Wronskian solutions for the nc eKP equation

Assume that $\theta_i = h_i + g_i C_i$ where C_i is an arbitrary constant, h_i and g_i satisfy the dispersion relations (4). Then it is clear that

$$\partial_y \theta_i = \partial_x^2 \theta_i, \quad \partial_t \theta_i = -4\partial_x^3 \theta_i. \tag{5}$$

By using variation of constants, we assume that $\theta_i = h_i + g_i C_i(t)$ with

$$C_i(t) = \begin{cases} c_i(t), & 1 \leq i \leq M \leq N \\ C_i, & \text{otherwise.} \end{cases} \tag{6}$$

Then we have

$$\partial_y \theta_i = \partial_x^2 \theta_i, \quad \partial_t \theta_i = -4\partial_x^3 \theta_i + g_i \dot{c}_i. \tag{7}$$

Let $\hat{\Theta}$ be the Wronskian matrix of $\theta_i, i = 1, 2, \cdots, N$ w.r.t. x and $\Theta = (\theta_1, \cdots, \theta_N)$. Denote

$$Q(i,j) = (-1)^j \begin{vmatrix} \hat{\Theta} & e_{N-j} \\ \Theta^{(N+i)} & \boxed{0} \end{vmatrix}, \tag{8}$$

It is obvious that $v = -2Q(0,0)$ will no longer satisfy the nc KP equation.

Denote $H^{(i)} = \left(g_1^{(i)}\dot{c}_1, \cdots, g_M^{(i)}\dot{c}_M, 0, \cdots, 0\right)$ and $\hat{H} = \left(H^{(0)}, H^{(1)}, \cdots, H^{(N-1)}\right)^T$.
Then $\Theta_t^{(i)} = -4\Theta^{(i+3)} + H^{(i)}$ and

$$\begin{aligned} v_t &= -2Q(0,0)_t \\ &= 8Q(3,0) - 8Q(0,3) + 8Q(0,2)Q(0,0) + 8Q(0,1)Q(1,0) + 8Q(0,0)Q(2,0) \\ &\quad -2\begin{vmatrix} \hat{\Theta} & e_N \\ H^{(N)} & \boxed{0} \end{vmatrix} - 2\sum_{i=1}^{M}\begin{vmatrix} \hat{\Theta} & e_i \\ \Theta^{(N)} & \boxed{0} \end{vmatrix}\begin{vmatrix} \hat{\Theta} & e_N \\ H^{(i-1)} & \boxed{0} \end{vmatrix} \end{aligned}$$

The nc extended KP becomes

$$(v_t + v_{xxx} + v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = -2\left(\sum_{i=1}^{M} q_i r_i\right)_x \tag{9}$$

with

$$q_i = (-1)^N \begin{vmatrix} \hat{\Theta} & \hat{g}_i \\ \Theta^{(N)} & \boxed{g_i^{(N)}} \end{vmatrix} \beta_i, \qquad r_i = (-1)^{N+i-1}\eta_i \begin{vmatrix} \hat{\Theta} & e_N \\ e_i^T & \boxed{0} \end{vmatrix} \tag{10}$$

Where $\hat{g}_i = \left(g_i^{(0)}, \cdots, g_i^{(N-1)}\right)^T$ is the column vector and $\dot{c}_i = \beta_i(t)\eta_i(t)$.

Besides

$$(v_t + v_{xxx} + v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = -2\left(\sum_{i=1}^{M} q_i r_i\right)_x, \tag{11}$$

we have the constraints

$$q_{i,y} = q_{i,xx} + v_x q_i, \tag{12}$$

$$r_{i,y} = -r_{i,xx} - r_i v_x. \tag{13}$$

H.X. Wu, J.X. Liu and C.X. Li. Quasideterminant solutions of the extended noncommutative Kadomtsev–Petviashvili hierarchy, *Theor. Math. Phys.* **192**(2017): 982–999. Darboux transformation!

In commutative case, we have actually

$$v = 2\ln(\tau)_x,$$
$$q_i = \frac{\Phi_i}{\tau},$$
$$r_i = \frac{\Psi_i}{\tau}$$

which gives Wronskian determinant solutions to the bilinear extended KP equation.

Quasi-Grammiann solutions for the nc eKP equation

Denote

$$R(i,j) = (-1)^j \begin{vmatrix} \Omega & P^{\dagger(j)} \\ \Theta^{(i)} & \boxed{0} \end{vmatrix} \tag{14}$$

Where $P = (\rho_1, \cdots, \rho_N)$ and $\Omega = \Omega(\Theta, P) = \Omega(\theta_j, \rho_i)_{N\times N}$ is the $N \times N$ Grammian matrix defined for $i, j \in \{1, \cdots, N\}$.

As is known, the nc KP equation (11) has the quasi-Grammian solutions

$$v = -2R(0,0) \tag{15}$$

where

$$\begin{cases} \Omega(\theta_j, \rho_i)_x = \rho_i^\dagger \theta_j, \\ \Omega(\theta_j, \rho_i)_y = \rho_i^\dagger \theta_{j,x} - \rho_{i,x}^\dagger \theta_j, \\ \Omega(\theta_j, \rho_i)_t = -4(\rho_i^\dagger \theta_{j,xx} - \rho_{j,x}^\dagger \theta_{i,x} + \rho_{i,xx}^\dagger \theta_j). \end{cases} \tag{16}$$

with θ_j satisfying the dispersion relations (4) and ρ_i satisfying

$$\begin{cases} \theta_{i,y} = \theta_{i,xx}, \\ \theta_{i,t} = -4\theta_{i,xxx}. \end{cases} \qquad \begin{cases} \rho_{j,y} = -\rho_{j,xx}, \\ \rho_{j,t} = -4\rho_{j,xxx}. \end{cases} \tag{17}$$

Rewrite $\Omega(\theta_j,\rho_i)_x = \rho_i^\dagger\theta_j$ as

$$\Omega(\theta_j,\rho_i) = C_{i,j} + \int \rho_i^\dagger\theta_j dx.$$

By using variation of constants, we assume

$$\Omega(\theta_j,\rho_i) = C_{i,j}(t) + \int \rho_i^\dagger\theta_j dx$$

where

$$C_{i,j}(t) = \begin{cases} C_i(t), & i=j \quad \text{and} \quad 1\le i\le M\le N, \quad M,N\in Z^+, \\ C_{i,j}, & \text{otherwise.} \end{cases} \tag{18}$$

Under these assumptions, we have

$$\begin{aligned} v_t =& -2R(0,0)_t \\ =& 8R(3,0)-8R(0,3)+8R(0,2)R(0,0)+8R(0,1)R(1,0) \\ &+8R(0,0)R(2,0)-2\sum_{i=1}^{M}\begin{vmatrix}\Omega & e_i \\ \Theta & \boxed{0}\end{vmatrix}\dot{c}_i\begin{vmatrix}\Omega & P^\dagger \\ e_i^\dagger & \boxed{0}\end{vmatrix} \end{aligned}$$

Actually, other derivative formulas of v are exactly the same as those appearing in the case of the nc KP equation.

Similar to the case of quasi-Wronskian solutions, we are able to construct the nc eKP equation

$$(v_t + v_{xxx} + v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = -2\left(\sum_{i=1}^{M} q_i r_i\right)_x,$$
$$q_{i,y} = q_{i,xx} + v_x q_i,$$
$$r_{i,y} = -r_{i,xx} - r_i v_x.$$

with q_i and r_i satisfying the constraints

$$q_i = (-1)^i \begin{vmatrix} \Omega & e_i \\ \Theta & \boxed{0} \end{vmatrix} \beta_i, \tag{19}$$

$$r_i = (-1)^i \eta_i \begin{vmatrix} \Omega & P^\dagger \\ e_i^T & \boxed{0} \end{vmatrix}. \tag{20}$$

where we have assumed $\dot{c}_i = \beta_i(t)\eta_i(t)$.

In commutative case, we have actually

$$v = 2\ln(\tau)_x,$$
$$q_i = \frac{\Phi_i}{\tau},$$
$$r_i = \frac{\Psi_i}{\tau}$$

which gives Grammian determinant solutions to the bilinear extended KP equation.

The nc extended mKP equation

The nc mKP equation

$$W_x - w_y + [w, W] = 0,$$
$$w_t + w_{xxx} - 6ww_xw + 3W_y + 3[w_x, W]_+ - 3[w_{xx}, w] - 3[W, w^2] = 0$$

becomes

$$\begin{aligned} &F(G_{xt} + 3G_{2y} + G_{4x} - G_tFG_x - 4G_{3x}FG_x - 3G_{2x}FG_{2x} + 6G_{2x}FG_xFG_x \\ &+ 3G_{2x}FG_y - 3G_yFG_y + 3G_yFG_{2x} - 6G_yFG_xFG_x) = 0 \end{aligned} \quad (21)$$

under the transformations

$$\begin{aligned} w = -F_xF^{-1}, \qquad W = -F_yF^{-1} \\ w = G^{-1}G_x, \qquad W = G^{-1}G_y. \end{aligned} \quad (22)$$

C. R. Gilson and J. J. C. Nimmo and C. M. Sooman On a direct approach to quasideterminant solutions of a noncommutative modified KP equation, *Journal of Physics A: Mathematical and Theoretical* **41**(2008): 3839-3850.

Lax pair

Lax pair for the nc extended modified KP equation

$$\phi_y = (\partial_x^2 + 2w\partial)\phi, \tag{23}$$

$$\phi_t = \left(-4\partial_x^3 - 12w\partial_x^2 - 6(w_x + w^2 + W)\partial_x + \sum_{i=1}^{M} \widetilde{q}_i \partial_x^{-1} \widetilde{r}_i \partial_x\right)\phi, \tag{24}$$

$$\widetilde{q}_{iy} = \widetilde{q}_{ixx} + 2w\widetilde{q}_{ix}, \quad \widetilde{r}_{iy} = -\widetilde{r}_{ixx} + 2\widetilde{r}_{ix}w, \tag{25}$$

whose compatibility condition gives the nc emKP equation

$$W_x - w_y + [w, W] = 0,$$

$$w_t + w_{xxx} - 6ww_xw + 3W_y + 3[w_x, W]_+ - 3[w_{xx}, w] - 3[W, w^2]$$

$$= -\sum_{i=1}^{M} \left([w, \widetilde{q}\widetilde{r}] + (\widetilde{q}\widetilde{r})_x\right).$$

Quasi-Wronskian solutions for the nc emKP equation

The nc emKP equation has quasi-Wronskian solutions given by

$$F = \begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & \\ \vdots & e_n \\ \Theta^{(n)} & \end{vmatrix}, \qquad G = F^{-1} = \begin{vmatrix} \hat{\Theta} & e_1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix} \tag{26}$$

and

$$\theta_{i,y} = \theta_{i,xx}, \quad \theta_{i,t} = -4\theta_{i,xxx}.$$

By choosing $\theta_i = h_i + g_i C_i$ where C_i is an arbitrary constant, h_i and g_i satisfy the dispersion relations (4), it is clear that

$$\partial_y \theta_i = \partial_x^2 \theta_i, \quad \partial_t \theta_i = -4\partial_x^3 \theta_i. \tag{27}$$

By appling variation of constants in the same way as the nc eKP eqquation, we assume that $\theta_i = h_i + g_i C_i(t)$ with

$$C_i(t) = \begin{cases} c_i(t), & 1 \le i \le M \le N \\ C_i, & \text{otherwise.} \end{cases} \tag{28}$$

Then we have

$$\partial_y \theta_i = \partial_x^2 \theta_i, \quad \partial_t \theta_i = -4\partial_x^3 \theta_i + g_i \dot{c}_i. \tag{29}$$

F and G no longer satisfy the nc mKP equation. Instead we have the nc emKP equation with quasi-Wronskian solutions given by

$$F = \begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & \\ \vdots & e_n \\ \Theta^{(n)} & \end{vmatrix}, \qquad G = F^{-1} = \begin{vmatrix} \hat{\Theta} & e_1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}, \tag{30}$$

$$\widetilde{q}_i = \begin{vmatrix} \Theta & \boxed{g_i} \\ \Theta^{(1)} & g_i^{(1)} \\ \vdots & \vdots \\ \Theta^{(N)} & g_i^{(N)} \end{vmatrix} \beta_i, \qquad \widetilde{r}_i = (-1)^i \eta_i \begin{vmatrix} \hat{\Theta} & e_1 \\ e_i^T & \boxed{0} \end{vmatrix}. \tag{31}$$

Quasi-Grammian solutions for the nc emKP equation

The nc mKP equation

$$W_x - w_y + [w, W] = 0,$$
$$w_t + w_{xxx} - 6ww_xw + 3W_y + 3[w_x, W]_+ - 3[w_{xx}, w] - 3[W, w^2] = 0$$

has quasi-Grammian solutions

$$\begin{aligned} w &= -F_xF^{-1}, \qquad & W &= -F_yF^{-1}, \\ w &= G^{-1}G_x, \qquad & W &= G^{-1}G_y \end{aligned} \tag{32}$$

where quasi-Grammians H and R are defined as

$$F = -\begin{vmatrix} \Omega' & P^{(-1)\dagger} \\ \Theta & \boxed{-I} \end{vmatrix} = I + R'(0,-1), \quad G = F^{-1} = I - R(0,-1) = \begin{vmatrix} \Omega & P^{(-1)\dagger} \\ \Theta & \boxed{I} \end{vmatrix}$$

Where $P^{\dagger}$ and Θ are column and row vector of length N respectively and obey the same dispersion relations as defined in the case of the nc KP equation. Ω and Ω' are $N \times N$ Grammian matrices with Ω' defined as

$$\begin{aligned} \Omega'(\Theta, P) &= \Omega(\Theta, P) - P^{(-1)\dagger}\Theta = (C_{i,j}) + \int P^{\dagger}\Theta dx - P^{(-1)\dagger}\Theta \\ &= (C_{ij}) - \int P^{(-1)\dagger}\Theta^{(1)}dx \end{aligned}$$

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By usingvariation of constants, we assume

$$\Omega(\theta_j, \rho_i) = C_{i,j}(t) + \int \rho_i^{\dagger}\theta_j dx$$

where

$$C_{i,j}(t) = \begin{cases} C_i(t), & i = j \quad \text{and} \quad 1 \le i \le M \le N, \quad M, N \in Z^+, \\ C_{i,j}, & \text{otherwise.} \end{cases} \tag{33}$$

Under these assumptions, we are able to construct the nc emKP equation whose quasi-Grammian solutions are given by

$$F = -\begin{vmatrix} \Omega' & P^{(-1)\dagger} \\ \Theta & \boxed{-I} \end{vmatrix} = I + R'(0, -1), \tag{34}$$

$$G = F^{-1} = I - R(0, -1) = \begin{vmatrix} \Omega & P^{(-1)\dagger} \\ \Theta & \boxed{I} \end{vmatrix}, \tag{35}$$

$$\widetilde{q}_i = (-1)^i \begin{vmatrix} \Omega' & e_i \\ \Theta^{(0)} & \boxed{0} \end{vmatrix} \beta_i, \quad \widetilde{r}_i = (-1)^{i-1}\eta_i \begin{vmatrix} \Omega & P^{(-1)^{\dagger}} \\ e_i^T & \boxed{0} \end{vmatrix}. \tag{36}$$

Miura transformation

The Miura transformation between the nc eKP (v, q_i, r_i) and the nc emKP $(w, W, \widetilde{q}_i, \widetilde{r}_i)$ is given by

$$-w_x - w^2 + W = F v_x F^{-1}, \tag{37}$$

$$\widetilde{q}_i = F q_i, \quad \widetilde{r}_i = -r_i G. \tag{38}$$

Remark

Both of the Miura transformations between the eKP hierarchy and the emKP hierarchy, the nc KP and the nc mKP have been obtained in literature. Here we establish the Miura transformation between the nc eKP and the nc emKP by using quasideterminant identities.

Conclusion and future work

- According to the Gauge transformation for the twisted derivation we proposed, we illustrate that determinant solutions and quasideterminant solutions can be constructed for some commutative integrable systems and nc integrable systems including the super KdV equation, respectively, by using Darboux transformations.

- Source generation procedure is generalized to construct nc integrable systems. Two types of quasideterminant solutions to the nc eKP equation and nc emKP equation are derived. Gauge transformation between the nc eKP equation and nc emKP equations are established from the viewpoint of solutions.

- Possible applications to other types of nc integrable systems.

- Other generalization of soliton theory to nc analogues.