Bogomolnyi-like Equations for Gravity Theories
[work in progress]

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Overview

Introduction

BPS Lagrangian Method

- Example: $SU(2)$ Yang-Mills-Higgs model

Four-dimensional Gravity Theories

- Schwarzschild black holes
- Static Electrovac
- V-scalar-vacuum

Einstein-Maxwell-Scalar gravity in $n-$dimensions

- Tangherlini Black Holes
- Three dimensional V-scalar vacuum

End

What are Bogomolnyi equations?

- In 1975 Prasad and Sommerfield attempted to find exact solutions of nonlinear second-order differential equations for 't Hooft monopoles and Julia-Zee dyons in the $SU(2)$ Yang-Mills-Higgs model.

$$\mathcal{L}_{sYMH} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \mathrm{Tr}\left(D_\mu\Phi D^\mu\Phi\right) - V(|\Phi|),$$

  with $|\Phi| = 2\mathrm{Tr}\Phi^2$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie\left[A_\mu, A_\nu\right]$, $D_\mu \equiv \partial_\mu - ie\left[A_\mu, \right]$, $A_\mu = \frac{1}{2}\tau^a A_\mu^a$, $\Phi = \frac{1}{2}\tau^a\Phi^a$, with $a = 1, 2, 3$ and τ^a are the Pauli matrices.
- They found in the limit where both mass and quartic-interaction coupling go to zero the exact solutions exist.
- In 1976 Bogomolnyi showed by rewriting its energy functional (Bogomolnyi's trick), those exact solutions are solutions to first-order differential equations known as Bogomolnyi equations.

$$E = \int d^3x \left[\mathrm{Tr}\left(B_i \mp \cos\alpha D_i\Phi\right)^2 + \mathrm{Tr}\left(E_i \mp \sin\alpha D_i\Phi\right)^2 + \mathrm{Tr}(D_0\Phi)^2 + V\right]$$
$$\pm 2\int d^3x \left[\cos\alpha\,\mathrm{Tr}\left(B_iD_i\Phi\right) + \sin\alpha\,\mathrm{Tr}\left(E_iD_i\Phi\right)\right],$$

  with $E_i = F_{0i}$, $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$, and α is an abitrary angle.
- Further he showed that these solutions are stable since their total energies are proportional to the topological charge.
- In supersymmetric theories the corresponding equations usually called BPS equations.

Methods for finding Bogomolnyi equations

- Strong Necessary Conditions method [Sokalski 1979]
- First-order formalism: pressureless condition [Bazeia et al.2006]
- On-Shell method [ANA & Ramadhan 2014]
- FOEL (First-Order Euler-Lagrange) formalism [Adam & Santamaria 2016]
- **BPS Lagrangian method** [ANA 2015]

The BPS Lagrangian method [ANA 2017, *Phys. Lett.* **B768** 351358 (*ArXiv* 1511.01620)]

- **Static energy of any static configuration** involving fields ϕ_i $(i = 1, ..., N)$ in a $(d+1)$-dimensional manifold $(M, g_{\mu\nu})$

$$E_{\text{static}} = -\int d^d x \sqrt{\det(g_{mn})}\mathcal{L}[\phi_1, \ldots, \phi_N, \partial\phi_1, \ldots, \partial\phi_N],$$

  with $\partial \equiv \frac{d}{dx^\mu}$ and $\mu = 0, 1, \ldots, d$.
- rewriting the Lagrangian density

$$\sqrt{\det(g_{mn})}\mathcal{L} = (\text{Squared terms}) + \mathcal{L}_{bps},$$

  where **BPS Lagrangian density** $\mathcal{L}_{bps}$ (normally) contains boundary terms and

$$(\text{Squared terms}) \propto \sum_{i=1, j\neq i}^{N} \left(\partial\phi_i - f(\phi_1, ..., \phi_N, \vec{x})\right)^2.$$

Continue...

- Define a BPS limit,

$$\sqrt{\det(g_{mn})}\mathcal{L} - \mathcal{L}_{bps} = 0 \Rightarrow \partial\phi_i = f(\phi_1, ..., \phi_N, \vec{x}).$$

  but what is the form of $\mathcal{L}_{bps}$ explicitly?

- **The On-Shell method** [ANA and Ramadhan 2014, *Phys. Rev.* **D90** 105009 (*ArXiv* 1406.6180)]: the static energy of **(spherically symmetric) BPS vortices** can be written as

$$E_{\text{bps}} = Q(r \to \infty) - Q(r \to 0) = \int_{r=0}^{r\to\infty} dQ,$$

  where $Q = Q(\phi_1, \cdots, \phi_N)$ is called **BPS energy functional (effectively one-dimensional space which is radial coordinate)**.

- $\mathcal{L}_{bps} \propto dQ$ is boundary terms $\longrightarrow$ its Euler-Lagrange equations are trivial.

Continue...

- Generalizing this (for $d = 3$), with $N = 3$, yields

$$\mathcal{L}_{bps} = -\sum_{m,n,p}\sum_{i,j,k} Q^{[mnp]}_{[ijk]}\partial_m\phi_i\partial_n\phi_j\partial_p\phi_k,$$

  with $Q^{[mnp]}_{[ijk]} \equiv= Q^{[mnp]}_{[ijk]}(\phi_1, \cdots, \phi_3)$ and m, n, and p are indices for the spatial coordinates. For more general possible boundary terms see [C. Adam and F. Santamaria, JHEP **1612**, 047 (2016)]
- In general $\mathcal{L}_{bps}$ **may also contain non-boundary terms**,and hence we have additional constraint equations from its Euler-Lagrange equations [ANA Eur.Phys.J.Plus 135 (2020) 8, 619]

$$\frac{\partial\mathcal{L}_{bps}}{\partial\phi_i} - \frac{\partial}{\partial x^m}\frac{\partial\mathcal{L}_{bps}}{\partial(\partial\phi_i/\partial x^m)} = 0.$$

Example: $SU(2)$ Yang-Mills-Higgs model

$$\mathcal{L} = \mathrm{Tr}\,(E_i)^2 - \mathrm{Tr}\,(B_i)^2 + \mathrm{Tr}\,(D_0\Phi)^2 - \mathrm{Tr}\,(D_i\Phi)^2 - V$$
$$\mathcal{L}_{BPS} = 2\alpha\ \mathrm{Tr}\,(E_iD_i\Phi) - 2\beta\ \mathrm{Tr}\,(B_iD_i\Phi) - \gamma\mathrm{Tr}\,(D_i\Phi)^2\,,$$

with α, β, and γ are arbitrary constants.

$$\begin{aligned} \mathcal{L} - \mathcal{L}_{BPS} &= \mathrm{Tr}\,(E_i - \alpha D_i\Phi)^2 - \mathrm{Tr}\,(B_i - \beta D_i\Phi)^2 + \mathrm{Tr}\,(D_0\Phi)^2 \\ &\quad - \left(1 - \gamma + \alpha^2 - \beta^2\right)\mathrm{Tr}\,(D_i\Phi)^2 - V. \end{aligned}$$

In BPS limit, $\mathcal{L} - \mathcal{L}_{BPS} = 0$, we obtain Bogomolnyi equations:

$$E_i = \alpha D_i\Phi, \qquad B_i = \beta D_i\Phi, \qquad D_0\Phi = 0,$$

while $V = 0$ and $\gamma = 1 + \alpha^2 - \beta^2$. There are additional constraint equations:

$$\begin{aligned} \left(1 - \beta^2\right) D_iD_i\Phi &= 0, \\ \left(1 - \alpha^2 - \beta^2\right)[D_i\Phi, \Phi] &= 0, \\ \alpha\ D_iD_i\Phi &= 0. \end{aligned}$$

- BPS monopoles: $\alpha = 0, \beta = \pm 1$
- BPS dyons: $\alpha \neq 0 \Longrightarrow D_iD_i\Phi = \beta D_iB_i = 0$ (Bianchi Identity) $\alpha^2 + \beta^2 = 1$.

- Bogomolnyi equations for monopoles and dyons in generalized $SU(2)$ Yang-Mills-Higgs model : [ANA Eur. Phys. J. C **82**, no.7, 602 (2022)].

Einstein-Hilbert Action

$$\begin{aligned} S =& S_{EH} + S_{GHY} \\ =& \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R - 2\Lambda \right) + \frac{1}{\kappa} \int_{\partial \mathcal{M}} d^3y \epsilon \sqrt{h} K - \frac{1}{\kappa} \int_{\partial \mathcal{M}} d^3y \epsilon \sqrt{h} K_0, \end{aligned}$$

Einstein equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0.$
ansatz for spherically symmetric static metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega_2^2,$$

with $A, B, C \geq 0.$
Einstein equations can be simplified to, with $' \equiv \frac{d}{dr}$,

$$\begin{aligned} 2CA'C' + A\left(C'^2 - 4BC\left(1 - \Lambda\ C\right)\right) &= 0, \\ 2CB'C' + B\left(C'^2 - 4CC''\right) + 4B^2C\left(1 - \Lambda\ C\right) &= 0, \\ B\left(-AC\left(A'C' + 2AC''\right) + C^2\left(A'^2 - 2AA''\right) + A^2C'^2\right) & \\ + ACB'\left(CA' + AC'\right) - 4\Lambda\ A^2B^2C^2 &= 0. \end{aligned}$$

The Reduced Action (effective Lagrangian density)

$$
\begin{aligned}
S_{RedEH} &\propto \int dr \frac{2C(r)A'(r)C'(r) + A(r)\left(4B(r)C(r)\left(1 - \Lambda\ C(r)\right) + C'(r)^2\right)}{\sqrt{A(r)B(r)C(r)^2}} \\
&= \int dr \mathcal{L}_{eff}.
\end{aligned}
$$

$\Longrightarrow$ The simplified Einstein equations = Euler-Lagrange equations of $\mathcal{L}_{eff}$ where fundamental fields are A, B, and C.

Define BPS Lagrangian density (linear in first-order derivative of fields):

$$
\mathcal{L}_{BPS} = X_0(A,B,C) + X_a(A,B,C)A'(r) + X_b(A,B,C)B'(r) + X_c(A,B,C)C'(r),
$$

where X_0, X_a, X_b, and X_c are auxilliary functions of A, B, and C, but not explicitly of r.

BPS Limit

Rewrite $\mathcal{L}_{eff} - \mathcal{L}_{BPS}$:

$$-\frac{C}{A^{3/2}\sqrt{B}}\left(A'(r) - \frac{\sqrt{AB}}{2C}\left(X_c C - X_a A\right)\right)^2 + \sqrt{\frac{A}{B}}\left(\frac{C'(r)}{C} + \frac{1}{A}\left(A'(r) - \frac{X_c}{2}\sqrt{AB}\right)\right)^2$$

$$-X_0 - B'(r)X_b + \frac{\sqrt{AB}}{4C}\left(X_a^2 A - 2C(X_a X_c - 8\left(1 - \Lambda\ C\right))\right)$$

In BPS limit, $\mathcal{L}_{eff} - \mathcal{L}_{BPS} = 0$, we obtain Bogomolnyi-like equations for A and C

$$\begin{aligned} A'(r) &= \frac{\sqrt{AB}}{2C}\left(X_c C - X_a A\right), \\ C'(r) &= -\frac{C}{A}\left(A'(r) - \frac{X_c}{2}\sqrt{AB}\right) \end{aligned}$$

requires $X_b = 0$ hence remaining terms

$$X_c = \frac{AB\left(X_a^2 A + 16C\left(1 - \Lambda\ C\right)\right) - 4X_0 C\sqrt{AB}}{2X_a ABC}.$$

Constraint Equations

Additional terms in $\mathcal{L}_{BPS}$ imply additional constraint equations:

- Constraint equation for B implies $X_0 = 0$.
- The remaining constraint equations, for A and C, can be simplified to

$$\frac{4C}{\sqrt{AB}}\frac{\partial X_a}{\partial B}X_a\; B'(r) = X_a^3 - 2C\frac{\partial X_a}{\partial C}X_a^2 - \left(16C\,(1-\Lambda\; C) - X_a^2 A\right)\frac{\partial X_a}{\partial A},$$

  which can be considered as Bogomolnyi-like equation for B.

The Bogomolnyi-like equations for A and C can simplified to

$$\begin{aligned} A'(r) &= \sqrt{AB}\left(\frac{4\,(1-\Lambda\; C)}{X_a(A,B,C)} - \frac{A}{4C}X_a(A,B,C)\right), \\ C'(r) &= \frac{\sqrt{AB}}{2}X_a(A,B,C), \end{aligned}$$

The constraint equation can be rewritten as

$$\frac{dX_a}{dr} = \frac{\sqrt{AB}}{4C}X_a^2 = \frac{X_a}{2C}\frac{dC}{dr} \Longrightarrow X_a = cx_a\sqrt{C}.$$

General Solutions

From Bogomolnyi-like equations for A and C

$$2C\frac{dA}{dC} = \frac{16}{cx_a^2}(1-\Lambda C) - A \Longrightarrow A = \frac{16}{3cx_a^2}(3-\Lambda\ C) + \frac{c_a}{\sqrt{C}}$$

From Bogomolnyi-like equation for C

$$B = \frac{4}{cx_a^2}\frac{1}{A\ C}C'(r)^2 \Longrightarrow B \propto \frac{1}{A}$$

- Schwarzschild black holes: $C = r^2,\ cx_a = 4$, and $c_a = -2M$

$$\begin{aligned} A &= 1 - \frac{r^2}{3}\Lambda - \frac{2M}{r}, \\ B &= \frac{1}{1 - \frac{r^2}{3}\Lambda - \frac{2M}{r}}, \end{aligned}$$

Effective and BPS Lagrangian Densities

$$S=\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}\left(R-2\Lambda\right)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right),$$

with $F_{\mu\nu}=D_\mu A_\nu-D_\nu A_\mu$. Taking $\kappa=1$ and using ansatz for electromagnetic field $A_\mu=\left(A_t(r),0,0,A_\phi\cos\theta\right)$.

$$\mathcal{L}_{eff} = \frac{2C\left(A'(r)C'(r)+CA_t'(r)^2\right)+A\left(C'(r)^2-2B\left(A_\phi^2+2C(\Lambda C-1)\right)\right)}{\sqrt{AB}C}$$

$$\begin{aligned}\mathcal{L}_{BPS} &= X_0(A,B,C,A_t)+X_a(A,B,C,A_t)A'(r)+X_b(A,B,C,A_t)B'(r)\\ &\quad +X_c(A,B,C,A_t)C'(r)+X_t(A,B,C,A_t)A_t'(r).\end{aligned}$$

- $\mathcal{L}_{eff}-\mathcal{L}_{BPS}=0$,

$$\begin{aligned}&\frac{2C}{\sqrt{AB}}\left(A_t'(r)-\frac{\sqrt{AB}}{4C}X_t\right)^2+\frac{1}{C}\sqrt{\frac{A}{B}}\left(C'(r)+\frac{1}{2}C\sqrt{\frac{B}{A}}\left(\frac{2A'(r)}{\sqrt{A}\sqrt{B}}-X_c\right)\right)^2\\ &-\frac{C}{A^{3/2}\sqrt{B}}\left(A'(r)-\frac{A^{3/2}\sqrt{B}}{2C}\left(\frac{C}{A}X_c-X_a\right)\right)^2\\ -\ &\frac{\sqrt{AB}}{8C}\left(4C(X_aX_c+8\Lambda C-8)+X_t^2+16A_\phi^2\right)-X_0+\frac{A^{3/2}\sqrt{B}}{4C}X_a^2-B'(r)X_b=0.\end{aligned}$$

Bogomolnyi-like and Constraint Equations

Similarly $X_b = 0$ and thus

$$
\begin{aligned}
X_c &= -\frac{\sqrt{AB}\left(-2A\,X_a^2 + X_t^2 + 16\left(A_\phi^2 + 2C(C\,\Lambda - 1)\right)\right) + 8C\,X_0}{4\sqrt{A}\sqrt{B}C\,X_a}, \\
A_t'(r) &= \frac{\sqrt{A\,B}}{4C}X_t, \\
C'(r) &= \frac{1}{2}\sqrt{A\,B}\,X_a, \\
A'(r) &= -\frac{\sqrt{A\,B}\left(2A\,X_a^2 + X_t^2 + 16\left(A_\phi^2 + 2C(C\Lambda - 1)\right)\right)}{8CX_a} + \frac{X_0}{X_a}.
\end{aligned}
$$

Constraint equations:

- constraint equation for $B \Longrightarrow X_0 = 0$.
- constraint equation for A_t: $\frac{dX_t}{dr} = 0$, or $X_t = cx_t$ is constant.
- constraint equations for A and C:

$$
\frac{dX_a}{dr} = \frac{\sqrt{A\,B}}{4C}X_a^2 = \frac{X_a}{2C}\frac{dC}{dr} \Longrightarrow X_a = cx_a\sqrt{C}
$$

Solutions

Bogomolnyi-like equations for A_t and C:

$$A_t = c_t - \frac{cx_t}{cx_a} C^{-1/2},$$

Bogomolnyi-like equations for A and C:

$$A = \frac{1}{6cx_a^2 C}\left(48A_\phi^2 - 32C(\Lambda\ C - 3) + 3cx_t^2\right) + \frac{c_a}{\sqrt{C}},$$

Bogomolnyi-like equation for C: $B = \frac{4}{cx_a^2}\frac{1}{A\ C}C'(r)^2$

- Reissner-Nordström black hole solutions: $C(r) = r^2, cx_a = 4, cx_t = 4Q$, and $c_a = -2M$

$$\begin{aligned} ds^2 &= -\left(1 - \frac{r^2}{3}\Lambda - \frac{2M}{r} + \frac{Q^2 + A_\phi^2}{2r^2}\right) dt^2 \\ &\quad + \left(1 - \frac{r^2}{3}\Lambda - \frac{2M}{r} + \frac{Q^2 + A_\phi^2}{2r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \end{aligned}$$

Effective and BPS Lagrangian Densities

$$S = \int dx^4 \sqrt{-g}\left(\frac{1}{2\kappa}(R-2\Lambda) - \frac{1}{2}D_\mu\phi D^\mu\phi - V(\phi)\right),$$

Ansatz for scalar field $\phi \equiv \phi(r)$.

$$\mathcal{L}_{eff} = \frac{2C\ A'(r)C'(r) + 4ABC(1-\Lambda C) + A\ C'(r)^2}{\sqrt{A\ B}\ C} - \sqrt{AB}C\left(\frac{2}{B}\phi'(r)^2 + 4V(\phi)\right)$$

$$\begin{aligned}\mathcal{L}_{BPS} &= X_0(A,B,C,\phi) + X_a(A,B,C,\phi)A'(r) + X_b(A,B,C,\phi)B'(r)\\ &\quad + X_c(A,B,C,\phi)C'(r) + X_\phi(A,B,C,\phi)\phi'(r)\end{aligned}$$

- $\mathcal{L}_{eff} - \mathcal{L}_{BPS} = 0$,

$$\begin{aligned}&-\frac{2\sqrt{A}C}{\sqrt{B}}\left(\phi'(r) + \frac{\sqrt{B}}{4\sqrt{A}C}X_\phi\right)^2 - \frac{C}{A^{3/2}B^{1/2}}\left(A'(r) - \frac{A^{3/2}B^{1/2}}{2C}\left(\frac{C}{A}X_c - X_a\right)\right)^2\\ &\qquad + \frac{A^{1/2}}{B^{1/2}\ C}\left(C'(r) + \frac{B^{1/2}C}{2A^{1/2}}\left(\frac{2}{\sqrt{A\ B}}A'(r) - X_c\right)\right)^2\\ &-B'(r)X_b - X_0 + \frac{\sqrt{B}}{8\sqrt{A}C}\left(-4AC(X_aX_c + 8C(\Lambda+V) - 8) + 2A^2X_a^2 + X_\phi^2\right) = 0.\end{aligned}$$

Bogomolnyi-like and Constraint Equations

Similarly $X_b = 0$ and thus

$$\begin{aligned} X_c &= \frac{1}{4ACX_a}\left(2A^2X_a^2 - \frac{8\sqrt{A}C}{\sqrt{B}}X_0 + X_\phi^2 - 32AC^2V - 32AC(\Lambda C - 1)\right), \\ \phi'(r) &= -\frac{1}{4C}\sqrt{\frac{B}{A}}X_\phi, \\ C'(r) &= \frac{\sqrt{A\ B}}{2}X_a, \\ A'(r) &= \frac{-2A^2\sqrt{B}X_a^2 - 8\sqrt{A}CX_0 + \sqrt{B}X_\phi^2 - 32A\sqrt{B}C(C(\Lambda + V) - 1)}{8\sqrt{A}CX_a}. \end{aligned}$$

Constraint equations:

- constraint equation for $B \Longrightarrow X_0 = 0$.
- constraint equation for A: $X_a{}'(r) = -\frac{\sqrt{B}}{8A^{3/2}C}\left(X_\phi^2 - 2A^2X_a^2\right)$
- constraint equations for ϕ: $X_\phi{}'(r) = -4\sqrt{AB}C\ V'(\phi)$

Solution: $C(r) = r^2 \Longleftrightarrow X_a = 4\sqrt{\frac{C}{A\ B}}$

$$
\begin{aligned}
A'(r) &= \frac{1}{32C^{3/2}}\left(BX_\phi^2 - 32AC(B(C\Lambda - 1) + BCV + 1)\right), \\
B'(r) &= \frac{B}{32AC^{3/2}}\left(BX_\phi^2 + 32AC(B(C\Lambda - 1) + BCV + 1)\right).
\end{aligned}
$$

Using these equations we can write

$$
\frac{d\log(AB)}{dr} = \sqrt{C}\phi'(r)^2.
$$

- Coulomb-like form: $\phi = \frac{Q_s}{r} \Longleftrightarrow X_\phi = 4\phi\sqrt{\frac{A\ C}{B}}$

$$
\frac{d\log(AB)}{dr} = \frac{\phi^2}{\sqrt{C}} = Q_s^2\frac{1}{r^3} \Longrightarrow A = c_a\frac{e^{-\frac{\phi^2}{2}}}{B}
$$

$$
\begin{aligned}
B'(\phi) &= -\frac{B}{2\phi^3}\left(2Q_s^2B(\Lambda + V) - 2(B-1)\phi^2 + \phi^4\right), \\
V'(\phi) &= \phi\left(-\frac{(B-1)\phi^2}{Q_s^2B} + \Lambda + V\right).
\end{aligned}
$$

Effective and BPS Lagrangian Densities

$$S = \int dx^n \sqrt{-g}\left(\frac{1}{2\kappa}(R-2\Lambda) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi D^\mu\phi - V(\phi)\right).$$

$$\begin{aligned}
ds^2 &= -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega_{n-2}^2, \quad d\Omega_{n-2}^2 = \sum_{i=1}^{n-2}\left(\prod_{l=1}^{i-1}\sin^2\theta_l \; d\theta_i^2\right), \\
A_\mu &= \left(A_t(r), 0, \ldots, 0\right), \quad \phi \equiv \phi(r). \\
\mathcal{L}_{eff} &= \frac{2(n-2)C\ A'(r)C'(r) + 4ABC\left((n-3)(n-2) - 2\Lambda C\right) + (n-3)(n-2)A\ C'(r)^2}{8\kappa\sqrt{A\ B}\ C^{3-\frac{n}{2}}} \\
&\quad + \sqrt{ABC^{n-2}}\left(\frac{A_t'(r)^2}{2AB} - \frac{\phi'(r)^2}{2B} - V(\phi)\right), \\
\mathcal{L}_{BPS} &= X_0(A,B,C,A_t,\phi) + X_a(A,B,C,A_t,\phi)A'(r) + X_b(A,B,C,A_t,\phi)B'(r) \\
&\quad + X_c(A,B,C,A_t,\phi)C'(r) + X_t(A,B,C,A_t,\phi)A_t'(r) + X_\phi(A,B,C,A_t,\phi)\phi'(r).
\end{aligned}$$

BPS limit $\mathcal{L}_{eff} - \mathcal{L}_{BPS} = 0$

$$\begin{aligned}
&\frac{C^{\frac{n}{2}-1}}{2\sqrt{A\,B}}\left(A_t'(r) - \frac{\sqrt{A\,B}}{C^{\frac{n}{2}-1}}X_t\right)^2 - \frac{\sqrt{A}C^{\frac{n}{2}-1}}{2\sqrt{B}}\left(\phi'(r) + \frac{\sqrt{B}}{\sqrt{A}C^{\frac{n}{2}-1}}X_\phi\right)^2 \\
&-\frac{(n-2)C^{\frac{n}{2}-1}}{8(n-3)A^{3/2}B^{1/2}}\left(A'(r) - \frac{4(n-3)A^{3/2}B^{1/2}}{(n-2)C^{\frac{n}{2}-1}}\left(\frac{C\,X_c}{(n-3)A} - X_a\right)\right)^2 \\
&+\frac{(n-3)(n-2)A^{1/2}}{8B^{1/2}\,C^{3-\frac{n}{2}}}\left(C'(r) + \frac{4B^{1/2}C^{3-\frac{n}{2}}}{(n-3)(n-2)A^{1/2}}\left(\frac{(n-2)C^{\frac{n}{2}-2}}{4\sqrt{A\,B}}A'(r) - X_c\right)\right)^2 \\
&-B'(r)X_b - X_0 + \frac{\sqrt{B}\,C^{-\frac{n}{2}-2}}{2(n-2)\sqrt{A}}\left(C^3\left(A\left(4(n-3)AX_a^2 - (n-2)X_t^2 - 8CX_aX_c\right) + (n-2)X_\phi^2\right)\right. \\
&\qquad\left. +(n-2)AC^n\left(-2C(\Lambda+V) + (n-3)(n-2)\right)\right) = 0.
\end{aligned}$$

Bogomolnyi-like Equations

$$
\begin{aligned}
X_c &= \frac{1}{8A\sqrt{B}C^4X_a}\left(\sqrt{B}\left(4A^2C^3(n-3)X_a^2+(n-2)\left(C^3\left(X_\phi^2-AX_t^2\right)+AC^n((n-3)(n-2)-2C\Lambda)\right)\right)\right.\\
&\qquad \left. -2\sqrt{A}(n-2)C^{\frac{n}{2}+2}\left(X_0+\sqrt{AB}C^{\frac{n}{2}+2}V\right)\right),\\
A_t'(r) &= \sqrt{A\,B}\,C^{1-\frac{n}{2}}X_t\,,\\
\phi'(r) &= -\sqrt{\frac{B}{A}}C^{1-\frac{n}{2}}X_\phi\,,\\
C'(r) &= \frac{4\sqrt{A\,B}}{(n-2)}C^{2-\frac{n}{2}}X_a\,,\\
A'(r) &= \frac{\sqrt{B}\left((n-2)AC^{-2+\frac{n}{2}}((n-3)(n-2)-2C(V+\Lambda))-C^{1-\frac{n}{2}}\left(4(n-3)A^2X_a^2+(n-2)\left(AX_t^2-X_\phi^2\right)\right)\right)}{2\sqrt{A}(n-2)X_a}-\frac{X_0}{X_a}\,.
\end{aligned}
$$

- constraint equations for B: $X_0=0$.
- constraint equation for A_t: $\frac{dX_t}{dr}=0$, or $X_t=cx_t$.
- constraint equation for A: $X_a{}'(r)=-\frac{\sqrt{B}C^{1-\frac{n}{2}}}{2(n-2)A^{3/2}}\left((n-2)X_\phi^2-4(n-3)A^2X_a^2\right)$.
- constraint equation for ϕ: $X_\phi{}'(r)=-\sqrt{AB}C^{-1+\frac{n}{2}}V'(\phi)$.
- constraint equation for C is trivial using above results.

Tangherlini Black Holes: $\phi = V = X_\phi = 0$

$$\frac{dX_a}{dr} = \frac{2(n-3)}{(n-2)}\sqrt{A\ B}\ C^{1-\frac{n}{2}} X_a^2 = \frac{(n-3)X_a}{2C}\frac{dC}{dr} \Longrightarrow X_a = cx_a C^{\frac{n-3}{2}}$$

$$A_t = c_t - \frac{(n-2)}{2(n-3)}\frac{cx_t}{cx_a} C^{-\frac{n-3}{2}},$$

$$A = \frac{(n-2)}{4(n-3)cx_a^2}\left(\frac{cx_t^2}{C^{n-3}} + \frac{(n-3)}{(n-1)}\left((n-2)(n-1) - 2C\Lambda\right)\right) + \frac{c_a}{C^{\frac{n-3}{2}}},$$

$$B = \frac{(n-2)^2}{16 cx_a^2}\frac{1}{A\ C} C'(r)^2.$$

- $C(r) = r^2$, $cx_a = \frac{(n-2)}{2}$, $cx_t = -Q$, and $c_a = -2M$

$$\begin{aligned} A_t &= \frac{Q}{(n-3)} C^{-\frac{n-3}{2}}, \\ ds^2 &= -\left(1 - \frac{2M}{C^{\frac{n-3}{2}}} + \frac{Q^2}{(n-3)(n-2)C^{n-3}} - \frac{2C\Lambda}{(n-2)(n-1)}\right) dt^2 \\ &\quad + \left(1 - \frac{2M}{C^{\frac{n-3}{2}}} + \frac{Q^2}{(n-3)(n-2)C^{n-3}} - \frac{2C\Lambda}{(n-2)(n-1)}\right)^{-1} dr^2 + r^2 d\Omega_{n-2}^2, \end{aligned}$$

Three Dimensional V-scalar Vacuum: $n = 3$ and $A_t = X_t = 0$

$$
\begin{aligned}
\phi'(r) &= -\sqrt{\frac{B}{A\ C}}X_\phi, \qquad C'(r) = 4\sqrt{A\ B\ C}\ X_a, \qquad A'(r) = \sqrt{\frac{B}{A\ C}}\frac{1}{2X_a}\left(X_\phi^2 - 2AC(V+\Lambda)\right),\\
X_a{}'(r) &= -\frac{1}{2A^{3/2}}\sqrt{\frac{B}{C}}X_\phi^2, \qquad X_\phi{}'(r) = -\sqrt{A\ B\ C}\ V'(\phi).
\end{aligned}
$$

- $C = r^2, \phi = \frac{c_\phi}{r}, \Lambda = 0$ as in $\Longleftrightarrow X_a = \frac{1}{2\sqrt{AB}}$ and $X_\phi = \phi\sqrt{\frac{A}{B}}$

$$
B'(r) = \frac{B}{\sqrt{C}}\left(2BC(V+\Lambda)+\phi^2\right) \Longrightarrow \frac{d\log(A\ B)}{dr} = -2\phi\frac{d\phi}{dr} \Longrightarrow A = \frac{c_a}{B}e^{-\phi^2}
$$

$$
V'(r) = -\frac{\phi^2}{\sqrt{C}}\left(\frac{1}{B\ C} + 2(V+\Lambda)\right).
$$

one of solutions [Karakasis'23]

$$
A(r) = \frac{r^2 e^{-\frac{c_\phi^2}{2r^2}}}{c_\phi^2}\left(c_\phi^2 c_V e^{\frac{c_\phi^2}{2r^2}} + c_b\right), \qquad B(r) = \frac{c_\phi^2 c_a e^{-\frac{c_\phi^2}{2r^2}}}{r^2\left(c_\phi^2 c_V e^{\frac{c_\phi^2}{2r^2}} + c_b\right)}, \qquad V(r) = \frac{c_\phi^2 c_V e^{\frac{c_\phi^2}{r^2}}}{2c_a r^2} - \frac{e^{\frac{c_\phi^2}{2r^2}}}{c_\phi^2 c_a}\left(c_\phi^2 c_V e^{\frac{c_\phi^2}{2r^2}} + c_b\right).
$$

Thank You