Non-Abelian ODEs and O∆Es

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Noncommutative Integrable Systems Workshop

A short overview of the topic

- S. P. Balandin and V. V. Sokolov. On the Painlevé test for non-Abelian equations. Physics letters A, 246(3-4):267–272, 1998.
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- H. Nagoya, B. Grammaticos, A. Ramani, et al. Quantum Painlevé equations: from Continuous to discrete. SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 4:051, 2008.
- V. S. Retakh and V. N. Rubtsov. Noncommutative Toda Chains, Hankel Quasideterminants and Painlevé II Equation. Journal of Physics. A, Mathematical and Theoretical, 43(50):505204, 2010. arXiv:1007.4168.
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- H. Kawakami. Matrix Painlevé systems. Journal of Mathematical Physics, 56(3):033503, 2015.
- M. Cafasso, D. Manuel, et al. The Toda and Painlevé systems associated with semiclassical matrix-valued orthogonal polynomials of Laguerre type. SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 14:076, 2018. arXiv:1801.08740.
- M. Bertola, M. Cafasso, and V. Rubtsov. Noncommutative Painlevé equations and systems of Calogero type. Communications in Mathematical Physics, 363(2):503–530, 2018. arXiv:1710.00736.
- V. E. Adler. Painlevé type reductions for the non-Abelian Volterra lattices. Journal of Physics A: Mathematical and Theoretical, 54(3):035204, 2020. arXiv:2010.09021.
- V. E. Adler and V. V. Sokolov. On matrix Painlevé II equations. Theoret. and Math. Phys., 207(2):188–201, 2021. arXiv:2012.05639.
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Outline

The Ablowitz-Ramani-Segur conjecture [Ablowitz et al., 1980]

> A nonlinear PDE is solvable by the inverse scattering method [Zakharov and Shabat, 1974] only if every nonlinear ODE obtained by an exact reduction has the Painlevé property.

(1) Motivating examples:

- A matrix KdV equation: integrability and symmetries. [Wadati and Kamijo, 1974], [Olver and Sokolov, 1998]
- A matrix first Painlevé equation as a reduction of the matrix KdV. [Olver and Sokolov, 1998]
- Discrete analogs for the matrix first Painlevé equation. [Adler, 2020]

(2) Non-commutative ODEs:

- Setting and main definitions. [Bobrova, 2023]
- First integrals and Lax pairs. [Mikhailov and Sokolov, 2000], [Bobrova, 2023]

(3) Non-commutative OΔEs:

- Setting and main definitions.
- First integrals and Lax pairs.
- Continuous limits.

(4) Methods for the derivation & more examples.

Motivating examples

A matrix KdV equation

$$w_t + 6\,ww_x + 6\,w_x w + w_{xxx} = 0, \qquad w = w(x,t) \in \mathrm{Mat}_n(\mathbb{C}), \qquad x,\ t \in \mathbb{C}. \qquad \text{KdV}$$

- The inverse scattering method. [Wadati and Kamijo, 1974]
- A hierarchy of commuting symmetries. [Olver and Sokolov, 1998], [Olver and Wang, 2000]
- The Zakharov-Shabat type pair

$$\begin{cases} \partial_x \Psi &= U\,\Psi, \\ \partial_t \Psi &= V\,\Psi, \end{cases} \qquad \Psi = \Psi(x,t) := (\psi_1 \ \ \psi_2)^T, \tag{1}$$

with 2×2-matrices $U = U(\mu, x, t)$ and $V = V(\mu, x, t)$ and the scalar spectral parameter μ:

$$U = \begin{pmatrix} 0 & \frac{1}{2}\mu\,\mathbb{I} + w \\ -2\,\mathbb{I} & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} -2w_x & 2\mu^2\,\mathbb{I} + 2w\mu - 4w^2 - w_{xx} \\ -8\mu\,\mathbb{I} + 8w & 2w_x \end{pmatrix}. \tag{2}$$

- The zero-curvature condition $\partial_t U - \partial_x V = [V, U]$ is equivalent to the KdV.
- Symmetries: [Olver and Sokolov, 1998]

shift along x	shift along t	Galilean transformation	self-similar transformation
$V_1 = \partial_x$	$V_2 = \partial_t$	$V_3 = 12t\,\partial_x + \partial_w$	$V_4 = x\,\partial_x + 3t\,\partial_t - 2w\,\partial_w$

A matrix P_1 equation (1)

$$y'' = 6y^2 + z\,\mathbb{I} + a, \qquad y(z),\ a \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \qquad P_1$$

Reduction of the equation

- Symmetry reduction of the matrix KdV equation:

$w_t + 6ww_x + 6w_xw + w_{xxx} = 0 \quad \Rightarrow$	$w(x,t) = -y(z) + t\,\mathbb{I}$, $z(x,t) = x - 6t^2$	$\Rightarrow \quad y'' = 6y^2 + z\,\mathbb{I} + a$
the KdV equation	**the Galilean transformation with the shift along t**	**the P_1 equation**

Reduction of the ZC representation

- Transformation of the spectral parameter:

$$\lambda(t) = \mu + 2t. \tag{3}$$

- The ZC representation $\partial_t U - \partial_x V = [V, U]$ becomes

$$\partial_z A - \partial_\lambda B = [B, A], \tag{4}$$

where $A(\lambda, z)$ and $B(\lambda, z)$ are

$$B(\lambda, z) = U(\lambda, z) = \begin{pmatrix} 0 & \frac{1}{2}\lambda\,\mathbb{I} - y \\ -2\,\mathbb{I} & 0 \end{pmatrix}, \tag{5}$$

$$A(\lambda, z) = \frac{1}{2}V(\lambda, z) + 6tU(\lambda, z) = \begin{pmatrix} y' & \lambda^2\,\mathbb{I} - \lambda y + y^2 + \frac{1}{2}z\,\mathbb{I} + \frac{1}{2}a \\ -4\lambda\,\mathbb{I} - 4y & -y' \end{pmatrix}. \tag{6}$$

- The compatibility condition (4) is equivalent to the matrix P_1 equation.

A matrix P_1 equation (1)

$$y'' = 6y^2 + z\,\mathbb{I} + a, \qquad y(z),\ a \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \qquad P_1$$

Reduction of the equation

- Symmetry reduction of the matrix KdV equation:

the KdV equation	the Galilean transformation with the shift along t	the P_1 equation
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A matrix P_1 equation (2)

$$y'' = 6y^2 + z\,\mathbb{I} + a, \qquad y(z),\ a \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \qquad P_1$$

Properties

- P_1 solves the matrix KdV equation.
- P_1 admits an isomonodromic representation.
- P_1 passes a matrix Painlevé-Kovalevskaya test [Balandin and Sokolov, 1998].
- P_1 is Hamiltonian:

$$H(u,v,z) = \mathrm{tr}\left(-2u^3 + \tfrac{1}{2}v^2 - au - zu\right), \qquad \left\{u_{ij}, v_{kl}\right\} = \delta_{il}\,\delta_{jk}; \tag{7}$$

$$\begin{cases} u' = v, \\ v' = 6u^2 + z\,\mathbb{I} + a, \end{cases} \quad \Leftrightarrow \quad P_1 \quad \text{for} \quad y(z) = u(z).$$

- P_1 as well as its Lax pair can be generalized to the case of an associative unital algebra $\mathcal{A}_{\mathbb{C}} = \langle u_i, v_i, a\rangle$, $i \geq 0$ equipped with a derivation $d_z : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule and

$$d_z(a) = 0, \quad d_z(z) = 1, \quad d_z(u_i) = u_{i+1} =: u^{(i+1)}, \quad d_z(v_i) = v_{i+1} =: v^{(i+1)}. \tag{8}$$

- Moreover, making the change $\bar{z} = z\,\mathbb{I} + a$ in P_1 and its Lax pair, we arrive at the so-called fully non-abelian version of the P_1 equation.

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Discrete analogs for the P_1 equation [Adler, 2020]

$$u_{m+1}u_m + u_m^2 + u_m u_{m-1} + x\,u_m + \gamma_m = 0, \qquad \mathrm{dP}_1^1$$
$$u_{m+1}^T u_m + u_m^2 + u_m u_{m-1}^T + x\,u_m + \gamma_m = 0, \qquad \mathrm{dP}_1^2$$
$$\gamma_m = m - \nu + (-1)^m \varepsilon, \quad u_m \in \mathrm{Mat}_n(\mathbb{C}), \quad x\,\nu, \varepsilon \in \mathbb{C}.$$

- They are results of a reduction of the matrix Volterra lattices for $u_m = u_m(x)$:

$$u_{m,x} = u_{m+1}\,u_m - u_m\,u_{m-1}, \qquad \mathrm{VL}^1$$
$$u_{m,x} = u_{m+1}^T\,u_m - u_m\,u_{m-1}^T. \qquad \mathrm{VL}^2$$

- The VL^1, VL^2 is equivalent to the compatibility condition of the given 2×2 matrix system

$$\begin{cases} \Psi_{m+1} = L_m(\lambda)\Psi_m, \\ \partial_x \Psi_m = M_m(\lambda)\Psi_m, \end{cases} \qquad \Psi_m = \Psi_m(x) := \begin{pmatrix} \psi_m & \psi_{m-1} \end{pmatrix}^T. \qquad (9)$$

- The reduction can be extended for the Lax pairs and leads to the system

$$\begin{cases} \partial_\lambda \Phi_m = A_m(\lambda)\,\Phi_m, \\ \Phi_{m+1} = B_m(\lambda)\,\Phi_m, \end{cases} \qquad \Phi_m = \Phi_m(\lambda) \in \mathrm{Mat}_2(\mathbb{C}). \qquad (10)$$

- E.g., for the dP_1^1, we have

$$A_m = \begin{pmatrix} \lambda^2 + \lambda(u_m + x) - \gamma_{m+1} & \lambda^2 u_m - \lambda(u_m u_{m-1} - \gamma_m) \\ -\lambda - u_m - u_{m-1} - x & -\lambda u_m - \gamma_m \end{pmatrix}, \quad B_m = \begin{pmatrix} \lambda & \lambda u_m \\ -1 & 0 \end{pmatrix}. \qquad (11)$$

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Some observations

- All considered matrix ODEs and OΔEs coincide with the well-known scalar analogs.
- All of them are the reductions of the integrable matrix PDEs or PΔEs.
- Thanks to the reductions, one can justify the integrability of the reduced matrix ODEs or OΔEs by using the Lax pairs.
- These systems might contain arbitrary matrix constants or even might be generalised to the fully non-commutative case.
- In the equations P_1, dP_1^1 and their Lax pairs we do not use the matrix setting explicitly. So, they can be extended to the case of an associative unital algebra $\mathcal{A}$ with a derivation.
- In order to deal with the dP_1^2, one needs to introduce an involution on $\mathcal{A}$.
- Regarding the discrete systems, it is natural to study continuous limits. Indeed, one can consider the change with the commutative parameter ε

$$z = \varepsilon m \tag{12}$$

supplemented by the maps

$$u_m \mapsto u, \qquad u_{m+k} \mapsto u + k\,\varepsilon u' + \tfrac{1}{2}k^2\,\varepsilon^2 u'' + O(\varepsilon^3). \tag{13}$$

The latter must be chosen in such a way that the limit $\varepsilon \to 0$ exists.

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Non-Abelian ODEs

Setting (1)

- Let G_i, $i = 0, 1, 2, \dots$ be a free group generated by the set $\bar{x}_i = \{x_{1,i}, x_{2,i}, \dots, x_{N,i}\}$:

$$G_i = \langle x_{1,i}, x_{2,i}, \dots, x_{N,i} \rangle. \tag{14}$$

  We set $x_{k,0} =: x_k$.

- Let $\mathcal{A}$ be a unital associative group algebra over the field $\mathbb{C}$ (or any other field of $char = 0$):

$$\mathcal{A} = \bigoplus_{i \geq 0} \mathbb{C} G_i. \tag{15}$$

Definition 1. An involution $\tau : \mathcal{A} \to \mathcal{A}$ defining by

$$\tau(x_k) = x_k, \qquad \tau(P\,Q) = \tau(Q)\,\tau(P), \qquad P, Q \in \mathcal{A} \tag{16}$$

is called a *transposition*. Its action on $M = (m_{i,j}) \in \mathrm{Mat}_n(\mathcal{A})$ is extended as follows

$$\tau(m_{i,j}) = (\tau(m_{j,i})). \tag{17}$$

- Let z be a central element of $\mathcal{A}$ and all parameters α_i belong to the field.

Remark 1. One can extend $\mathcal{A}$ in order to include z, α_i.

Example 1. Let $N = 3$ and $P = x_1\,x_2^2\,x_3$. Then

$$\tau(P) \equiv \tau\left(x_1\,x_2^2\,x_3\right) = \tau(x_3)\,\tau(x_2)^2\,\tau(x_1) = x_3\,x_2^2\,x_1. \tag{18}$$

Setting (2)

Remark 2. We identify the unit of $\mathcal{A}$ with the unit of the field $\mathbb{C}$.

Definition 2. A $\mathbb{C}$-linear map $d_z : \mathcal{A} \to \mathcal{A}$ satisfying the properties

$$d_z(\alpha_i) = 0, \qquad d_z(z) = 1, \qquad d_z(x_{k,i}) = x_{k,i+1}, \tag{19}$$

$$d_z(P\,Q) = d_z(P)\,Q + P\,d_z(Q) \quad P, Q \in \mathcal{A} \tag{20}$$

is called a *derivation* of $\mathcal{A}$. We denote $d_z(x_k) = x_k'$, $d_z^2(x_k) = x_k''$, and so on.

Remark 3. τ and d_z commute with each other.

Example 1. Consider $N = 2$ and $P = x_1 x_2^2 x_1$. Then $d_z(P)$ is

$$d_z(P) = d_z\left(x_1\,x_2^2\,x_1\right) = x_1'\,x_2^2\,x_1 + x_1\,x_2'\,x_2\,x_1 + x_1\,x_2\,x_2'\,x_1 + x_1\,x_2^2\,x_1'. \tag{21}$$

Example 2. Let $N = 1$. Find $d_z\left(x_1^{-1}\right)$. Since $x_1\,x_1^{-1} = x_1^{-1}\,x_1 = 1$, we have

$$d_z\left(x_1\,x_1^{-1}\right) = d_z\left(x_1\right)\,x_1^{-1} + x_1\,d_z\left(x_1^{-1}\right) \equiv d_z\left(1\right) = 0. \tag{22}$$

Therefore,

$$d_z\left(x_1^{-1}\right) = -x_1^{-1}\,x_1'\,x_1^{-1}. \tag{23}$$

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Main definition

Definition 3. A set of relations of the form

$$d_z(x_k) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \dots, N \tag{24}$$

we call a *system of non-abelian ODEs*. If for some k the element F_k depends on z explicitly, the system is *non-autonomous*, otherwise – *autonomous*.

Remark 4. The system (24) is a non-abelian generalization of a system of first order ODEs. It is also easy to introduce a non-abelian analog for a system of the higher order ODEs.

Remark 5. Note that we can introduce a set of derivations $d_{z_1}, d_{z_2}, \dots$. Then, the system it is easy to define a system of non-abelian PDEs just by considering different d_{z_l} in (24).

Example 3. Let $N = 1$ in (24). Then the following equations

$$x_1' = x_1, \qquad x_1' = z\, x_1 \tag{25}$$

are autonomous and non-autonomous, respectively. These equations are invariant under the τ-action, since, for instance,

$$\tau(x_1') = (\tau(x_1))' = x_1' \equiv \tau(z\, x_1) = \tau(x_1)\, \tau(z) = x_1\, z = z\, x_1. \tag{26}$$

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we call a *system of non-abelian ODEs*. If for some k the element F_k depends on z explicitly, the system is *non-autonomous*, otherwise – *autonomous*.

Remark 4. The system (24) is a non-abelian generalization of a system of first order ODEs. It is also easy to introduce a non-abelian analog for a system of the higher order ODEs.

Remark 5. Note that we can introduce a set of derivations $d_{z_1}, d_{z_2}, \ldots$. Then, the system it is easy to define a system of non-abelian PDEs just by considering different d_{z_l} in (24).

Example 3. Let $N = 1$ in (24). Then the following equations

$$x_1' = x_1, \qquad x_1' = z\,x_1 \tag{25}$$

are autonomous and non-autonomous, respectively. These equations are invariant under the τ-action, since, for instance,

$$\tau(x_1') = (\tau(x_1))' = x_1' \equiv \tau(z\,x_1) = \tau(x_1)\,\tau(z) = x_1\,z = z\,x_1. \tag{26}$$

First integrals (1)

$$d_z(x_k) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \ldots, N. \tag{24}$$

Definition 4. An element $I \in \mathcal{A}$ is a *first integral* for system (24) if

$$d_z(I) = 0. \tag{27}$$

Example 4. Consider $N = 2$ and set $I = x_1\,x_2 - x_2\,x_1$. For the system

$$\begin{cases} x_1' = x_1\,x_2\,x_1, \\ x_2' = -x_2\,x_1\,x_2 \end{cases} \tag{28}$$

the element I is a first integral:

$$\begin{aligned} d_z(I) &= x_1'\,x_2 + x_1\,x_2' - x_2'\,x_1 - x_2\,x_1' \\ &= x_1\,x_2\,x_1\,x_2 - x_1\,x_2\,x_1\,x_2 + x_2\,x_1\,x_2\,x_1 - x_2\,x_1\,x_2\,x_1 = 0, \end{aligned} \tag{29}$$

while for the system

$$\begin{cases} x_1' = x_1^2\,x_2, \\ x_2' = -x_1\,x_2^2 \end{cases} \tag{30}$$

we have

$$\begin{aligned} d_z(I) &= x_1'\,x_2 + x_1\,x_2' - x_2'\,x_1 - x_2\,x_1' = x_1^2\,x_2^2 - x_1^2\,x_2^2 + x_1\,x_2^2\,x_1 - x_2\,x_1^2\,x_2 \\ &= x_1\,x_2^2\,x_1 - x_2\,x_1^2\,x_2 \neq 0. \quad (= 0 \text{ up to } [x_1, x_2]) \end{aligned} \tag{31}$$

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First integrals (2)

$$d_z(x_k) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \dots, N. \qquad (24)$$

Definition 5. If $P - Q \in [\mathcal{A}, \mathcal{A}]$ for P, $Q \in \mathcal{A}$, then we write $P \sim Q$.

Example 5. $x_1 \, x_2^2 \, x_1 \sim x_2 \, x_1^2 \, x_2$.

Definition 6. A class of $P \in \mathcal{A}$ in the space $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ is denoted by $\operatorname{Tr} P$.

Definition 7. An element $I \in \mathcal{A}$ is a *first trace-integral* for system (24) if

$$d_z\,(\operatorname{Tr} I) = 0. \qquad (32)$$

Example 6. Under Example 4, the element $I = x_1 \, x_2 - x_2 \, x_1$ is a trivial trace-integral for both systems (moreover, for any non-abelian system), since $I \sim 0$.

Remark 6. The trace-integrals are necessary for introducing a non-abelian Hamiltonian formalism. We will not consider such a formalism in this series of lectures. For more details, see the original paper [Kontsevich, 1993] where this formalism was introduced for the first time. See also [Olver and Sokolov, 1998] and [Mikhailov and Sokolov, 2000].

Lax pairs (1)

$$d_z(x_k) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \dots, N. \qquad (24)$$

- In addition to d_z, consider a derivation d_λ and $\lambda \in \mathcal{Z}(\mathcal{A})$ such that
$$d_\lambda(\lambda) = 1, \qquad d_\lambda(z) = 0, \qquad d_\lambda(\alpha_i) = 0, \qquad d_\lambda(x_k) = 0. \qquad (33)$$
The parameter λ is a *spectral parameter*.
- Let $\mathrm{A} = \mathrm{A}(\lambda, z)$, $\mathrm{B} = \mathrm{B}(\lambda, z)$ and $\mathrm{L} = \mathrm{L}(\lambda, z)$, $\mathrm{M} = \mathrm{M}(\lambda, z)$ be $n \times n$ matrices over $\mathcal{A}$.

Definition 8. If the non-autonomous system (24) is equivalent to the equation
$$d_z\mathrm{A} - d_\lambda\mathrm{B} = \mathrm{B}\,\mathrm{A} - \mathrm{A}\,\mathrm{B}, \qquad (34)$$
then the matrices A, B and condition (34) are called an *isomonodromic Lax pair* and an *isomonodromic representation* for system (24).

Definition 9. If the autonomous system (24) is equivalent to the equation
$$d_z\mathrm{L} = \mathrm{M}\,\mathrm{L} - \mathrm{L}\,\mathrm{M}, \qquad (35)$$
then the matrices L, M and condition (35) are called an *isospectral Lax pair* and a *Lax equation* for system (24).

Remark 7. The existence of a Lax pair is invariant under the τ-action. Note that the matrices change as follows
$$\mathrm{A} \mapsto -\mathrm{A}, \qquad \mathrm{B} \mapsto -\mathrm{B}; \qquad \mathrm{L} \mapsto \mathrm{L}, \qquad \mathrm{M} \mapsto -\mathrm{M}. \qquad (36)$$

Lax pairs (2)

Example 7. Let $N = 2$. The system

$$\begin{cases} x_1' = x_1\, x_2\, x_1, \\ x_2' = -x_2\, x_1\, x_2 \end{cases} \tag{37}$$

has the following isospectral Lax pair

$$\mathrm{L} = \begin{pmatrix} 0 & -x_1 \\ -x_2\, x_1\, x_2 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} x_2 & -1 \\ x_2^2 & -x_2 \end{pmatrix} \lambda^{-2}, \qquad \mathrm{M} = \frac{1}{2} \begin{pmatrix} x_1\, x_2 & -x_1 \\ -x_2\, x_1\, x_2 & -x_2\, x_1 \end{pmatrix}. \tag{38}$$

Definition 10. A non-autonomous system turns to be autonomous by replacing z with $t \in \mathfrak{Z}(\mathcal{A})$ in all right-hand sides F_k and assuming $d_z(t) = 0$. We call this procedure an *autonomization*.

Proposition 1. [Bobrova, 2023] If a non-autonomous system has an isomonodromic Lax pair, then the corresponding autonomous system has an isospectral Lax pair.

Example 8. An autonomous version of the P_1 system

$$\begin{cases} u' = v, \\ v' = 6u^2 + z, \end{cases} \qquad \Leftrightarrow \qquad u'' = 6u^2 + z \tag{39}$$

has the following isospectral Lax pair

$$\mathrm{L} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & -2u \\ -2 & 0 \end{pmatrix} \lambda + \begin{pmatrix} v & 2u^2 + t \\ -2u & -v \end{pmatrix}, \quad \mathrm{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & -2u \\ -1 & 0 \end{pmatrix}. \tag{40}$$

Remark 8. An autonomous version of the P_1 equation is solved in terms of the $\wp$-function.

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Non-Abelian OΔEs

Setting and definitions (1)

- Let $\mathcal{A}$ be as before a unital associative group algebra over $\mathbb{C}$:

$$\mathcal{A} = \bigoplus_{i \geq 0} \mathbb{C} G_i, \qquad i = 0, 1, 2, \ldots, \tag{41}$$

where $G_i = \langle x_{1,i}, x_{2,i}, \ldots, x_{N,i} \rangle$.

- Instead of a derivation of $\mathcal{A}$, we introduce a translation operator on $\mathcal{A}$.

Definition 1. A homomorphism $T : \mathcal{A} \to \mathcal{A}$ satisfying the properties

$$T(z) = z, \qquad T(\alpha_i) = f(\alpha_i), \qquad T(x_{k,i}) = x_{k,i+1}, \tag{42}$$

where $f(\alpha_i)$ is a certain function, is called a *shift operator* on $\mathcal{A}$.

Definition 2. A set of relations of the form

$$T(x_{k,i}) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \ldots, N \tag{43}$$

we call a *discrete non-abelian system*. It can be classified into three types:

- if $f(\alpha_i) = \alpha_i$ for any i, then (43) is autonomous;
- if $f(\alpha_i) = \alpha_i \pm 1$ for some i, then (43) is non-autonomous and of additive type (d);
- if $f(\alpha_i) = q^{\pm 1} \alpha_i$ for some i, then (43) is non-autonomous and of multiplicative type (q).

Remark 1. In abelian case, there exist discrete elliptic systems. We do not consider this case, since we are not aware of examples of such systems (yet).

Setting and definitions (2)

$$T(x_{k,i}) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \dots, N \tag{43}$$

Remark 2. Considering the notation

$$T^m(x_k) \equiv T(T \dots T(T(x_k)) \dots) =: x_{k,m}, \tag{44}$$

(43) can be rewritten in a difference form that we will call a *system of non-abelian OΔEs*.

Example 1. Let $N = 1$ in (43). Then the following equations

$$x_{m+1} = \alpha\, x_m, \qquad x_{m+1} = (\alpha + m)\, x_m, \qquad x_{m+1} = \alpha\, q^m\, x_m \tag{45}$$

are autonomous and non-autonomous of additive and multiplicative type respectively.

Remark 3. A discrete dynamic might be considered as a map

$$\varphi : \mathcal{A}^N \to \mathcal{A}^N. \tag{46}$$

In particular, considering the precious example, we have for the autonomous system the map

$$\varphi : \mathcal{A} \to \mathcal{A}, \qquad x \mapsto \alpha\, x, \tag{47}$$

where $x := x_1$.

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First integrals

$$T(x_{k,i}) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \dots, N \tag{43}$$

Definition 3. An element $I \in \mathcal{A}$ is a *first integral* for system (43) if

$$\varphi(I) = I. \tag{48}$$

Example 4. Let $N = 4$. Consider the discrete map

$$y_{m+4} = y_{m+1} + y_{m+2} \left(y_m^{-1} - y_{m+3}^{-1} \right) y_{m+2}. \tag{49}$$

The map $\varphi : \mathcal{A}^4 \to \mathcal{A}^4$

$$(y_1, y_2, y_3, y_4) \mapsto \left(y_2, y_3, y_4, y_2 + y_3 \,(y_1^{-1} - y_4^{-1})\, y_3 \right) \tag{50}$$

preserves the function

$$I = y_2\, y_3^{-1} + y_3\, y_1^{-1} + y_4\, y_2^{-1}. \tag{51}$$

Remark 4. (49) is a non-abelian analog [Bobrova et al., 2023] for the Somos-4 equation:

$$x_{m+4}\, x_m = x_{m+3}\, x_{m+1} + x_{m+2}^2. \tag{52}$$

In this case, $I = x_2^2(x_1x_3)^{-1} + x_3^2(x_2x_4)^{-1} + x_1x_4(x_2x_3)^{-1} + x_2x_3(x_1x_4)^{-1}$.

Lax pairs (1)

$$T(x_{k,i}) = F_k, \qquad F_k \in \mathcal{A}, \qquad k = 1, \ldots, N \qquad (43)$$

- λ, q are central elements of $\mathcal{A}$.

Definition 4. If the autonomous system (43) is equivalent to the equation

$$\mathsf{L}_{m+1}(\lambda)\,\mathsf{M}_m(\lambda) = \mathsf{M}_m(\lambda)\,\mathsf{L}_m(\lambda), \qquad (53)$$

then the matrices $\mathsf{L}_m = \mathsf{L}_m(\lambda)$, $\mathsf{M}_m = \mathsf{M}_m(\lambda)$ and condition (53) are called a *discrete Lax pair* and a *discrete Lax equation* for system (43).

Definition 5. If the non-autonomous d-system (43) is equivalent to the equation

$$d_\lambda \mathsf{B}_m(\lambda) = \mathsf{A}_{m+1}(\lambda)\,\mathsf{B}_m(\lambda) - \mathsf{B}_m(\lambda)\,\mathsf{A}_m(\lambda), \qquad (54)$$

then the matrices $\mathsf{A}_m = \mathsf{A}_m(\lambda)$, $\mathsf{B}_m = \mathsf{B}_m(\lambda)$ and condition (54) are called an *isomonodromic d-pair* and an *isomonodromic d-representation* for system (43).

Definition 6. If the non-autonomous q-system (43) is equivalent to the equation

$$\mathsf{B}_m(q\,\lambda)\,\mathsf{A}_m(\lambda) = \mathsf{A}_{m+1}(\lambda)\,\mathsf{B}_m(\lambda), \qquad (55)$$

then the matrices $\mathsf{A}_m = \mathsf{A}_m(\lambda)$, $\mathsf{B}_m = \mathsf{B}_m(\lambda)$ and condition (55) are called an *isomonodromic q-pair* and an *isomonodromic q-representation* for system (43).

Lax pairs (2)

$$\mathsf{L}_{m+1}(\lambda)\,\mathsf{M}_m(\lambda) = \mathsf{M}_m(\lambda)\,\mathsf{L}_m(\lambda), \tag{53}$$

Example 5. Let $N = 4$ and $a_m = y_{m+2}\,y_m^{-1}$, $b_m = y_m\,y_{m+1}^{-1}$. Consider the matrices

$$\mathsf{L}_m = \begin{pmatrix} \lambda\,(\lambda^2 + b_{m+1} + a_{m+1}) & (\lambda^2 + b_{m+1})\,a_m \\ \lambda^2 + b_m & \lambda\,a_m \end{pmatrix}, \qquad \mathsf{M}_m = \begin{pmatrix} \lambda & a_m \\ 1 & 0 \end{pmatrix}. \tag{56}$$

Then, the compatibility condition (53) is equivalent to the non-abelian Somos-4 equations:

$$y_{m+4} = y_{m+1} + y_{m+2}\left(y_m^{-1} - y_{m+3}^{-1}\right)y_{m+2}. \tag{57}$$

Example 6. Let $N = 5$, $a_m = y_{m+3}\,y_m^{-1}$ and $b_m = y_m\,y_{m+1}^{-1}$. The matrices [Bobrova et al., 2023]

$$\mathsf{L}_m = \begin{pmatrix} \lambda^2 & \lambda\,(b_{m+2} + a_{m+1}) & b_{m+2}\,a_m \\ b_{m+1} & \lambda^2 & \lambda\,a_m \\ \lambda & b_m & 0 \end{pmatrix}, \qquad \mathsf{M}_m = \begin{pmatrix} 0 & \lambda & a_m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{58}$$

leads to a non-abelian version of the Somos-5 equation:

$$y_{m+5} = y_{m+1} + y_{m+3}\left(y_m^{-1} - y_{m+4}^{-1}\right)y_{m+2}. \tag{59}$$

Continuous limits

Commutative case

- Set $z = \varepsilon\, m$ and $x_m = x(z)$.
- Then, $x_{m+k} = x(z + \varepsilon\, k)$ and one can consider the formal Taylor series near $\varepsilon = 0$.
- Under the limit $\varepsilon \to 0$ (if it exists), the discrete equation becomes a continuous one.

Example 7. Consider the so-called q-P_1 equation

$$u_{m+1}\, u_m^2\, u_{m-1} = \alpha\, q^m\, u_m + \beta. \qquad q\text{-}\mathrm{P}_1$$

After the change

$$u_m = 1 - \varepsilon^2\, y(z), \qquad z = \varepsilon\, m, \qquad \alpha = 4, \qquad \beta = -3, \qquad q = 1 - \tfrac{1}{4}\varepsilon^5, \tag{60}$$

we can take the limit $\varepsilon \to 0$ and, thus, it becomes the first Painlevé equation:

$$y'' = 6y^2 + z. \tag{61}$$

Non-commutative case

- Similar to the commutative case, we set $z = \varepsilon\, m$ and $x_m = x$.
- Instead of the Taylor series, we use the change $x_{m+k} = x + k\,\varepsilon x' + \frac{1}{2}k^2\,\varepsilon^2 x'' + O(\varepsilon^3)$.

Example 9. Consider a non-abelian analog for the q-P_1 [Bobrova et al., 2023]

$$u_{m+1} u_m - u_{m-1} u_{m-2} = \alpha_m\, u_{m-1}^{-1} - u_m^{-1}\, \alpha_{m-1}, \qquad \alpha_m = \alpha\, q^m \qquad q\text{-}\mathrm{P}_1[1]$$

and a straightforward generalisation of change (60). Taking the limit $\varepsilon \to 0$, it turns to

$$y''' = 6\, y\, y' + 6\, y'\, y + 1, \tag{62}$$

or, after the integration, to the P_1.

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or, after the integration, to the P_1.

Methods & Examples

Statement of the problem

The matrix P_1 equation

$$y'' = 6y^2 + z\,\mathbb{I} + a, \qquad y(z),\ a \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \qquad \mathrm{P}_1$$

How to detect non-abelian integrable analogs for the Painlevé equations?

Classification steps

(i) Construct a criterion allowing to select a finite list of non-abelian analogs such that under the commutative reduction the generalizations coincide with a given Painlevé equation.

(ii) For the obtained analogs find their zero-curvature representation.

Definition 1. A matrix or a non-abelian generalization of a Painlevé equation is integrable, if it satisfies a criterion from item (i) and admits the zero-curvature representation.

Some methods

- Matrix Painlevé-Kovalevskaya test $\Rightarrow$ ${}_{\text{mat}}\text{P}_1$, ${}_{\text{mat}}\text{P}_2$ [Balandin and Sokolov, 1998]:

$${}_{\text{mat}}\text{P}_2: \quad y'' = 2y^3 + zy + \alpha\,\mathbb{I},$$

$${}_{\text{mat}}\text{P}_1: \quad y'' = 6y^2 + z\,\mathbb{I} + a; \qquad y(z),\ a \in \text{Mat}_n(\mathbb{C}), \quad z,\ \alpha \in \mathbb{C}.$$

- Quantization of Poisson brackets $\Rightarrow$ ${}_{\text{q}}\text{P}_2$, ${}_{\text{q}}\text{P}_4$, ${}_{\text{q}}\text{P}_5$ [Nagoya et al., 2008]:

$${}_{\text{q}}\text{P}_4: \quad y'' = \tfrac{1}{2}y'\,y^{-1}\,y' + \tfrac{3}{2}y^3 - 2zy^2 + \left(\tfrac{1}{2}z^2 + 1 - 2\alpha_0 - \alpha_1\right)y - \tfrac{1}{2}\left(\alpha_1^2 - \hbar^2\right)y^{-1},$$

$$y \in \mathcal{A}_{\mathbb{C}}, \quad z,\ \alpha_i \in \mathbb{C}.$$

- An infinite ncToda system $\Rightarrow$ ${}_{\text{nc}}\text{P}_2$ [Retakh and Rubtsov, 2010]:

$${}_{\text{nc}}\text{P}_2: \quad y'' = 2y^3 + \tfrac{1}{2}z\,y + \tfrac{1}{2}y\,z + \alpha, \qquad y,\ z \in \mathcal{R}_{\mathbb{F}}, \quad \alpha \in \mathbb{F}.$$

- Matrix Schlesinger deformation $\Rightarrow$ ${}_{\text{mat}}\text{P}_6^H$ [Kawakami, 2015].
  Also ${}_{\text{mat}}\text{P}_5^H$, ${}_{\text{mat}}\text{P}_4^H$, ${}_{\text{mat}}\text{P}_3^H(D_6)$, ${}_{\text{mat}}\text{P}_3^H(D_7)$, ${}_{\text{mat}}\text{P}_3^H(D_8)$, ${}_{\text{mat}}\text{P}_2^H$, ${}_{\text{mat}}\text{P}_1^H$ systems.

Recent results

- Matrix P_2 type systems with matrix coefficients [Adler and Sokolov, 2021].
- Matrix P_4 type systems with matrix coefficients [Bobrova and Sokolov, 2022].
- A fully non-commutative P_4 system [Bobrova et al., 2022].
- Hamiltonian non-abelian Painlevé type systems [Bobrova and Sokolov, 2023a].
- Non-abelian Painlevé systems with Okamoto integral [Bobrova and Sokolov, 2023b].
- A symmetry approach to non-abelian Painlevé systems [Bobrova and Sokolov, 2023c].
- Reductions of a non-abelian Hirota equation [Bobrova et al., 2023].

Non-Abelian Okamoto integrals [Bobrova and Sokolov, 2023b]

Description of the method

- Construct non-abelian ansatz for the auxiliary system and the Okamoto integral J.
- Require that the generalized Okamoto integral $J \in \mathcal{A}$ should be a first integral of the system. This leads to the restrictions on the unknown coefficients.
- For a given (finite) list of non-abelian systems reconstruct non-abelian Painlevé systems:
  (a) replace t by z,
  (b) reconstruct $f(z)$ in the system.

Example 1.

- The commutative Hamiltonian P_2 system:

$$H = -u^2 v + \tfrac{1}{2} v^2 - \kappa u - \tfrac{1}{2} z v, \qquad \{u, v\} = 1, \quad \{u, u\} = \{v, v\} = 0; \qquad \left\{ \begin{array}{lcl} u' & = & -u^2 + v - \tfrac{1}{2} z, \\ v' & = & 2uv + \kappa, \end{array} \right. \quad u(z),\ v(z), \quad z,\ \kappa \in \mathbb{C}.$$

- Non-abelian Okamoto integral:

$$J(u, v) = a_1 u^2 v + a_2 uvu + (-1 - a_1 - a_2) vu^2 + \tfrac{1}{2} v^2 - \kappa u - \tfrac{1}{2} t\, v. \tag{63}$$

- Non-abelian autonomous system:

$$\left\{ \begin{array}{lcl} u' & = & -u^2 + v - \tfrac{1}{2} t, \\ v' & = & 2vu + \beta [v, u] + \kappa, \end{array} \right. \qquad \beta \in \mathbb{C}. \tag{64}$$

- $d_z\,(J(u, v)) = 0 \quad \Longleftrightarrow \quad \beta = 0,\ a_1 = 0,\ a_2 = -1 \quad \text{or} \quad \beta = -2,\ a_1 = -1,\ a_2 = 0.$

2d dToda → Somos-N → q-Painlevé

Commutative case

discrete Toda equations $\xrightarrow{\text{[Hone et al., 2017]}}$ Somos-N equations $\xrightarrow{\text{[Hone and Inoue, 2014]}}$ discrete Painlevé equations.

Non-commutative case [Bobrova et al., 2023]

- Consider the non-abelian 2ddTL:

$$\theta_{l+1,m+1,n} = \theta_{l,m,n+1} + \theta_{l+1,m,n} \left(\theta^{-1}_{l,m,n} - \theta^{-1}_{l+1,m+1,n-1} \right) \theta_{l,m+1,n}. \qquad \text{2ddTL}$$

- One may introduce a non-autonomous constant into the 2ddTL by a scaling:

$$\theta_{l+1,m+1,n} = \alpha_{l,m,n} \theta_{l,m,n+1} + \theta_{l,m+1,n} \left(\theta^{-1}_{l,m,n} - \theta^{-1}_{l+1,m+1,n-1} \alpha_{l,m,n-1} \right) \theta_{l+1,m,n}, \quad \alpha_{l,m,n} = \alpha\, q^{k_1 l + k_2 m + k_3 n}, \tag{65}$$

  where α is a non-abelian constant parameter and k_i, q are commutative ones.

- By a plane-wave reduction, (65) reduces to a non-abelian Somos-N like equation:

$$y_{M+N} = \alpha_M\, y_{M+r} + y_{M+s} \left(y_M^{-1} - y^{-1}_{M+N-r}\, \alpha_{M-r} \right) y_{M+N-s}, \quad \alpha_M = \alpha\, q^M, \quad N \in \mathbb{N}_{>3}, \quad 1 \leq r < s \leq \left[\tfrac{N}{2} \right]. \tag{66}$$

- Let $r = 1$ and $s = 2$. Then, for even $N \geq 4$ and odd $N \geq 5$ consider the changes

$$u_M = y_{M+3}\, y^{-1}_{M+2}, \qquad u_M = y_{M+4}\, y^{-1}_{M+2}. \tag{67}$$

  They lead to q-$\mathrm{P}_1[\mathrm{n}]$ and q-$\mathrm{P}_2[\mathrm{n}]$ hierarchies, respectively.

2d dToda $\to$ Somos-N $\to$ q-Painlevé

Commutative case

$$\text{discrete Toda equations} \quad \xrightarrow{[\text{Hone et al., 2017}]} \quad \text{Somos-}N\text{ equations}$$

$$\xrightarrow{[\text{Hone and Inoue, 2014}]} \quad \text{discrete Painlevé equations.}$$

Non-commutative case [Bobrova et al., 2023]

- Consider the non-abelian 2ddTL:

$$\theta_{l+1,m+1,n} = \theta_{l,m,n+1} + \theta_{l+1,m,n} \left(\theta_{l,m,n}^{-1} - \theta_{l+1,m+1,n-1}^{-1} \right) \theta_{l,m+1,n}. \qquad \text{2ddTL}$$

- One may introduce a non-autonomous constant into the 2ddTL by a scaling:

$$\theta_{l+1,m+1,n} = \alpha_{l,m,n} \theta_{l,m,n+1} + \theta_{l,m+1,n} \left(\theta_{l,m,n}^{-1} - \theta_{l+1,m+1,n-1}^{-1} \alpha_{l,m,n-1} \right) \theta_{l+1,m,n}, \qquad \alpha_{l,m,n} = \alpha\, q^{k_1 l + k_2 m + k_3 n}, \tag{65}$$

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- Let $r = 1$ and $s = 2$. Then, for even $N \ge 4$ and odd $N \ge 5$ consider the changes

$$u_M = y_{M+3}\, y_{M+2}^{-1}, \qquad u_M = y_{M+4}\, y_{M+2}^{-1}. \tag{67}$$

  They lead to q-P_1[n] and q-P_2[n] hierarchies, respectively.

A non-abelian q-P_1

- Recall the non-abelian Somos-4 equation:

$$y_{M+4} = \alpha_M \, y_{M+1} + y_{M+2} \left(y_M^{-1} - y_{M+3}^{-1} \, \alpha_{M-1} \right) y_{M+2}. \tag{68}$$

- It can be rewritten as

$$y_{M+4} y_{M+2}^{-1} - y_{M+2} y_M^{-1} = \alpha_M \, y_{M+1} y_{M+2}^{-1} - y_{M+2} y_{M+3}^{-1} \, \alpha_{M-1}. \tag{69}$$

- Consider the change $u_M = y_{M+3} \, y_{M+2}^{-1}$. Then, the latter becomes

$$u_{M+1} u_M - u_{M-1} u_{M-2} = \alpha_M \, u_{M-1}^{-1} - u_M^{-1} \, \alpha_{M-1}. \qquad q\text{-P}_1[1]$$

- The second member of the hierarchy:

$$u_{M+3} \, u_{M+2} - u_{M-1} \, u_{M-2} = \alpha_M \; u_{M-1}^{-1} \, u_M^{-1} \, u_{M+1}^{-1} - u_M^{-1} \, u_{M+1}^{-1} \, u_{M+2}^{-1} \, \alpha_{M-1}. \qquad q\text{-P}_1[2]$$

Remark 1. In the abelian case, the q-$\mathrm{P}_1[1]$ can be derived as follows. Let us take two q-P_1:

$$u_{M+1} \, u_M^2 \, u_{M-1} = \beta + \alpha_M \, u_M, \qquad u_M \, u_{M-1}^2 \, u_{M-2} = \beta + \alpha_{M-1} u_{M-1}, \tag{70}$$

or, equivalently,

$$u_{M+1} \, u_M = \beta \, u_M^{-1} \, u_{M-1}^{-1} + \alpha_M \, u_{M-1}^{-1}, \qquad u_{M-1} \, u_{M-2} = \beta \, u_M^{-1} \, u_{M-1}^{-1} + \alpha_{M-1} \, u_M^{-1}. \tag{71}$$

Then their difference leads to q-$\mathrm{P}_1[1]$.

A non-abelian q-P_1

- Recall the non-abelian Somos-4 equation:

$$y_{M+4} = \alpha_M\, y_{M+1} + y_{M+2}\left(y_M^{-1} - y_{M+3}^{-1}\,\alpha_{M-1}\right) y_{M+2}. \tag{68}$$

- It can be rewritten as

$$y_{M+4}y_{M+2}^{-1} - y_{M+2}y_M^{-1} = \alpha_M\, y_{M+1}y_{M+2}^{-1} - y_{M+2}y_{M+3}^{-1}\,\alpha_{M-1}. \tag{69}$$

- Consider the change $u_M = y_{M+3}\, y_{M+2}^{-1}$. Then, the latter becomes

$$u_{M+1}u_M - u_{M-1}u_{M-2} = \alpha_M\, u_{M-1}^{-1} - u_M^{-1}\,\alpha_{M-1}. \qquad q\text{-}\mathrm{P}_1[1]$$

- The second member of the hierarchy:

$$u_{M+3}\, u_{M+2} - u_{M-1}\, u_{M-2} = \alpha_M\; u_{M-1}^{-1}\, u_M^{-1}\, u_{M+1}^{-1} - u_M^{-1}\, u_{M+1}^{-1}\, u_{M+2}^{-1}\;\alpha_{M-1}. \qquad q\text{-}\mathrm{P}_1[2]$$

Remark 1. In the abelian case, the q-$\mathrm{P}_1[1]$ can be derived as follows. Let us take two q-P_1:

$$u_{M+1}\, u_M^2\, u_{M-1} = \beta + \alpha_M\, u_M, \qquad u_M\, u_{M-1}^2\, u_{M-2} = \beta + \alpha_{M-1}u_{M-1}, \tag{70}$$

or, equivalently,

$$u_{M+1}\, u_M = \beta\, u_M^{-1}\, u_{M-1}^{-1} + \alpha_M\, u_{M-1}^{-1}, \qquad u_{M-1}\, u_{M-2} = \beta\, u_M^{-1}\, u_{M-1}^{-1} + \alpha_{M-1}\, u_M^{-1}. \tag{71}$$

Then their difference leads to q-$\mathrm{P}_1[1]$.

Many thanks!

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