Classical and Quantum Integrability in Self-Dual Gravity

R. Bittleston

Perimter Institute

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Overview

Recently there's been significant progress in the understanding of integrability in four space-time dimensions. **[Costello, 21; Costello, Paquette, 22; Costello, Paquette, Sharma, 22; . . .]**

This is closely related to developments in the celestial holography program. **[Guevara et al., 21; Strominger, 21; Ball et al., 22; . . .]**

Please see overview talk for more details.

This talk concerns the application of similar twistor methods in the context of self-dual gravity. Based on arXiv:2208.12701 **[RB, Sharma, Skinner, 22]**, arXiv:2211.06417 **[RB, 22]** and arXiv:2305.09451 **[RB, Heuveline, Skinner, 23]**.

Self-dual Gravity

Why self-dual gravity?

- General relativity in four dimensions both interesting and complicated.
- Self-dual gravity is less interesting but also much simpler: it's classically integrable, 1-loop exact and finite.
- It retains some important features of full general relativity: it's 4-dimensional and has two propagating degrees of freedom.
- It can be deformed to full general relativity, and the simplicity of the self-dual sector can be leveraged to understand this deformation.

Let g be a Riemannian metric on the 4d manifold $\mathcal{M}$. The metric is self-dual Ricci flat if

$$C = *C \,, \qquad \mathrm{Ric} = 0 \,,$$

where C denotes the Weyl tensor and Ric the Ricci tensor. Equivalent to $\mathcal{M}$ being hyperkähler.

Introducing vierbeins

$$g = e^{\dot{\alpha}\alpha} \odot e_{\dot{\alpha}\alpha} \,,$$

the self-dual Ricci flat condition can be written as

$$\frac{1}{2}\mathrm{d}\big(e^{\dot{\alpha}\alpha} \wedge e_{\dot{\alpha}}{}^{\beta}\big) = 0 \,.$$

These equations are slightly easier to understand from the perspective of the action: the self-dual Palatini action is

$$S_{\mathrm{gr}}[\Gamma, e] = \frac{1}{2}\int_{\mathcal{M}} e^{\dot{\alpha}\alpha} \wedge e_{\dot{\alpha}}{}^{\beta} \wedge \left(\mathrm{d}\Gamma_{\alpha\beta} + \kappa^2 \Gamma_{\alpha}{}^{\gamma} \wedge \Gamma_{\gamma\beta}\right).$$

It differs from the tetradic Palatini action by a topological Nieh-Yan term.

In the weak coupling limit $\kappa^2 \to 0$ we obtain an action for the self-dual vacuum Einstein equations

$$S_{\mathrm{sdgr}}[\Gamma, e] = \frac{1}{2}\int_{\mathcal{M}} e^{\dot{\alpha}\alpha} \wedge e_{\dot{\alpha}}{}^{\beta} \wedge \mathrm{d}\Gamma_{\alpha\beta}\,.$$

Deforming around flat space $e^{\dot{\alpha}\alpha} = \mathrm{d}x^{\dot{\alpha}\alpha} + \delta e^{\dot{\alpha}\alpha}$, we can use this to define self-dual gravity (sdgr) as a perturbative quantum field theory.

Twistor formulation

Penrose's non-linear graviton provides an identification between:

- 4-dimensional Riemannian manifolds $\mathcal{M}$ with self-dual Ricci flat metric.
- 3-dimensional complex manifolds $\mathcal{PT}$ admitting a holomorphic fibration over $\mathbb{CP}^1$ and an $\mathcal{O}(2)$-valued symplectic form on the fibres (+ extra qualifiers).

$\mathcal{PT}$ is the twistor space of $\mathcal{M}$. **[Penrose, 76]**

Points $x \in \mathcal{M}_{\mathbb{C}}$ correspond to rational curves $\mathcal{L}_x \subset \mathcal{PT}$. Rational curves corresponding to points in the Riemannian slice are fixed by a certain antiholomorphic involution σ. This provides a non-holomorphic fibration $\mathcal{PT} \to \mathcal{M}$ with fibre over x given by $\mathcal{L}_x$.

Example

The twistor space of $\mathbb{R}^4$, denoted $\mathbb{PT}$, is the total space of the vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1 \,.$$

Using coordinates $v^{\dot\alpha}$ for $\dot\alpha = \dot 0, \dot 1$ on fibres and z on base the symplectic form is $\mathrm{d}v^{\dot 0} \wedge \mathrm{d}v^{\dot 1}$.

Can view $\mathcal{PT}$ as a deformation of $\mathbb{PT}$. An almost complex structure deformation

$$\bar\partial \mapsto \bar\nabla = \bar\partial + \mathcal{L}_V$$

is encoded in a Beltrami differential $V \in \Omega^{0,1}(T^{1,0}_{\mathbb{PT}})$. It is integrable when

$$\bar\nabla^2 = \bar\partial V + \frac{1}{2}[V, V] = 0 \,.$$

Almost complex structure deformations of this type preserve the 2-form $\mathrm{d}v^{\dot{0}} \wedge \mathrm{d}v^{\dot{1}}$ if V is Hamiltonian in the sense

$$V = \{h,\ \} = \epsilon^{\dot{\alpha}\dot{\beta}} \partial_{v^{\dot{\beta}}} h \, \partial_{v^{\dot{\alpha}}}$$

for $h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$. Integrability is implied by

$$T(h) = \bar{\partial} h + \frac{1}{2}\{h, h\} = 0\,.$$

A natural twistor action is then Poisson-BF theory **[Mason, Wolf, 09]**

$$S_{\mathrm{PBF}}[g, h] = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} g \wedge T(h)$$

for $g \in \Omega^{3,1}(\mathbb{PT}, \mathcal{O}(-2))$. Classically equivalent to our perturbative action for sdgr on spacetime. **[Sharma, 21; RB et al., 22]**

Hidden Symmetries

The equations of motion of sdgr are integrable.

Self-dual Ricci flat metrics are acted upon by an infinite dimensional hidden symmetry group. **[Penrose, 76; Park, 90; Dunajski, Mason, 00]**

Recently this has been identified as the chiral algebra of positive-helicity asymptotic symmetries arising from soft theorems in gr. **[Guevara et al., 21; Strominger, 21]**

It naturally arises as the universal holomorphic surface defect on twistor space.

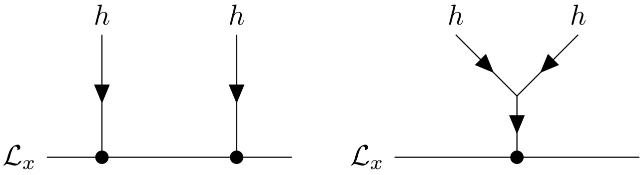
Consider the most general coupling between Poisson BF theory, and a surface defect supported on the twistor line $\mathcal{L}_x$

$$\sum_{m,n\in\mathbb{Z}_{\geq 0}} \frac{1}{m!n!}\int_{\mathcal{L}_x}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\left(w[m,n](z)\partial_{v^{\dot{0}}}^m\partial_{v^{\dot{1}}}^n h+\tilde{w}[m,n](z)\partial_{v^{\dot{0}}}^m\partial_{v^{\dot{1}}}^n g\right).$$

Only holomorphic derivatives are allowed, as the antiholomorphic derivatives are exact in BRST cohomology.

BRST invariance of this defect imposes conditions on the operator products of the $w[m,n](z)$, $\tilde{w}[m,n](z)$.

At tree level the BRST variation of the defect receives contributions from these diagrams.



For these to cancel the generators must have operator products

$$w[p,q](z)w[r,s](0) \sim \frac{ps-qr}{z} w[p+r-1, q+s-1](0)\,.$$

We denote this hidden symmetry algebra, or celestial chiral algebra (CCA), by $\mathcal{W}_{\mathrm{sdgr}} = \mathcal{L}\mathfrak{ham}(\mathbb{C}^2)$.

Here $\mathfrak{ham}(\mathbb{C}^2)$ is simply the (complexified) phase space of a classical particle in 1-dimension. Generated by polynomials in X, P with Poisson bracket

$$\{X, P\} = 1\,.$$

$\mathcal{L}\mathfrak{ham}(\mathbb{C}^2)$ then the Lie algebra of loops into $\mathfrak{ham}(\mathbb{C}^2)$.

The $\tilde{w}$ states have trivial operator products among themselves, and transform in the adjoint of $\mathcal{W}_{\mathrm{sdgr}}$

$$w[p,q](z)\tilde{w}[r,s](0) \sim \frac{ps-qr}{z}\tilde{w}[p+r-1,q+s-1](0)\,.$$

Celestial Interpretation

Interpret z as an inhomogeneous coordinate on $\mathbb{CP}^1$ with $\lambda = (1, z)$. Then let

$$w(\tilde{\lambda}, \lambda) = \sum_{m,n \in \mathbb{Z}_{\geq 0}} \frac{(\tilde{\lambda}_{\dot{0}})^m (\tilde{\lambda}_{\dot{1}})^n}{m!n!} w[m,n](z)\,,$$

Interpreting $\lambda, \tilde{\lambda}$ as spinor helicity variables, we can form the null momentum $p_i^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha$

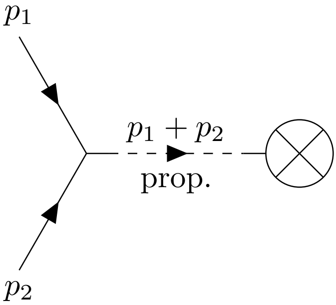
$$w(\tilde{\lambda}_1, \lambda_1) w(\tilde{\lambda}_2, \lambda_2) \sim -\frac{[12]}{\langle 12 \rangle} \tau^{-2} w(\tau \tilde{\lambda}_1 + \tilde{\lambda}_2, \lambda_2)\,.$$

Here $\langle 12 \rangle = z_1 - z_2$, $[12] = \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_1^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}}$. τ absorbs weights, and vanishes in inhomogeneous coordinates.

We recognise the tree graviton splitting amplitude

$$\mathrm{Split}_{-}^{\mathrm{tree}}(1^{+},2^{+}) = -\frac{[12]}{\langle 12\rangle}$$

encoding the singularity in an amplitude as the external momenta of two positive helicity gravitons become collinear.



The CCA of sdYM describes the universal collinear singularities of amplitudes in the presence of local operators, i.e., form factors. What is the CCA of sdgr describing?

There are no local operators in gravitational theories. Next best thing is a first order deformation:

$$S_{\mathrm{sdgr}} \mapsto S_{\mathrm{sdgr}} + \epsilon \int_{\mathbb{R}^4} \mathcal{O} \,.$$

BRST variation of $\mathcal{O}$ must be de Rham exact $\delta\mathcal{O} = \mathrm{d}\mathcal{O}'$.

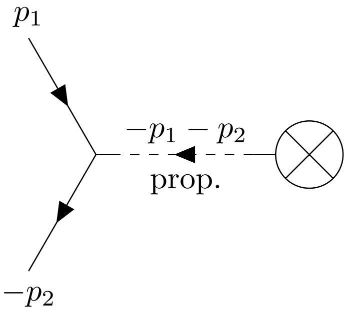
A particularly interesting example is the first order deformation to full gr

$$\mathcal{O} = \frac{1}{2} e^{\dot{\alpha}\alpha} \wedge e_{\dot{\alpha}}{}^{\beta} \wedge \Gamma_{\alpha\gamma} \wedge \Gamma^{\gamma}{}_{\beta} \,.$$

Can consider amplitudes in the presence of such a deformation.

The only non-vanishing tree amplitudes in sdgr are the 3-point vertices.

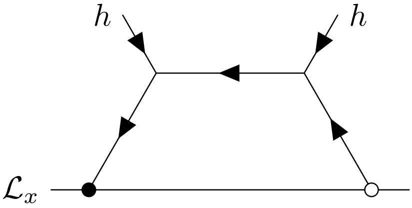
Hence, the collinear singularities of tree amplitudes in the presence of a first order deformation are universal, i.e., they do not depend on $\mathcal{O}$.



Crossed dot represents a tree amplitude in the deformation by $\mathcal{O}$.

Quantum corrections

The BRST variation of the defect is modified at 1-loop, forcing corrections to the operator products of the CCA.



The above diagram necessitates a correction

$$w[4,0](z)w[0,4](0) \sim \frac{2}{5\pi^2}\frac{1}{z^2}\tilde{w}[0,0](0) + \frac{7}{5\pi^2}\frac{1}{z} :\{w[1,0], w[0,1]\}:(0)\,.$$

There are also further simple poles.

What is the space-time interpretation?

There are no 1-loop splitting amplitudes in gravity, at least in the true collinear limit **[Bern et al., 98]**

$$p_2 \xrightarrow{\parallel} p_1 \,.$$

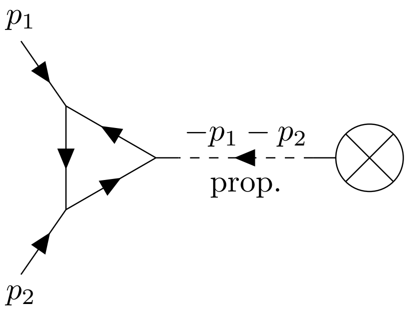
However the chiral algebra encodes the singularities in holomorphic collinear limits

$$\lambda_2 \xrightarrow{\parallel} \lambda_1$$

at fixed $\tilde{\lambda}_1, \tilde{\lambda}_2$. These get new contributions in generic loop amplitudes. **[Brandhuber et al., 07; Dunbar et al., 10]**

The 1-loop amplitudes present in sdgr do not acquire new collinear singularities; however those in generic first order deformations do see these new contributions.

The leading singularity at 1-loop is generated by the following diagram:



Loop integral is described by the effective graviton vertex

$$\mathcal{M}_3^{1-\mathrm{loop}}(1^+, 2^+, 3^+) = -\frac{\mathrm{i}}{180(4\pi)^2} \frac{[12]^2[23]^2[31]^2}{(p_1 + p_2)^2} \,.$$

Including propagator gives a 1-loop graviton 'holomorphic splitting amplitude'

$$\mathrm{Split}_+^{1-\mathrm{loop}}(1^+,2^+) = \frac{\mathrm{i}\mathcal{M}_3^{1-\mathrm{loop}}(1^+,2^+,P_{12}^+)}{P_{12}^2} = \frac{1}{180(4\pi)^2}\frac{[12]^4}{\langle 12\rangle^2}\,,$$

where $P_{12} = p_1 + p_2$. This vanishes in the true limit $p_2 \xrightarrow{\parallel} p_1$.

Introduces a new term in the operator product

$$\begin{aligned} w(\tilde{\lambda}_1,\lambda_1)w(\tilde{\lambda}_2,\lambda_2) &\sim \mathrm{Split}_+^{1-\mathrm{loop}}(1^+,2^+)\tau^2\tilde{w}(\tau\tilde{\lambda}_1+\tilde{\lambda}_2,\lambda_2) \\ &\sim \frac{1}{180(4\pi)^2}\frac{[12]^4}{\langle 12\rangle^2}\tau^2\tilde{w}(\tau\tilde{\lambda}_1+\tilde{\lambda}_2,\lambda_2)\,. \end{aligned}$$

There are also subleading simple poles including some which are non-linear in generators.

Decomposing into soft modes gives

$$
\begin{aligned}
&w[p,q](z)w[r,s](0) \\
&\sim \frac{2}{5\pi^2}\frac{R_4(p,q,r,s)}{z^2}\tilde{w}[p+r-4,q+s-4](0)+\dots\,,
\end{aligned}
$$

where

$$
R_\ell(p,q,r,s)=\frac{1}{\ell!^2}\sum_{k=0}^{\ell}(-)^k\binom{\ell}{k}[p]_{\ell-k}[q]_k[r]_k[s]_{\ell-k}\,.
$$

Remark

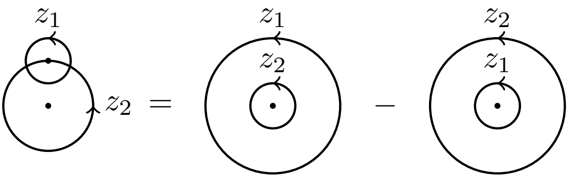
$R_\ell(p,q,r,s)$ intertwines $\mathfrak{sl}_2(\mathbb{C})$ representations

$$
(\mathbf{p}+\mathbf{q}+\mathbf{1})\otimes(\mathbf{r}+\mathbf{s}+\mathbf{1})\to(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s}+\mathbf{1}-\mathbf{2}\ell)\,.
$$

This does not define a consistent chiral algebra: associativity of the operator product necessitates

$$\oint_{|z_2|=2} \mathrm{d}z_2 \left(\oint_{|z_{12}|=1} \mathrm{d}z_{12}\, \mathcal{O}_1(z_1)\mathcal{O}_2(z_2) \right) \mathcal{O}_3(0)$$
$$= \oint_{|z_1|=2} \mathrm{d}z_1\, \mathcal{O}_1(z_1) \left(\oint_{|z_2|=1} \mathrm{d}z_2\, \mathcal{O}_2(z_2)\mathcal{O}_3(0) \right)$$
$$- \oint_{|z_2|=2} \mathrm{d}z_2\, \mathcal{O}_2(z_2) \left(\oint_{|z_1|=1} \mathrm{d}z_1\, \mathcal{O}_1(z_1)\mathcal{O}_3(0) \right)$$

Here we're using the following equivalence of contours.



Choosing

$$\mathcal{O}_1(z) = w[3,0](z)\,, \quad \mathcal{O}_2(z) = zw[0,3](z)\,, \quad \mathcal{O}_3(z) = w[2,2](z)$$

gives

$$\frac{3}{5\pi^2}\tilde{w}[0,0](0) = 0\,.$$

Remark

Tempting to throw away $\tilde{w}[0,0]$, but repeating calculation with

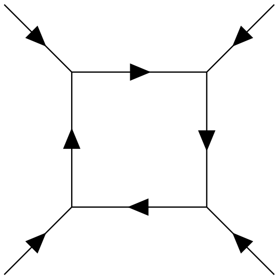
$$\mathcal{O}_1(z) = w[4,0](z)\,, \quad \mathcal{O}_2(z) = zw[0,3](z)\,, \quad \mathcal{O}_1(z) = w[2,2](z)$$

gives

$$\frac{12}{5\pi^2}\tilde{w}[1,0] = 0\,.$$

Anomalies

What is going wrong? The twistor uplift of sdgr suffers from an anomaly which can be attributed to the failure of the following diagram to be BRST invariant.

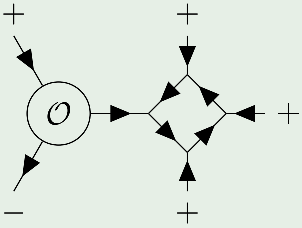


Can be identified with the 4-point 1-loop all-plus amplitude on spacetime. It represents a global anomaly in integrability.

This anomaly is signalling that the collinear singularities of amplitudes in first order deformations are not universal. The 1-loop all-plus amplitudes are the source of this non-universality.

Example

In the first order deformation to full Einstein gravity the 1-loop 5-point mostly-plus amplitudes acquires a non-universal collinear singularity.



To obtain a consistent chiral algebra we must cancel the twistorial anomaly, or equivalently eliminate the non-vanishing 1-loop all-plus amplitudes in sdgr.

There are multiple ways of doing this:

- Couple to scalars, fermions, gauge bosons and gravitinos so that a count of the degrees of freedom weighted by Grassmann parity gives zero. This occurs in self-dual supergravity and chiral higher-spin gravity.
- Couple to an exotic 4^{th}-order gravitational axion on spacetime, cancelling the twistorial anomaly by a Green-Schwarz mechanism.

Let's concentrate on the latter.

On twistor space couple to a field $\eta \in \Omega^{2,1}(\mathbb{PT})$ obeying $\partial\eta = 0$. Action is

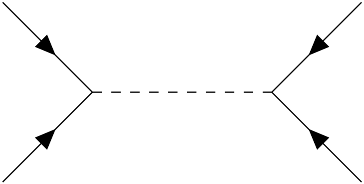
$$S_\eta[\eta; h] = \frac{1}{4\pi\mathrm{i}} \int_{\mathbb{PT}} \left(\partial^{-1}\eta\, \bar{\nabla}\eta + \mu\, \eta\, \partial^{\dot\beta}\partial_{\dot\alpha} h\, \partial^{\dot\alpha}\partial_{\dot\beta}\partial h \right).$$

η descends to a scalar field ρ on spacetime, and the action becomes

$$S_\rho[\rho; g] = \int_{\mathbb{R}^4} \left(\mathrm{vol}_g\, \frac{1}{2}(\triangle_g \rho)^2 + \frac{\mu}{\sqrt{2}}\, \rho\, R^\mu{}_\nu \wedge R^\nu{}_\mu \right).$$

Here $R^\mu{}_\nu \wedge R^\nu{}_\mu$ is the Pontryagin class, revealing ρ to be a 4$^{\text{th}}$-order gravitational axion.

The twistorial anomaly, or equivalently the 1-loop all-plus amplitudes, are cancelled by tree level axion exchange



if the coupling constant μ is tuned so that

$$\mu^2 = \frac{1}{5!}\left(\frac{\mathrm{i}}{2\pi}\right)^2 .$$

This relies on the following trace identity for the fundamental of $\mathfrak{sl}_2(\mathbb{C})$

$$\mathrm{tr}(X^4) = \frac{1}{2}\mathrm{tr}(X^2)^2 .$$

Expect that sdgr coupled to this 4^{th}-order gravitational axion admits a quantum chiral algebra governing the holomorphic collinear limits of amplitudes in its first order deformations.

If so, the previously identified associativity failure should no longer be present. Are careful calculation shows that this is indeed the case precisely if the double pole has the normalization matching the 1-loop holomorphic splitting amplitude.

In conclusion, we get a kind of quantum group for sdgr + 4^{th}-order gravitational axion which plays a role analogous to the Yangian for the principal chiral model.

Self-Dual Gravity on Eguchi-Hanson

Now let's change direction slightly and revisit the classical hidden symmetry, but now on a curved background.

Consider the twistor uplift of classical self-dual gravity in the presence of the simplest holomorphic surface defect supported on a twistor line over the origin

$$S_{\text{defect}}[g] \propto \frac{c^2}{2\pi\mathrm{i}} \int_{\mathcal{L}_0} \mathrm{d}z\, z^2\, g\,.$$

Here we're interpreting $g \in \Omega^{3,1}(\mathbb{PT}, \mathcal{O}(-2)) \cong \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-6))$. The factor of z^2 should be interpreted as a section of $\mathcal{O}(4)$.

This defect sources the PBF equation of motion

$$T(h) = \bar{\partial} h + \frac{1}{2}\{h, h\} \propto c^2 z^2 \bar{\delta}^2(v)\,.$$

This is solved by

$$h \propto c^2 z^2 \frac{\bar{v}^{\dot{0}} \mathrm{d}\bar{v}^{\dot{1}} - \bar{v}^{\dot{1}} \mathrm{d}\bar{v}^{\dot{0}}}{(|v^{\dot{0}}|^2 + |v^{\dot{1}}|^2)^2}\,,$$

which deforms the complex structure on twistor space to

$$\bar{\nabla} = \bar{\partial} + \{h,\ \} = \bar{\partial} - c^2 z^2 \frac{\bar{v}^{\dot{0}} \mathrm{d}\bar{v}^{\dot{1}} - \bar{v}^{\dot{1}} \mathrm{d}\bar{v}^{\dot{0}}}{(|v^{\dot{0}}|^2 + |v^{\dot{1}}|^2)^3} (\bar{v}^{\dot{0}} \partial_{v^{\dot{1}}} - \bar{v}^{\dot{1}} \partial_{v^{\dot{0}}})\,.$$

We find that, along with z, the coordinates

$$X^{\dot\alpha\dot\beta} = v^{\dot\alpha}v^{\dot\beta} - \frac{c^2z^2}{(|v^{\dot 0}|^2 + |v^{\dot 1}|^2)^2}\epsilon^{\dot\alpha}{}_{\dot\gamma}\epsilon^{\dot\beta}{}_{\dot\delta}\bar{v}^{\dot\gamma}\bar{v}^{\dot\delta}$$

are holomorphic in this deformed complex structure. These are redundant, obeying the constraint

$$X^{\dot\alpha\dot\beta}X_{\dot\alpha\dot\beta} + 2c^2z^2 = 0\,.$$

This is Hitchin's description of the twistor space of the Eguchi-Hanson gravitational instanton. **[Hitchin, 79]**

In Euclidean signature the Eguchi-Hanson metric has a singularity at the origin. This can be eliminated if we quotient by the $\mathbb{Z}_2$ action

$$v^{\dot\alpha} \mapsto -v^{\dot\alpha}\,.$$

To understand scattering processes it is more convenient to work in split signature. To recover the space-time metric well follow Penrose's instructions. (Construct the curved twistor lines, pull back the symplectic structure and then extract the vierbeins.)

The resulting space-time metric is

$$\mathrm{d}s^2 = 2[\mathrm{d}u \odot \mathrm{d}\tilde{u}] + \frac{2c^2[\tilde{u}\,\mathrm{d}u]^{\odot 2}}{[u\,\tilde{u}]^3}\,,$$

where $u^{\dot{\alpha}}, \tilde{u}^{\dot{\alpha}}$ are real coordinates. This is the Kerr-Schild form of the Eguchi-Hanson metric. **[Sparling, Tod, 81]**

We can write down twistor representatives for scattering states with null momentum p_i and tending to $\cos(x \cdot p_i)$ at infinity

$$\delta h_i \propto \cos\sqrt{X^{\dot\alpha\dot\beta}X_{\dot\alpha\dot\beta}}\,\bar\delta(z - z_i)\,.$$

Penrose transforming gives the corresponding states on space-time

$$\delta\Theta_i \propto \cos\sqrt{(x \cdot p_i)^2 - c^2 z_i^2\frac{[\tilde u\, i]^2}{[u\,\tilde u]^2}}\,.$$

The $\mathbb{Z}_2$ quotient restricts the indices on the soft modes, denoted by $W[m, n](z)$, so that $m + n$ is even.

In the absence of the defects ($c = 0$), these generate the fixed point subalgebra of $\mathcal{W}_{\mathrm{sdgr}}$ under the $\mathbb{Z}_2$ action, denoted

$$\mathfrak{L}\mathfrak{ham}(\mathbb{C}^2)^{\mathbb{Z}_2}\,.$$

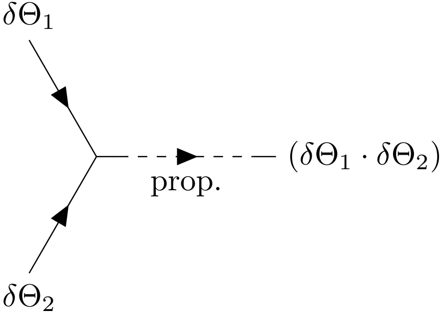
The celestial OPE is determined by requiring BRST invariance of

$$\epsilon_1 \delta h_1 + \epsilon_2 \delta h_2 + \epsilon_1 \epsilon_2 (\delta h_1 \cdot \delta h_2)$$

at order $\epsilon_1 \epsilon_2$ on twistor space, or by requiring that

$$\epsilon_1 \delta \Theta_1 + \epsilon_2 \delta \Theta_2 + \epsilon_1 \epsilon_2 (\delta \Theta_1 \cdot \delta \Theta_2)$$

obeys sdgr equations of motion at order $\epsilon_1 \epsilon_2$ on space-time.



Using the twistor approach, we find that the celestial operator product on the Eguchi-Hanson background takes the form

$$W[p,q](z_1)W[r,s](z_2) \sim -\frac{1}{2z_{12}}\sum_{\ell\geq 0}(2cz_2)^{2\ell}R_{2\ell+1}(p,q,r,s)$$
$$\psi_{2\ell+1}\left(\frac{p+q}{2},\frac{r+s}{2}\right)W[p+r-2\ell-1,q+s-2\ell-1](z_2)\,.$$

where

$$\psi_{2\ell+1}(m,n) = (-)^\ell\frac{[\ell+1/2]_\ell}{4^{2\ell}[m-1/2]_\ell[n-1/2]_\ell[m+n-1/2\ell]}\,.$$

This can be obtained as a scaling limit of a family of algebras which we shall denote by $\mathcal{L}W(\mu)$ when sending $\mu\to\infty$. Explicitly

$$W(\mu) = U(\mathfrak{sl}_2(\mathbb{C}))/\langle C-\mu\rangle\,.$$

Can also perform the analogous computation on space-time, which is much more involved. At leading order in c^2 we find

$$(\delta\Theta_1 \cdot \delta\Theta_2)(x) \sim \frac{[12]^3}{\langle 12\rangle} \int_{[0,1]^2} \mathrm{d}s\, \mathrm{d}t\ \min(s,t)(1-\max(s,t))$$
$$\left(\delta\Theta(x;\lambda_2, s\tilde{\lambda}_1 + t\tilde{\lambda}_2) - \delta\Theta(x;\lambda_2, s\tilde{\lambda}_1 - t\tilde{\lambda}_2)\right).$$

Expanding in soft modes this leads to precisely the structure constants obtained using twistor methods.

Non-Commutativity

Self-dual gravity admits a natural non-commutative deformation.

On $\mathbb{PT}$ this is obtained by replacing the Poisson bracket with the corresponding Moyal bracket

$$\{f,g\} = \partial^{\dot\alpha} f \vee \partial_{\dot\alpha} g \mapsto [f,g]_\star = \frac{2}{\mathfrak{q}z} m \circ \sinh\left(\frac{\mathfrak{q}z}{2}\,\partial^{\dot\alpha}\vee\partial_{\dot\alpha}\right)(f\otimes g)$$

where $m : f\otimes g \mapsto fg$.

At the level of the CCA this deforms $\mathcal{L}\mathfrak{ham}(\mathbb{C}^2) \to \mathcal{L}\mathfrak{diff}(\mathbb{C})_{\mathfrak{q}}$. **[Bu et al., 22]** $\mathfrak{diff}(\mathbb{C})_{\mathfrak{q}}$ is the Weyl algebra generated by

$$X\star P - P\star X = \mathfrak{q}\,.$$

We can repeat the backreaction computation in the non-commutative setting. The complex structure is now deformed in the non-commutative sense to

$$\bar{\nabla} = \bar{\partial} + [h,\ \ ]_\star\,.$$

The coordinates $X^{\dot\alpha\dot\beta} = X^{\dot\beta\dot\alpha}$ are still holomorphic for this $\bar{\nabla}$ operator, and obey

$$[X^{\dot\alpha\dot\beta}, X^{\dot\gamma\dot\delta}]_\star = \epsilon^{\dot\alpha\dot\gamma} X^{\dot\beta\dot\delta} + \epsilon^{\dot\alpha\dot\delta} X^{\dot\beta\dot\gamma} + \epsilon^{\dot\beta\dot\gamma} X^{\dot\alpha\dot\delta} + \epsilon^{\dot\beta\dot\delta} X^{\dot\alpha\dot\gamma}\,.$$

If the $X^{\dot\alpha\dot\beta}$ were unconstrained then they would generate the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, but instead they obey

$$X^{\dot\alpha\dot\beta} \star X_{\dot\alpha\dot\beta} = X^{\dot\alpha\dot\beta} X_{\dot\alpha\dot\beta} + \frac{3\mathfrak{q}^2 z^2}{2} = z^2\left(\frac{3\mathfrak{q}^2}{2} - 2c^2\right).$$

Can show that the CCA is isomorphic to $\mathcal{L}W(\mu)$ where $W(\mu) \cong U(\mathfrak{sl}_2(\mathbb{C}))/\langle C-\mu\rangle$, for C the quadratic Casimir. (This algebra is often written as $\mathrm{hs}[\lambda]$ for $\lambda^2 = 1+4\mu$.)

In our case

$$\mu = \frac{c^2}{4\mathfrak{q}^2} - \frac{3}{16}\,,$$

consistent with

$$\mathfrak{diff}(\mathbb{C})^{\mathbb{Z}_2} \cong W(-3/16)\,.$$

Sending $\mathfrak{q}\to 0$ at fixed $c\neq 0$ takes us back to self-dual gravity on Eguchi-Hanson. At the level of the CCA

$$\lim_{\mathfrak{q}\to\infty} \mathcal{L}W\left(\frac{c^2}{4\mathfrak{q}^2} - \frac{3}{16}\right) \to \mathcal{L}W(\infty)\,.$$

This includes the Lie algebra of loops into the wedge subalgebras of W_∞, $W_{1+\infty}$.

Summary

- Considered the hidden symmetry, or celestial chiral algebra, of self-dual gravity. It describes the holomorphic collinear singularities of amplitudes in first order deformations of sdgr.
- At tree level have a consistent chiral algebra, but 1-loop corrections violate associativity.
- Failure can be traced to an anomaly on twistor space, or equivalently to the 1-loop all-plus amplitudes.
- Cancelling the anomaly cures previously identified failure. Get quantum groups for sdgr + stuff!
- On the simplest non-trivial self-dual Ricci flat background, the tree level CCA is modified to $\mathcal{L}W(\infty)$.
- Furthermore switching on non-commutativity can get $\mathcal{L}W(\mu)$ for all μ.

Future work

- Can we better understand these quantum groups? Simplest example probably arises for self-dual $\mathcal{N} = 1$ supergravity.
- In case of sdYM there is a correspondence:

$$\begin{array}{ccc} \text{local operators} & \leftrightarrow & \text{conformal blocks} \\ \text{form factors} & \leftrightarrow & \text{chiral algebra correlators} \end{array}$$

  Is there an analogous statement in gravity?
- Poisson-Chern-Simons theory on twistor space conjecturally describes the $\mathcal{N} = 2$ string on spacetime. What is its chiral algebra?
- Recently a remarkable new holographic duality has been obtained using twistor methods. **[Costello et al., 22, 23]** Is there a sdgr counterpart?

Thank you for listening.