The Construction of Monopoles

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Topological Solitons
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Based on collaborations with Sergey Cherkis, Linden Disney-Hogg and Victor Enolski.

Outline

Timeline of Monopoles, Instantons & the Baker-Akhiezer f^n

1974

- 't Hooft, Magnetic monopoles in unified gauge theories (31 May)
- Polyakov, Particle Spectrum in the Quantum Field Theory (5 July)

1975

- Prasad & Sommerfield, An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon (16 June)
- Belavin, Polyakov, Schwartz & Tyupkin, Pseudoparticle solutions of the Yang-Mills equations (19 Aug.)

1976

- Bogomolny, The stability of classical solutions
- Krichever, An algebraic-geometric construction of the Zakharov-Shabat equations and their periodic solutions

1977

- Ward, On self-dual gauge fields (26 Feb.)
- Cervero, Exact Monopole Solution and Euclidean Yang-Mills Field (Feb.)
- Manton, The Force Between 't Hooft-Polyakov Monopoles (25 May)
- Atiyah & Ward, Instantons and algebraic geometry (31 May)
- Lohe, Two-Dimensional and Three-Dimensional Instantons (ICTP/76/14 6 June)

1978

- Manton, Complex Structure of Monopoles (16 Nov.)
- Corrigan, Fairlie, Yates, & Goddard, The construction of self-dual solutions to $\mathrm{SU}(2)$ gauge theory (25 Nov. 1977)
- Atiyah, Hitchin, Drinfeld & Manin, Construction of instantons (16 Dec. 1977)

1980

- Ward, A Yang-Mills-Higgs monopole of charge 2 (8 Sep.)
- Prasad & Rossi, Construction of exact Yang-Mills-Higgs multimonopoles of arbitrary charge (12 Dec.)

1981

- Corrigan & Goddard, An n monopole solution with $4n - 1$ degrees of freedom (10 Apr.)
- Nahm, All self-dual multimonopoles for arbitrary gauge groups (24 Sep.)

1982

- Hitchin, Monopoles and geodesics (27 Oct. 1981)

1983

- Hitchin, On the construction of monopoles (3 Jan.)

- t'H-P sought time independent solutions for the model with

$$\mathcal{L} = -\frac{1}{8}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\mathrm{Tr}\, D_\mu \Phi\, D^\mu \Phi + V(\Phi), \quad V(\Phi) = \frac{\lambda}{4}(1 - |\Phi|^2)$$

- P&S found the hedgehog solution with $\lambda = 0$, $|\Phi| \to_{r\to\infty} 1$

$$\phi = [\frac{1}{2r} - coth(2r)]\frac{x^i T_i}{r}, \quad A_j = \frac{1}{2}[1 - \frac{2r}{\sinh(2r)}]\frac{\epsilon_{jkl}x^k T^l}{r^2}, \quad T_j = -i\sigma_j$$

- Bogomolny bound $E \geq 2\pi|n|$, $n \in \mathbb{Z}$

$$E = \frac{1}{4}\int_{\mathbb{R}^3}\left(|F|^2 + |D\phi|^2\right) d^3x = \frac{1}{4}\int_{\mathbb{R}^3} |F \mp \star D\phi|^2\, d^3x \mp S.T.$$

$$S.T. = \lim_{R\to\infty}\frac{1}{2}\int_{S_R^2}\langle \phi \wedge (d\phi \wedge d\phi)\rangle \in 2\pi\mathbb{Z}$$

- $*_{\mathbb{R}^3} F = \pm D\Phi$, $\qquad B_i = \frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}F^{jk} = \pm D_i\Phi$

- Cervero, Lohe $A = A_\mu dx^\mu = \Phi dt + A_j dx^j$ ind. of t $\;*_{\mathbb{R}^4} F = \mp F \Leftrightarrow *_{\mathbb{R}^3} F = \pm D\Phi$

$$F = \pm *_{\mathbb{R}^4} F: \qquad F_{12} = \pm F_{34} \qquad F_{13} = \pm F_{42} \qquad F_{14} = \pm F_{23}$$

$$\substack{z=(x_4+ix_3)/\sqrt{2} \\ w=-(ix_1+x_2)/\sqrt{2}} \qquad F_{zw} = 0 \qquad F_{\bar{z}\bar{w}} = 0 \qquad F_{z\bar{z}} + F_{w\bar{w}} = 0$$

$$[D_z + \zeta D_{\bar{w}}, D_w - \zeta D_{\bar{z}}] = 0$$

$$*_{\mathbb{R}^3} F = \pm D\Phi \qquad [D_i, D_j] = \epsilon_{ijk}[D_k, \pm A_4]$$

$$\Phi = \pm A_4 \qquad \left[\frac{1}{2}(D_3 - i\Phi) + \zeta D_z, D_{\bar{z}} - \zeta D_3 - \zeta^2 D_z\right] = 0$$

A *monopole* of charge n

$$\left.\sqrt{-\frac{1}{2}\mathrm{Tr}\,\Phi(r)^2}\right|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

ADHM data for n-instanton solution $F = *_{\mathbb{R}^4} F$ with $G = SU(2)$

$$\Delta = a + bx,\ {\scriptstyle a,b\in \mathrm{Mat}([n+1]\times n,\mathbb{H}),\ x\in\mathbb{H}},\ \ \Delta^\dagger v = 0,\ \ v^\dagger v = 1,\ \ A = v^\dagger dv$$

$\Delta^\dagger\Delta$ real and invertible, $\dim_{\mathbb{H}} \ker \Delta^\dagger = 1$

Nahm data for n-monopole solution $*_{\mathbb{R}^3} F = D\Phi$ with $G = SU(2)$

$$\Delta = 1_{2n}\left(i\frac{d}{ds} + x_4\right) - iT_4(s) + \sum_{j=1}^{3} (T_j(s) + ix_j) \otimes \sigma_j, \qquad {\scriptstyle T_a(s)\in \mathrm{Mat}(n\times n,\mathbb{C})}$$

N1 Nahm's equation $\qquad \dot{T}_i = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j(z), T_k(z)]$

N2 $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0, 2$. Residues form $\mathfrak{su}(2)$ irreducible n-dimensional representation.

N3 $T_i(s) = -T_i^\dagger(s), \qquad T_i(s) = T_i^t(2 - s)$

$$\Delta^\dagger \boldsymbol{v} = i\left(1_{2n}\frac{d}{ds} - T_4(s) + i\sum_{j=1}^{3} T_j(s)\otimes\sigma_j - \sum_{j=1}^{3} x_j 1_n \otimes \sigma_j\right)\boldsymbol{v}$$

$$\mathcal{D}\Psi = i\left(\sigma\cdot D_A + s + i\Phi\right)\Psi \quad 0 = \not{D}_A \Psi_D = \begin{pmatrix} 0 & -\mathcal{D}^\dagger \\ \mathcal{D} & 0 \end{pmatrix}\begin{pmatrix}\Psi \\ \Psi'\end{pmatrix} e^{ix_4 s}$$

$\Delta^\dagger \boldsymbol{v} = 0$	$\mathcal{D}\Psi = 0$
$\boldsymbol{v}_a = \boldsymbol{v}_a(\boldsymbol{x}, s)$	$\Psi_a = \Psi_a(\boldsymbol{x}, s)$ $\quad a,b=1,2 \quad j=1,2,3$
$\delta_{ab} = \int_I \boldsymbol{v}_a^\dagger \boldsymbol{v}_b ds$	$\delta_{ab} = \int_{\mathbb{R}^3} \Psi_a^\dagger \Psi_b d^3x$
$\Phi(\boldsymbol{x})_{ab} = i\int_I s\, \boldsymbol{v}_a^\dagger \boldsymbol{v}_b ds$	$T_4 = -\int_{\mathbb{R}^3} \Psi_a^\dagger \partial_s \Psi_b d^3x$
$A_j(\boldsymbol{x})_{ab} = \int_I \boldsymbol{v}_a^\dagger \frac{\partial}{\partial x_j} \boldsymbol{v}_b ds$	$T_j = -i\int_{\mathbb{R}^3} x_j\, \Psi_a^\dagger \Psi_b d^3x$

Integrals computed in closed form

$$\int \mathbf{v}_a^\dagger(\mathbf{x},s)\mathbf{v}_b(\mathbf{x},s)\mathrm{d}s = \mathbf{v}_a^\dagger(\mathbf{x},s)\mathcal{Q}^{-1}(\mathbf{x},s)\mathbf{v}_b(\mathbf{x},s)$$

$$\mathcal{Q}(\mathbf{x},s) = \frac{1}{r^2}\mathcal{H}(\mathbf{x})\mathcal{T}(s)\mathcal{H}(\mathbf{x}) - \mathcal{T}(s)$$

$$\mathcal{H}(\mathbf{x}) = \sum_{j=1}^{3} 1_n \otimes x_j\sigma_j, \qquad \mathcal{T}(s) = i\sum_{j=1}^{3} T_j(s)\otimes\sigma_j$$

$$\int s\mathbf{v}_a^\dagger\mathbf{v}_b\mathrm{d}s = \mathbf{v}_a^\dagger\,\mathcal{Q}^{-1}\left[s + \mathcal{H}(\mathbf{x})\frac{x_k}{r^2}\frac{\partial}{\partial x_k}\right]\mathbf{v}_b$$

$$\int \mathbf{v}_a^\dagger\frac{\partial}{\partial x_j}\mathbf{v}_b\mathrm{d}s = \mathbf{v}_a^\dagger\;\mathcal{Q}^{-1}\left[\frac{\partial}{\partial x_j} + \mathcal{H}(\mathbf{x})\frac{s\,x_j + i(\mathbf{x}\times\nabla)_j}{r^2}\right]\mathbf{v}_b$$

Observe: Dependence only on boundary values and their derivatives

$$L := \beta - (\alpha+\alpha^\dagger)\zeta - \beta^\dagger\zeta^2, \quad M := -\alpha - \beta^\dagger\zeta, \ {\scriptstyle \alpha = T_4 + iT_3,\ \beta = T_1 + iT_2}$$

$$\dot{T}_i = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j(s), T_k(s)]$$

$$\Longleftrightarrow \dot{L} = [L, M] \Longleftrightarrow \quad \begin{cases} \left[\dfrac{d}{ds} - \alpha, \beta\right] = 0 \\ \dfrac{d(\alpha+\alpha^\dagger)}{ds} = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] \end{cases}$$

$$\beta g = g\nu, \quad \left(\frac{d}{ds} - \alpha\right) g = 0 \Longleftrightarrow \beta = g\nu g^{-1}, \quad \alpha = \dot{g} g^{-1},$$

$$h = g^\dagger g, \qquad \frac{d}{ds}\left(\dot{h}h^{-1}\right) = \left[h\nu h^{-1}, \nu^\dagger\right]$$

- $\mathcal{C} \subset T\mathbb{P}^1 : \ 0 = \det(\eta 1_n - L(\zeta)) := P(\eta, \zeta)$
- $\mathcal{C}$ has real structure $\tau : (\zeta, \eta) \to (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$
- genus given by Riemann Hurwitz formula $g_{\mathcal{C}} = (n-1)^2$

$$\begin{array}{cccccc} \mathbb{R}^4 & \longleftrightarrow & \mathbb{T} & F=\pm *_{\mathbb{R}^4} F & \mathcal{A}_n \text{ ansatz} \\ \frac{\partial}{\partial x^0}\downarrow & & V\downarrow & & \\ \mathbb{R}^3 & \longleftrightarrow & \mathbb{MT} & *_{\mathbb{R}^3} F=\pm D\Phi & \mathcal{A}_n \text{ ansatz} \end{array}$$

$\mathbb{MT} := T\mathbb{P}^1 = \{{}^{\text{oriented}}_{\text{lines}}\ \ell \in \mathbb{R}^3\}$ $\quad (\eta,\zeta) \leftrightarrow \eta \frac{d}{d\zeta} \in T\mathbb{P}^1$

$\tau : (\zeta,\eta) \to (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$ anti-holomorphic involution reverses orientation of lines

Theorem (Hitchin) (A,ϕ), satisfy $*_{\mathbb{R}^3} F = D\Phi$ for $\mathrm{SU}(2) \Leftrightarrow\ \exists$ hol. v.b. $E \to T\mathbb{P}^1$ s.t.

- E is trivial along the image of a real section of $T\mathbb{P}^1 \to \mathbb{P}^1$,
- E has a symplectic structure, i.e. a symp. form ω_z on each fibre E_z
- E has a quaternionic structure $\sigma : E \to \tau^* \overline{E}$, $\sigma^2 = -1$

$L^\lambda \to T\mathbb{P}^1$ line bundle with transition function $\exp(-\lambda\eta/\zeta)$
$L^\lambda(m) := L^\lambda \otimes \pi^*\mathcal{O}(m) \to T\mathbb{P}^1$ with $\zeta^m \exp(-\lambda\eta/\zeta)$

$$*_{\mathbb{R}^3} F = D\Phi \implies [D_3 - i\Phi, D_{\bar{z}}] = 0 \qquad \underset{\text{line}}{\text{oriented}}\ \ell \in \mathbb{R}^3 \quad (D_\ell - i\Phi)u = 0$$

$$\mathcal{C} := \{\ell \mid u \in L^2(-\infty, \infty)\} \subset \mathbb{MT}$$

$$0 \to L^+ \to E \to L^- = (L^+)^* \to 0 \qquad L^+ \cong L(-n)$$

Trans. Fn. $\begin{pmatrix} \zeta^n e^{\eta/\zeta} & \Gamma \\ 0 & \zeta^{-n} e^{-\eta/\zeta} \end{pmatrix}$, $\quad \Gamma = \left[\dfrac{e^{\eta/\zeta + \chi} + (-1)^n e^{-\eta/\zeta - \chi^\tau}}{\zeta^{-n} P(\eta, \zeta)} \right]$, $\chi^\tau := \overline{\tau^* \chi}$

$\chi(\zeta,\eta) = \sum_{i=0}^{n-1} \eta^i \chi_i(\zeta)$ $\chi_i(\zeta)$ holomorphic on $\tilde{U}_0$

Hitchin Data

- H1 $\mathcal{C}$ has real structure τ
- H2 L^2 is trivial on $\mathcal{C}$ and $L^1(n-1)$ is real
  L^2 is trivial $\implies$ $\exists$ nowhere-vanishing holomorphic section
- H3 $H^0(\mathcal{C}, L^s(n-2)) = 0$ for $s \in (0, 2)$ $\implies H^0(\mathcal{C}, L^s) = 0$ $_{s \in (0,2)}$

Euclidean Monopoles $\longleftrightarrow$ Nahm Data $\longleftrightarrow$ Hitchin Data $\mathcal{C} \Leftrightarrow (A, \Phi)$

- $L^2 \to \mathcal{C}$ trivial $\Longleftrightarrow f_0(\eta,\zeta) = \exp\left\{-2\frac{\eta}{\zeta}\right\} f_1(\eta,\zeta)$

$$\mathrm{dlog}\, f_0 = \mathrm{d}\left(-2\frac{\eta}{\zeta}\right) + \mathrm{dlog} f_1,\quad \exp\oint_\gamma \mathrm{dlog}\, f_0 = 1 \quad \forall\gamma\in H_1(\mathcal{C},\mathbb{Z})$$

- $\gamma_\infty(P) = \frac{1}{2}\,\mathrm{dlog}\, f_0(P) + i\pi\sum_{j=1}^{g} m_j\,\omega_j(P)$, $\quad \oint_{\mathfrak{a}_k}\omega_j = \delta_{jk}, \quad \oint_{\mathfrak{a}_k}\gamma_\infty = 0$

$$\oint_{\mathfrak{b}_k}\gamma_\infty = i\pi n_k + i\pi\sum_{l=1}^{g} m_l\tau_{lk} \in i\pi\Lambda \qquad -\frac{\eta}{\zeta}\sim_{P\to\infty_j}\int_{P_0}^{P}\gamma_\infty$$

- Ercolani-Sinha Constraints: The following are equivalent:
  1. L^2 is trivial on $\mathcal{C}$.
  2. If $\boldsymbol{U} = \frac{1}{2i\pi}\left(\oint_{\mathfrak{b}_1}\gamma_\infty,\ldots,\oint_{\mathfrak{b}_g}\gamma_\infty\right)^T$ then $2\boldsymbol{U}\in\Lambda$.
  3. $\exists$ 1-cycle $\mathfrak{es} = \boldsymbol{n}\cdot\mathfrak{a} + \boldsymbol{m}\cdot\mathfrak{b}$ s.t. for every holomorphic differential
  $$\Omega = \frac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\,d\zeta,\quad \oint_{\mathfrak{es}}\Omega = -2\beta_0$$
  4. $$(\mathrm{n},\mathrm{m})\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix} = -2(0,\ldots,0,1), \qquad du_g = \frac{\eta^{n-2}}{\frac{\partial\mathcal{P}}{\partial\eta}}\,d\zeta,$$
- ES constraints impose *g transcendental constraints* on curve

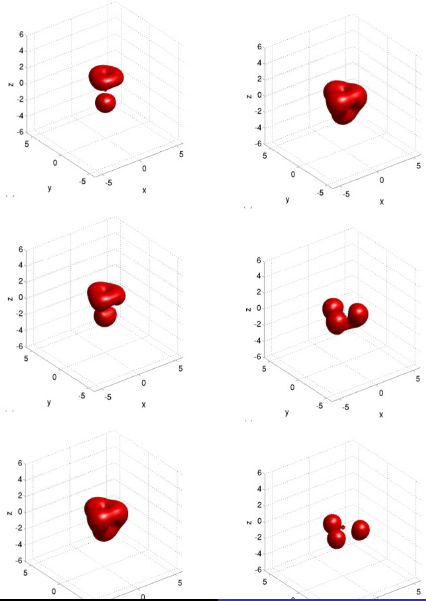
Theorem (B. 2018) *BPS monopole spectral curves are transcendental.*

$P(\eta,\zeta)$	G	
η	$SO(3)$	[PS75]
$\eta\prod_{l=1}^{m}(\eta^2+l^2\pi^2\zeta^2)$	$SO(2)$	[Hit83]
$\prod_{l=0}^{m}(\eta^2+[l+\frac{1}{2}]^2\pi^2\zeta^2)$	$SO(2)$	[Hit83]
$\eta^2+\frac{\boldsymbol{K}(k)^2}{4}\left(\zeta^4+2(k^2-k'^2)\zeta^2+1\right)$	C_2	[Hur83]
$\eta^3+a_3(\zeta^6+5\sqrt{2}\zeta^3-1)$	A_4	[HMM95]
$\eta^4+a_4(\zeta^8+14\zeta^4+1)$	S_4	[HMM95]
$\eta\left(\eta^2-\boldsymbol{K}(k)^2\left(\zeta^4+2(k^2-k'^2)\zeta^2+1\right)\right)$	$C_2(inv)$	[HS96b]
$\eta\left(\eta^4-4a_4(\zeta^8+14\zeta^4+1)\right)$	S_4	[HS96d]
$\eta\left(\eta^6+a_7\zeta(\zeta^{10}+11\zeta^5-1)\right)$	A_5	[HS96d]
$\eta^4+36ia\kappa^3\eta\zeta(\zeta^4-1)+3\kappa^4(\zeta^8+14\zeta^4+1)$	A_4	[HS96a]
$\eta^3-6(a^2\pm 4)^{1/3}\kappa^2\eta\zeta^2+2i\kappa^3a(\zeta^5-\zeta)$	C_4	[HS96c]
$\eta^3+\alpha\eta\zeta^2+\beta\zeta^6+\gamma\zeta^3-\beta$	C_3	[BDE11]

$a_3=\pm\frac{1}{48\sqrt{3}\pi^{3/2}}\,\Gamma\left(\frac{1}{6}\right)^3\Gamma\left(\frac{1}{3}\right)^3=\pm\frac{1}{48\sqrt{3}}\,B\left(\frac{1}{6},\frac{1}{3}\right)^3$ $\quad a_4=\frac{3}{1024\pi^2}\,\Gamma\left(\frac{1}{4}\right)^8=\frac{3}{256}\,B\left(\frac{1}{4},\frac{1}{2}\right)^4$

$a_7=\frac{1}{64\pi^3}\,\Gamma\left(\frac{1}{6}\right)^6\Gamma\left(\frac{1}{3}\right)^6=\frac{1}{64}\,B\left(\frac{1}{6},\frac{1}{3}\right)^6$

BPS Monopoles
Example: Cyclic 3-Monopole Scattering



Rational Maps

- Donaldson Rational Map $\dfrac{q(\eta)}{p(\eta)} := \dfrac{c_1\eta^{n-1} + \ldots + c_n}{P(\eta, 0)}$ Breaks symmetry $\zeta = 0$
  $q(\eta)$ may also be specified.
- Jarvis Rational Map. Retains $\mathrm{SO}(3)$ symmetry
- Moduli space of solutions

How do we reconstruct the fields?

- Numerical Solutions
  - Solve Nahm equations. Assumes Nahm data and so curve.
  - (Jarvis) Rational Map. Take what you get.
- Analytical Solutions
  - $n = 1$
  - $n \geq 2$ coincident monopoles; axially symmetric
  - $n = 2$ Pre 2019: on (portions of) various axes and one plane. BE19 all $\mathbb{R}^3$
- Aim: to show how to use integrability to solve $\mathcal{C} \longrightarrow (\Phi, A)$

Equations $\xrightarrow[\dot{L}=[L,M]]{}$ *Zero Curvature/Lax* Spectral Curve $\mathcal{C}$
$0=\det(\eta 1_n - L(\zeta))$

$\uparrow$ $\downarrow$

Reconstruction $\longleftarrow$ Baker-Akhiezer Function

$s\mathrm{U} + \mathrm{C} \in \mathrm{Jac}(\mathcal{C})$ (possibly Prym) $\theta\left(s\mathrm{U} + \mathrm{C}|\tau\right)$

"*θ-functions are still far from being a spectator sport.*" (L.V. Ahlfors)

For linear flow we require a cohomological condition: true in the case of Nahm's equations (Griffiths).
$\boldsymbol{U} = \frac{1}{2i\pi}\left(\oint_{\mathfrak{b}} \gamma_\infty\right)$ $H^0(\mathcal{C}, L^s) = 0 \Longleftrightarrow \theta\left(s\mathrm{U} + \mathrm{C}|\tau\right) \neq 0, \ s \in (0, 2)$
$\mathcal{C} \longrightarrow (\Phi, A)$ Analogy with Instantons: Solve ADHM data $\longrightarrow A_\mu = v^\dagger \partial_\mu v$ Our programme: based on integrability.

1. We do not need to solve a differential eqn. $\leftrightarrow$ gauge choice
2. Fields algebraic with one transcendent $\int \gamma_\infty$ as $\mathcal{A}_n$ ansatz

- Determine fundamental solution $\boldsymbol{V}_{2n\times 2n}$: $\quad \Delta^\dagger \boldsymbol{V} = 0$
- Project: $\boldsymbol{V}\boldsymbol{\mu} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$, $\int_I \boldsymbol{v}_a^\dagger(\boldsymbol{x}, s)\boldsymbol{v}_b(\boldsymbol{x}, s)ds = \delta_{ab}$

$$\Delta^\dagger \boldsymbol{v} = i\left(1_{2n}\frac{d}{ds} + i\sum_{j=1}^{3} T_j(s)\otimes\sigma_j - \sum_{j=1}^{3} x_j 1_n \otimes \sigma_j\right)\boldsymbol{v}(\boldsymbol{x}, s)$$

$$\Delta \boldsymbol{w} = i\left(1_{2n}\frac{d}{ds} - i\sum_{j=1}^{3} T_j(s)\otimes\sigma_j + \sum_{j=1}^{3} x_j 1_n \otimes \sigma_j\right)\boldsymbol{w}(\boldsymbol{x}, s)$$

$$\boldsymbol{W} = \boldsymbol{V}^{\dagger\,-1}$$

Problems:

- We don't know T_i's (needed to construct $\Delta^\dagger$ and $\mathcal{Q}$),
- How do we solve $\Delta^\dagger \boldsymbol{v} = 0$ or $\Delta \boldsymbol{w} = 0$?

- Let $\boldsymbol{y} = \left(\frac{1+\zeta^2}{2i}, \frac{1-\zeta^2}{2}, -\zeta \right)$ $\quad \widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}(\zeta) := i\, \frac{\boldsymbol{y} \times \overline{\boldsymbol{y}}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}}$

$$\boldsymbol{w} = (1_2 + \widehat{\boldsymbol{u}}(\zeta) \cdot \boldsymbol{\sigma})\, e^{-is[(x_1 - ix_2)\zeta - ix_3]} |\chi > \otimes\, \widehat{\boldsymbol{w}}(s)$$

$$\Delta \boldsymbol{w} = 0 \Longleftrightarrow \begin{cases} 0 = (L(\zeta) - \eta)\, \widehat{\boldsymbol{w}}(s), & \eta = 2\boldsymbol{y} \cdot \boldsymbol{x} \\ 0 = \left(\frac{\mathrm{d}}{\mathrm{d}s} + M \right) \widehat{\boldsymbol{w}}(s) & \\ \text{Nahm eqn's.} & \end{cases}$$

- Need $\boldsymbol{W}$ $(2n \times 2n)$ for $\boldsymbol{V} = (\boldsymbol{W}^\dagger)^{-1}$
  1. Given $\boldsymbol{x}$ solve $P(2\boldsymbol{y} \cdot \boldsymbol{x}, \zeta) = 0$ of degree $2n$ in ζ
     Atiyah-Ward constraint
  2. Let P_i be the corresponding $2n$ points on $\mathcal{C}$.
  3. Construct the $2n \times 2n$ matrix $\boldsymbol{W} = (\boldsymbol{w}(P_i))$

Programme

Given the T_i's

1. Construct the $2n \times 2n$ matrix $\boldsymbol{W} = (\boldsymbol{w}(P_i))$
2. $\boldsymbol{V} = (\boldsymbol{W}^\dagger)^{-1}, \quad \Delta^\dagger \boldsymbol{V} = 0$
3. Construct the projector $\boldsymbol{V}\boldsymbol{\mu} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$

$$0 = (L(\zeta) - \eta)\, \widehat{\boldsymbol{w}}(s)$$

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}s} + M\right) \widehat{\boldsymbol{w}}(s)$$

Solve $\hat{w}$ in terms of Baker-Akhiezer function

Theorem $\mathcal{C}$ a smooth algebraic curve (genus $g_{\mathcal{C}}$) with $n \geqslant 1$ punctures $\{P_j\}_{j=1}^n$. Then for each set $\{\delta_k\}_{k=1}^{g_{\mathcal{C}}+n-1}$ of $g_{\mathcal{C}} + n - 1$ pts in gen. posn. $\exists!\ \Psi_j\,(z, P)$ & loc. coords $w_j(P)$ ($w_j(P_j) = 0$), s.t.

1. $\Psi_j(z, P)$ is mero. $P \in \mathcal{C} \setminus \{P_j\}_{j=1}^n$ & at most simple poles at δ_s (if all of them are distinct);
2. $\Psi_j\,(z, P) = e^{z\, w_l{}^{-m}} \left(\delta_{jl} + \sum_{k=1}^{\infty} \alpha_{jl}^k\,(z)\, w_l^k\right)$ $\qquad w_l = w_l\,(P),\ m \in \mathbb{N}^+$

Standard Scattering & BA function

Recall: $L := \beta - (\alpha + \alpha^\dagger)\zeta - \beta^\dagger\zeta^2,\ M := -\alpha - \beta^\dagger\zeta,$ $\alpha = T_4 + iT_3,\ \beta = T_1 + iT_2$

$$(L - \eta)U = 0, \quad \left[\frac{d}{ds} + M\right] U = 0, \quad P(\eta, \zeta) := \det(\eta - L(\zeta)) = 0$$

$\det(\eta/\zeta^2 - L/\zeta^2) \sim \prod_{i=1}^{n} (\eta/\zeta^2 + \nu_i^\dagger)$ and so $\eta/\zeta \sim -\nu_i^\dagger \zeta$

$U = g^{\dagger\,-1}\Phi, \qquad \left[\frac{d}{ds} - g^\dagger(\alpha + \alpha^\dagger)g^{\dagger\,-1}\right] \Phi = \zeta\nu^\dagger\Phi.$

From BA fn. Φ we can recover:

- $g^\dagger L g^{\dagger\,-1} = h\nu h^{-1} - \dot{h}h^{-1}\zeta - \nu^\dagger\zeta^2$
- Ercolani-Sinha $g^\dagger T_i g^{\dagger\,-1}$
- $h = g^\dagger g$

To recover the T_i's and not just a gauge transform we need to solve a Matrix Factorization for g

Reconstructing the fields

$$W = \left(1 \otimes g^{\dagger\,-1}\right)\ \hat{W}, \qquad V = (1 \otimes g)\ \hat{V}, \qquad \hat{V} = \hat{W}^{\dagger\,-1}$$

where

$$\hat{\boldsymbol{w}} = (1_2 + \widehat{\boldsymbol{u}}(\boldsymbol{x}) \cdot \boldsymbol{\sigma})\, e^{-is[(x_1 - ix_2)\zeta - ix_3 - x_4]} |s> \otimes \Phi_{BA}$$

To reconstruct fields

$$\begin{aligned}\mu^\dagger V^\dagger \mathcal{Q}^{-1} \mathcal{O} V \mu &= \mu^\dagger \hat{V}^\dagger \left[\left(1 \otimes g^\dagger\right) \mathcal{Q}^{-1} (1 \otimes g) \right] \mathcal{O} \hat{V} \mu \\ &= \mu^\dagger \hat{V}^\dagger \mathcal{Q}'^{-1} (1 \otimes h)\, \mathcal{O} \hat{V} \mu \qquad h = g^\dagger g\end{aligned}$$

$$Q' = \left(1 \otimes g^\dagger\right) \mathcal{Q} \left(1 \otimes g^{\dagger\,-1}\right) := \frac{1}{r^2} \mathcal{H} \mathcal{F}' \mathcal{H} - \mathcal{F}'$$

$$\mathcal{F}' = i \sum_{j=1}^{3} \sigma_j \otimes g^\dagger T_j g^{\dagger\,-1} = \begin{pmatrix} \frac{1}{2} \dot{h} h^{-1} & -i\nu^\dagger \\ i\, h \nu h^{-1} & -\frac{1}{2} \dot{h} h^{-1} \end{pmatrix}$$

Some Further Remarks

Theorem: The extraction of norm. solns./construction of the projector μ is algebraic.

Theorem: $\left(V^\dagger Q^{-1}\mathcal{H}V\right)(s) = \text{constant}, \quad \left(W^\dagger Q\mathcal{H}V\right)(s) = \text{constant},$

Simplifications with Symmetry

Let $G \leq \mathrm{Aut}(\mathcal{C})$, $\pi : \mathcal{C} \to \mathcal{C}' := \mathcal{C}/G \Rightarrow \mathrm{Jac}(\mathcal{C}) \sim_{\mathrm{isog.}} \mathrm{Jac}(\mathcal{C}') \times \mathcal{A}$. If $\eta^{n-2}d\zeta/\partial_\eta P$ is

- G-invt. $\Leftrightarrow U$ G-invt. $\Leftrightarrow U = \pi^* U'$, $U' \in \mathrm{Jac}(\mathcal{C}')$ Reduce to $\mathcal{C}'$
- not G-invt. Relate flow to $\mathcal{A}$

Example $\omega = \exp(2\pi i/n)$, $(\eta, \zeta) \to (\omega\eta, \omega\zeta)$

$\mathtt{C}_n$ symmetric (centred) charge-n monopole curve of form

$\hat{\mathcal{C}}: \quad \eta^n + a_2\eta^{n-2}\zeta^2 + \ldots + a_n\zeta^n + \beta\zeta^{2n} + (-1)^n\beta = 0,$ $\quad {}^{a_i,\beta\in\mathrm{R}}_{g_{\mathrm{monopole}}=(n-1)^2}$

$\hat{\mathcal{C}}$ a $n:1$ unbranched cover Aff. Toda Spectral Curve $\hat{\mathcal{C}}/\mathtt{C}_n$ $\quad g_{\mathrm{Toda}}=(n-1)$

$\mathcal{C} := \hat{\mathcal{C}}/\mathtt{C}_n : \quad y^2 = (x^n + a_2x^{n-2} + \ldots + a_n)^2 - 4(-1)^n\beta^2$

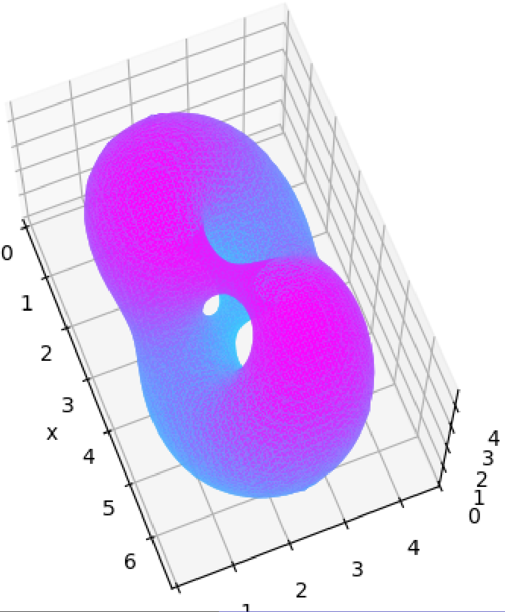
Theorem: Any $\mathtt{C}_n$ symmetric monopole is gauge equivalent to Nahm data given by Sutcliffe's ansatz, and so obtained from the affine Toda eqns.

Theorem (BD-H): The charge-3 monopole spectral curves with $H \leq \mathrm{Aut}(\mathcal{C})$ such that the quotient genus $g(\mathcal{C}/H) = 1$ may be classified.

$$
\begin{tikzcd}
 & \eta^3 + \alpha_2\eta\zeta^2 + \alpha_3\zeta^3 + \beta(\zeta^6 - 1), [BDE11] \arrow[dl, "\alpha_3=0"'] \arrow[d, "\alpha_2=0 \text{ and rotation}"] \\
\eta^3 + \alpha_2\eta\zeta^2 + \beta(\zeta^6 - 1), [BDH23] \arrow[d, "\beta=0"'] & \eta^3 + c\zeta(\zeta^4 - 1), [HMM95\ A_4] \\
\eta[\eta^2 + \pi^2\zeta^2], [Hit83] & \eta^3 + b\eta\zeta^2 + c\zeta(\zeta^4 - 1), [HS96c\ C_4] \arrow[l, "c=0"] \arrow[u, "b=0"'] \\
\eta\left[\eta^2 + a(\zeta^4 + 1) + b\zeta^2\right], [HS96b] \arrow[u, "a=0"] & \eta^3 + \eta[a(\zeta^4 + 1) + b\zeta^2] + c\zeta(\zeta^4 - 1) \arrow[l, "c=0"] \arrow[u, "a=0"]
\end{tikzcd}
$$

$H97$, $BDH23$ V_4

Charge-3 $V4$ Monopole Energy Density



Conclusion

- Reviewed the Nahm and Hitchin constructions of monopoles and the associated spectral curve.
- Shown how one may reconstruct the fields from the spectral curve.
- Indicated how symmetry may be used to reduce the curve and the problem.
- Described the possible forms of all charge-3 monopole spectral curves $\mathcal{C}$ with non-trivial automorphism group H such that $g(\mathcal{C}) = 1$.
- Described a new class of such curves with V_4 symmetry solving the transcendental constraints.
- The Challenge: how to find monopole spectral curves?