Dualities and Discretizations of Integrable Quantum Field Theories from 4d Chern-Simons Theory

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Based on "Dualities and Discretizations of Integrable Quantum Field Theories from 4d Chern-Simons Theory" - M. Ashwinkumar, J. Sakamoto, and M. Yamazaki - arXiv : 2309.14412

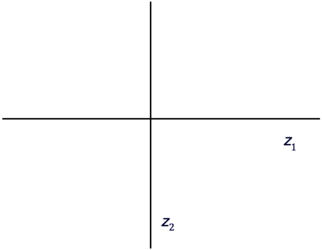
Introduction

- There exists a four-dimensional variant of Chern-Simons theory with the action

$$S = \frac{1}{\hbar} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big). \qquad (1.2)$$

  Here, $\mathcal{A}$ is a complex-valued gauge field, Σ is a 2-manifold, and C is a Riemann surface endowed with a meromorphic one-form $\omega = \omega(z)dz$.
- Topological along Σ (modulo a framing anomaly), but has holomorphic dependence on C.
- It has a complex gauge group, denoted G.

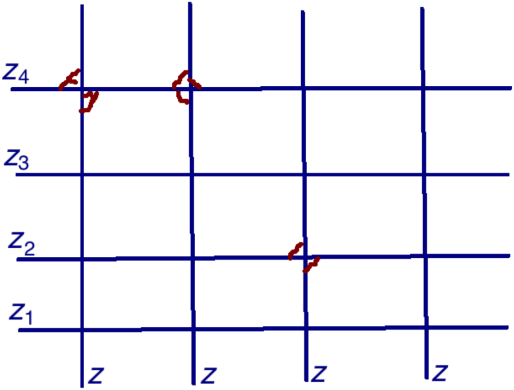
- Studied by Costello, Witten and Yamazaki,[1] who showed that **correlation functions of Wilson lines** realize solutions of the **Yang-Baxter equation with spectral parameters**.
- E.g., the rational R-matrix for Wilson lines on $\Sigma = \mathbb{R}^2$:



$$= I + \frac{\hbar\, c_{\rho,\rho'}}{z_1 - z_2} + \mathcal{O}(\hbar^2)$$

- Here the Wilson lines at z_1 and z_2 are respectively in representations ρ and ρ', with $c_{\rho,\rho'} = \sum_a T_{a,\rho} \otimes T_{a,\rho'}$.

[1] K. Costello, E. Witten, M. Yamazaki, *Gauge Theory and Integrability, I, II*, arXiv:1709.09993, 1802.01579

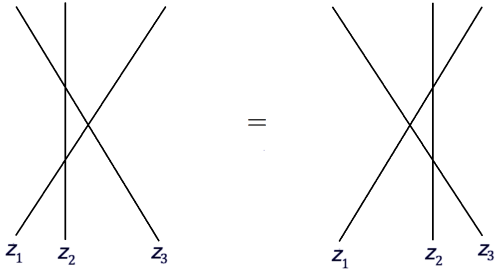


A correlation function of net of Wilson lines thus leads to the partition function of the lattice model.

- The YBE

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2)$$

is also realized due to the topological symmetry along Σ.



- No singular behaviour arises in moving a Wilson line, as long as z_1, z_2 and z_3 are distinct.

- For integrable lattice models one considers C to be one of the following possibilities:

$$\begin{aligned} C &= \mathbb{C} \,, & \omega &= dz \,, \quad \text{(rational)} \,, \\ C &= \mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \,, & \omega &= \frac{dz}{z} \,, \quad \text{(trigonometric)} \,, \\ C &= E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \,, & \omega &= dz \,, \quad \text{(elliptic)} \,. \end{aligned} \tag{1.3}$$

- The three choices of C lead to rational, trigonometric and elliptic integrable lattice models.

Two-dimensional integrable field theories can also be realized via 4d CS.[2] There are two essential classes of defects we consider:

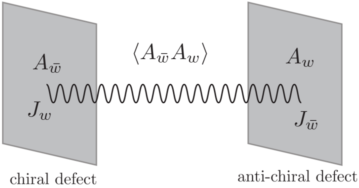
- **Order defects**, where new degrees of freedom are introduced on the defect which are coupled to the bulk gauge theory.
- **Disorder defects**, where the four-dimensional gauge field is required to have some singularities.

We shall mostly focus on the study of **order defects** in this talk, but will eventually observe the **emergence of disorder defects** due to anomalies.

[2] K. Costello, M. Yamazaki, *Gauge Theory and Integrability, III*, arXiv:1908.02289

$$S_{4\mathrm{d}-2\mathrm{d}} = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2\times C} \mathrm{d}z\, \mathrm{CS}(A) + \sum_{\alpha=1}^{n} \frac{1}{\hbar}\int_{\mathbb{R}^2\times z_\alpha} \mathcal{L}_\alpha\left(\phi_\alpha; A_w|_{z_\alpha}, A_{\overline{w}}|_{z_\alpha}\right)$$

In realizing integrable field theories, we usually explicitly break the topological invariance along Σ and employ complex coordinates $w = x + iy$ and $\bar{w} = x - iy$.



We are usually interested in surface defects with global G-symmetry. A large class of interesting IFTs come from a combination of chiral and antichiral defects, which support the currents associated with the G symmetry denoted as $J_w^\alpha = J_w^{\alpha a} T_a$ and $\overline{J}_{\bar{w}}^\beta = \overline{J}_{\bar{w}T_b}^{\beta b}$, respectively.

$$\mathcal{L}_\alpha\left(\phi_\alpha; A_{\overline{w}}\right) = \mathcal{L}_\alpha\left(\phi_\alpha\right) + J_w^\alpha A_{\overline{w}} \quad \left(\alpha = 1, \ldots n_+\right),$$
$$\overline{\mathcal{L}}_\beta\left(\phi_\beta; A_w\right) = \overline{\mathcal{L}}_\beta\left(\phi_\beta\right) + \overline{J}_{\overline{w}}^\beta A_w \quad \left(\beta = 1, \ldots n_-\right).$$

The effective two-dimensional theory takes the form,

$$S_{2\mathrm{d}}^{\mathrm{eff}} = \frac{1}{\hbar}\int_{\mathbb{R}^2}\left[\sum_{\alpha=1}^{n_+}\mathcal{L}_\alpha^c\left(\phi_\alpha\right) + \sum_{\beta=1}^{n_-}\mathcal{L}_\beta^a\left(\phi_\beta\right) + \sum_{\alpha=1}^{n_+}\sum_{\beta=1}^{n_-} r_{ab}(z_\alpha - z'_\beta) J_a^\alpha \overline{J}_b^\beta\right]$$

where

$$r_{ab}(z, z') = \frac{\delta_{ab}}{z - z'}. \tag{1.12}$$

For a point z away from the surface defects, we can define the $\mathfrak{g}$-valued one-form

$$\mathscr{L}^{4d}(z) = A_w(z)\mathrm{d}w + A_{\overline{w}}(z)\mathrm{d}\overline{w} \, .$$

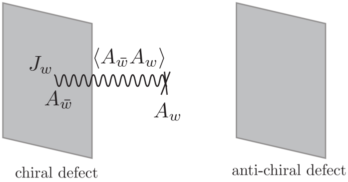
It satisfies a zero-curvature equation

$$\mathrm{d}\mathscr{L}^{4\mathrm{d}}(z) + \mathscr{L}^{4\mathrm{d}}(z) \wedge \mathscr{L}^{4\mathrm{d}}(z) = 0$$

thanks to the 4d CS equation of motion

$$\partial_{\overline{w}} A_w - \partial_w A_{\overline{w}} + [A_{\overline{w}}, A_w] = 0$$

The one-form can thus be identified as the two-dimensional Lax operator, which indicates classical integrability.



The Lax operator can be computed explicitly to be of the form:

$$\mathscr{L}^{2\mathrm{d}}(z) = \sum_{\alpha=1}^{n_+} J^{\alpha}(z_\alpha) r(z_\alpha, z) dw + \sum_{\beta=1}^{n_-} \overline{J}^{\beta}(z_\beta) r(z, z_\beta) d\overline{w}$$

E.g., if we specialize to the case $n_+ = n_- = 1$, with chiral (anti-chiral) defect located at $z = 1$ ($z = -1$), and the rational case ($C = \mathbb{C}$), then

$$\mathcal{L}^{\mathrm{2d}}(z) = \frac{J(1)dw}{1-z} + \frac{\overline{J}(-1)d\bar{w}}{z+1} = \frac{j + z \star j}{1 - z^2} , \qquad (1.13)$$

where the $\mathfrak{g}$-valued one-form j (current) is

$$j := J(1)dw + \overline{J}(-1)d\bar{w} , \qquad (1.14)$$

and $\star$ is the Hodge-star operator in the two-dimensional space so that $\star j = J(1)dw - \overline{J}(-1)d\bar{w}$.

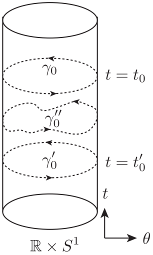
Let us put the 2d theory on a cylinder $\mathbb{R} \times S^1$ with coordinates t, θ, where we choose the period of θ to be 2π. Choose a representation V of g. We can construct an operator

$$W(t_0, z, V) = Tr_V P \exp \int_{\theta=0}^{2\pi} \mathcal{L}(z) \tag{1.17}$$

as the trace in V of path-ordered exponential of the Lax operator along the circle $t = t_0$ in the cylinder.
The zero-curvature equation implies immediately that the operator $W(t_0, V)$ is independent of the value of t_0:

$$\partial_{t_0} W(t_0, z, V) = 0 \, . \tag{1.18}$$

That is, $W(t_0, z, V)$ is conserved.



By expanding in z, say at around $z = \infty$, we find that the Lax operator provides an infinite number of conserved (non-local) charges acting on the Hilbert space of the theory on a circle:

$$W(t_0, z, V) = \exp\left(\sum_{n=0}^{\infty} \frac{Q_n}{z^n}\right) . \tag{1.19}$$

The operator $W(t_0, z, V)$ originates from 4d CS as a Wilson loop.

Example : Chiral and anti-chiral fermions.

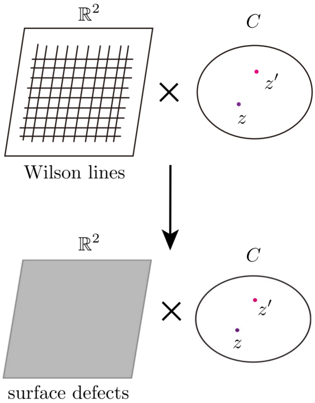
$$S_{\mathrm{4d-2d}} = \frac{1}{\hbar}\left[\frac{1}{2\pi}\int dz\, \mathrm{CS}(A) + \int_{\mathbb{C}\times z_0}\langle\psi, \bar{\partial}_A\psi\rangle + \int_{\mathbb{C}\times z_1}\langle\overline{\psi}, \partial_A\overline{\psi}\rangle\right] .$$

$$\begin{aligned} J_a(\psi) &= \langle\psi, t_a\psi\rangle = \psi t_a \psi \ , \\ \overline{J}_a(\overline{\psi}) &= \left\langle\overline{\psi}, t_a\overline{\psi}\right\rangle = \overline{\psi} t_a \overline{\psi} \ , \end{aligned} \quad (1.20)$$

where $\psi t_a \psi = \psi_i (t_a)_{ij}\psi_j$.
Integrating out the four-dimensional gauge field gives a classical coupling $r_{ab}(z_0 - z_1)J_a\overline{J}_b$. Effective two-dimensional theory is

$$S_{\mathrm{2d}}^{\mathrm{eff}} = \frac{1}{\hbar}\left[\int \psi\bar{\partial}\psi + \int \overline{\psi}\partial\overline{\psi} + r_{ab}(z_0 - z_1)\int J_a(\psi)\overline{J}_b(\overline{\psi})\right] . \quad (1.21)$$

The relationship to integrable lattice models is expected to be the thermodynamic limit of a large number of Wilson lines. The aim of our work is to show this relationship explicitly, and explore its implications.



Discretization of integrable field theories in terms of lattice models is a rather old technique, and is in fact one of the approaches to quantization of these field theories, first developed by Faddeev and Reshetikhin in studying the $SU(2)$ principal chiral model[5]

$$\mathcal{L} = \frac{\kappa}{2} \operatorname{Tr}\left(\partial_\mu g^{-1} \partial^\mu g\right)$$

Their approach was to modify the Poisson brackets of the PCM

$$\left\{j_0^a(x), j_0^b(y)\right\} = f^{abc} j_0^c(x)\delta(x-y)$$

$$\left\{j_0^a(x), j_1^b(y)\right\} = f^{abc} j_1^c(x)\delta(x-y) + \kappa\delta^{ab}\delta'(x-y)$$

$$\left\{j_1^a(x), j_1^b(y)\right\} = 0$$

such that the obstruction to discretization (non-ultralocality) was removed.

[5] L. D. Faddeev, N. Yu. Reshetikhin, Integrability of the Principal Chiral Field Model in 1 + 1 Dimension

Another class of IFTs studied by Costello-Yamazaki arise from disorder defects, singular gauge configurations at **zeroes** of $\omega = \varphi(z)dz$. As explained by Costello-Yamazaki, and Delduc et al.,[6] these can be obtained via a formal gauge transformation

$$A = -d\hat{g}\hat{g}^{-1} + \hat{g}\mathcal{L}\hat{g}^{-1} \tag{1.31}$$

where $\hat{g} : \Sigma \times C \to G$, leading to the unifying action

$$S\left[\{g_x\}_{x\in\boldsymbol{z}}\right] = \frac{1}{2}\sum_{x\in\boldsymbol{z}}\int_{\Sigma}\left\langle \mathrm{res}_x\,\omega \wedge \mathcal{L}, g_x^{-1}dg_x\right\rangle - \frac{1}{2}\sum_{x\in\boldsymbol{z}}\left(\mathrm{res}_x\,\omega\right) I_{\mathrm{WZ}}\left[g_x\right] \tag{1.32}$$

where g_x is the restriction of $\hat{g}$ to each pole $x \in \boldsymbol{z}$.

[6]A Unifying 2D Action For Integrable -Models from 4D Chern-Simons Theory, F. Delduc, S. Lacroix, M. Magro, B. Vicedo

The Poisson brackets of these (nonultralocal) integrable field theories were derived by Vicedo[7]

$$\begin{aligned}
&\left\{A_{\sigma 1}(z, \sigma), A_{\sigma 2}\left(z^{\prime}, \sigma^{\prime}\right)\right\}^{\star} \\
=&\left[\mathcal{R}_{12}\left(z, z^{\prime}\right), A_{\sigma 1}(z, \sigma)\right] \delta_{\sigma \sigma^{\prime}}-\left[\mathcal{R}_{21}\left(z^{\prime}, z\right), A_{\sigma 2}\left(z^{\prime}, \sigma\right)\right] \delta_{\sigma \sigma^{\prime}} \\
&-\left(\mathcal{R}_{12}\left(z, z^{\prime}\right)+\mathcal{R}_{21}\left(z^{\prime}, z\right)\right) \delta_{\sigma \sigma^{\prime}}^{\prime}
\end{aligned}$$

where $\mathcal{R}_{12}(z, z^{\prime}) := 2\pi \frac{C_{12}}{z^{\prime}-z} \varphi(z^{\prime})^{-1}$.

[7] Holomorphic Chern-Simons Theory and Affine Gaudin Models - Vicedo

Discretization of Order Surface Defects

In general, we may discretize chiral and anti-chiral surface operators of the form

$$\begin{aligned}
&\frac{1}{\hbar_{2\mathrm{d}}}\int_{\mathcal{M}\times\{z^+_\alpha\}} \mathcal{L}_\alpha\left(\phi^\alpha,\partial_-\phi^\alpha\right)|_+ + \mathcal{J}^\alpha A_- \quad (\alpha=1,\dots n_+)\ ,\\
&\frac{1}{\hbar_{2\mathrm{d}}}\int_{\mathcal{M}\times\{z^-_{\overline{\alpha}}\}} \overline{\mathcal{L}}_{\overline{\alpha}}\left(\overline{\phi}^{\overline{\alpha}},\partial_+\overline{\phi}^{\overline{\alpha}}\right)|_- + \bar{\mathcal{J}}^{\overline{\alpha}} A_+ \quad (\overline{\alpha}=1,\dots n_-)\ .
\end{aligned} \tag{1.56}$$

where we have analytically continued $w, \bar{w}$ to lightcone coordinates σ^+, σ^-. Here, the notation $|_+$ and $|_-$ denotes quantities that transform as components of one-forms defined along the $+$ and $-$ lightcone directions, respectively. Let us try to discretize one of these surface operators, which has the chiral form given in the first line.

For such a surface operator located at a point $z_+ \in \mathbb{CP}^1$, and with dependence on a set of fields ϕ^α and partial derivatives of these fields, but only with respect to σ^-, we shall discretize it along the σ^+ direction. We achieve this discretization by replacing the integral over σ^+ in the defect action

$$\frac{1}{\hbar_{2\mathrm{d}}}\int_{\mathcal{M}\times\{z_+\}} d\sigma^+ d\sigma^- \mathcal{L}\left(\phi^\alpha, \partial_-\phi^\alpha; A_-|_{z_+}\right) \tag{1.57}$$

by a Riemann sum over lattice points $\sigma^+ = \sigma_n^+ := \Delta n$ $(n \in \mathbb{Z})$ that are equidistant, with lattice spacing Δ.

The resulting action is then a sum over those of the 1d defects along the lightcone direction, which we denote at $\mathbb{R}_-$:

$$\frac{1}{\hbar_{1\mathrm{d}}}\sum_i \int_{\{\sigma_i^+\}\times\mathbb{R}_-\times\{z_+\}} d\sigma^- \mathcal{L}\left(\phi_i^\alpha, \partial_-\phi_i^\alpha; A_{-,i}\right)$$
$$= \frac{1}{\hbar_{1\mathrm{d}}}\sum_{i=1}^{N} \int_{\{\sigma_i^+\}\times\mathbb{R}_-\times\{z_+\}} d\sigma^- \left(\mathcal{L}\left(\phi_i^\alpha, \partial_-\phi_i^\alpha\right) + J_i^\alpha A_{-,a}\big|_{\sigma_i^+, z_+}\right), \tag{1.58}$$

where we defined

$$\phi_n^\alpha := \phi^\alpha\big|_{\sigma^+=\sigma_n^+}, \quad A_{-,n} := A_-\big|_{\sigma^+=\sigma_n^+}, \quad J_n^{\alpha,a} := \mathcal{J}^{\alpha,a}\big|_{\sigma^+=\sigma_n^+}, \tag{1.59}$$

and the one-dimensional Planck constant $\hbar_{1\mathrm{d}}$ by

$$\frac{1}{\hbar_{1\mathrm{d}}} = \frac{\Delta}{\hbar_{2\mathrm{d}}}. \tag{1.60}$$

We now repeat the same procedure for an anti-chiral surface operator located at a point z_- and with dependence on a set of fields denoted $\overline{\phi}$, i.e.,

$$\frac{1}{\hbar_{2\mathrm{d}}}\int_{\mathcal{M}\times\{z_-\}} d\sigma^+ d\sigma^- \mathcal{L}\left(\overline{\phi}^{\overline{\alpha}}, \partial_+\overline{\phi}^{\overline{\alpha}}; A_+|_{z_-}\right), \tag{1.61}$$

but discretize the σ^- direction instead into $\sigma^- = \sigma^-_n := n\Delta$. Thereby, we arrive at another infinite set of line defects oriented along the σ^+ direction, i.e.,

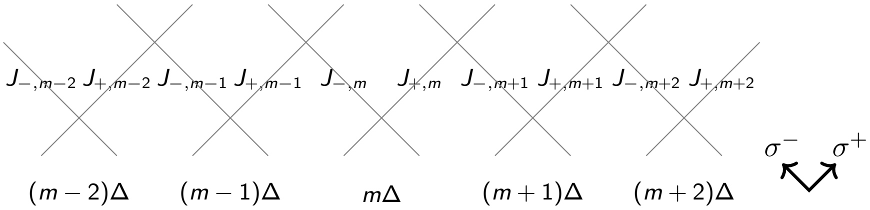
$$\frac{1}{\hbar_{1\mathrm{d}}}\sum_{j=1}^{N}\int_{\mathbb{R}_+\times\{\sigma_j^-\}\times\{z_-\}} d\sigma^+\left(\mathcal{L}\left(\overline{\phi}_j^{\overline{\alpha}}, \partial_+\overline{\phi}_j^{\overline{\alpha}}\right) + J\left(\overline{\phi}_j^{\overline{\alpha}}\right) A_+|_{\sigma_j^-, z_-}\right), \tag{1.62}$$

where we defined $\overline{\phi}_I = \overline{\phi}_I|_{\sigma_j^-}$.

Since the defect actions are exponentiated in the path integral, the complete path integral now takes the form

$$\int DA \exp\left(\frac{i}{2\pi\hbar}\int_{T^2\times C} dz\wedge \mathrm{CS}(A)\right)\times$$
$$\prod_{i=1}^{N}\int [D\phi_k|_{\sigma_i^+}]\exp\left(i\int_{\{\sigma_i^+\}\times S^1\times\{z_+\}} d\sigma^-\mathcal{L}\left(\phi_k|_{\sigma_i^+},\partial_-\phi_k|_{\sigma_i^+};A_-|_{\sigma_i^+,z_+}\right)\right)$$
$$\prod_{j=1}^{N}\int [D\tilde{\phi}_l|_{\sigma_j^-}]\exp\left(i\int_{S^1\times\{\sigma_j^-\}\times\{z_-\}} d\sigma^+\mathcal{L}\left(\tilde{\phi}_l|_{\sigma_j^-},\partial_+\tilde{\phi}_l|_{\sigma_j^-};A_+|_{\sigma_j^-,z_-}\right)\right), \tag{1.72}$$

where $[D\phi_k|_{\sigma_i^+}]$ represents the product of path integral measures for the fields denoted $\phi_k|_{\sigma_i^+}$.



The lightcone lattice where the discrete modes of the currents are supported along the null segments.

We thus arrive at a 4d CS theory path integral with insertions of line operators described by path integrals of 1d theories. As we shall see in concrete examples, these 1d systems can be identified with gauge-invariant Wilson lines using a convenient quantization scheme. In such cases, we find the path integral to be of the form

$$
\begin{aligned}
&\int DA \exp\left(\frac{i}{2\pi\hbar}\int_{T^2\times C} dz \wedge \mathrm{CS}(A)\right) \\
&\times \prod_{i=1}^{N} \mathrm{Tr}_{\Lambda_+}\mathcal{P}\exp\left(i\int_{\{\sigma_i^+\}\times S^1\times\{z_+\}} d\sigma^- A_-|_{\sigma_i^+,z_+}\right) \\
&\times \prod_{j=1}^{N} \mathrm{Tr}_{\Lambda_-}\mathcal{P}\exp\left(i\int_{S^1\times\{\sigma_j^-\}\times\{z_-\}} d\sigma^+ A_+|_{\sigma_j^-,z_-}\right).
\end{aligned}
\tag{1.73}
$$

Here, the traces are taken in representations denoted Λ_+ and Λ_-, with the details of representation governed by the specific form of the surface defect.

Monodromy Matrix and 1d Spin Chain

Recall that the Lax operator gives rise to an infinite number of conserved nonlocal charges by defining the operator

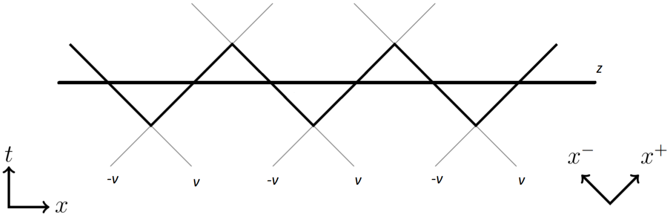
$$W(t_0, z, A) = \mathrm{Tr}_A \mathrm{P}\,\exp\,\int dx \mathcal{L}(x, z). \tag{1.74}$$

From the perspective of 4d Chern-Simons theory, this operator arises from another Wilson line that lies along the spatial direction, intersecting the lightcone lattice that results from discretization. In the quantum theory, the Lax operator in (1.74) would be replaced by an off-shell gauge field.

As a result, one finds via 4d Chern-Simons perturbation theory that the horizontal Wilson line gives rise to a product of R-matrices, i.e.,

$$R_{\Lambda A}(z-v)R_{\Lambda A}(z+v)R_{\Lambda A}(z-v)\ldots, \tag{1.75}$$

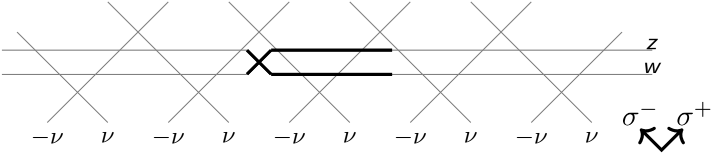
which is the total monodromy matrix of an inhomogeneous spin chain with alternating inhomogeneities.



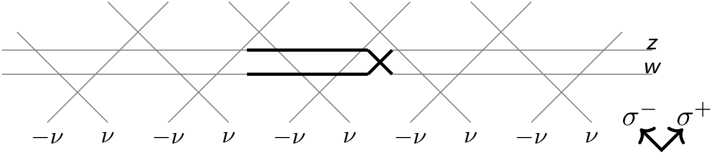
Let us consider two parallel Wilson lines W_A and $W_{A'}$ along the spatial directions, and let us compactify the spatial direction so that we are taking the trace of the monodromy matrix along an auxiliary Hilbert space V_A.

We can use the topological invariance along Σ to exchange the two Wilson lines, realizing the Yang-Baxter equation along the way. This implies that

$$[\, \mathrm{Tr}_A T(z),\ \mathrm{Tr}_{A'} T(z')] = 0\ . \tag{1.76}$$



The configuration that realizes the LHS of (1.80).



The configuration that realizes the RHS of (1.80).

Considering two parallel Wilson lines, we can use the topological invariance along $\mathcal{M}$ to exchange them, realizing the RLL relation

$$R_{12}(z-w)L_{n,1}(z)L_{n,2}(w) = L_{n,2}(w)L_{n,1}(z)R_{12}(z-w)\ , \quad (1.80)$$

which leads to the discretized Poisson bracket in the small $\hbar$ limit

$$\{\mathcal{L}_n(z) \overset{\otimes}{,} \mathcal{L}_n(w)\} = \frac{1}{\Delta}[r_{12}(z-w), \mathcal{L}_n(z) \otimes 1 + 1 \otimes \mathcal{L}_{n,2}(w)] \, . \quad (1.81)$$

Examples of Discretization

Our first example concerns "coadjoint orbit" defects used to realize the Faddeev-Reshetikhin model

$$S_{\mathrm{FR}}\left[\mathcal{G}_{(\pm)}\right] = 2\int_{\Sigma} d^2\sigma\ \mathrm{Tr}\Big(\Lambda\mathcal{G}_{(+)}^{-1}\partial_{-}\mathcal{G}_{(+)} + \Lambda\mathcal{G}_{(-)}^{-1}\partial_{+}\mathcal{G}_{(-)} - \frac{1}{2\nu}\mathcal{G}_{(+)}\Lambda\mathcal{G}_{(+)}^{-1}\mathcal{G}_{(-)}\Lambda\mathcal{G}_{(-)}^{-1}\,. \tag{1.82}$$

This is obtained[9] from

$$S[A, \mathcal{G}_{(\pm)}] = S_{CS}[A] - \int_{\mathcal{M}\times\{z_+\}} \mathrm{Tr}(\Lambda \mathcal{G}_{(+)}^{-1} D_- \mathcal{G}_{(+)})\, d\sigma^+ \wedge d\sigma^- \\ - \int_{\mathcal{M}\times\{z_-\}} \mathrm{Tr}(\Lambda \cdot \mathcal{G}_{(-)}^{-1} D_+ \mathcal{G}_{(-)})\, d\sigma^+ \wedge d\sigma^- \,, \tag{1.83}$$

Looks very similar in form to the description of Wilson loops as quantum mechanics based on an action that depends only on the degrees of freedom describing the coadjoint orbit $\mathcal{G}_{\pm}\Lambda\mathcal{G}_{\pm}^{-1}$.

[9] O. Fukushima, J. Sakamoto, and K. Yoshida, The Faddeev-Reshetikhin model from a 4D Chern-Simons theory

Applying the discretization procedure leaves us with

$$\sum_{i=1}^{N} \int_{\{\sigma_i^+\}\times S^1\times\{z_+\}} \mathrm{Tr}\left(\Lambda \mathcal{G}_{i(+)}^{-1} D_- \mathcal{G}_{i(+)}\right) \, d\sigma^- . \tag{1.89}$$

Repeating the same procedure for the other surface defect located at z_-, we arrive at another infinite set of line defects

$$\sum_{j=1}^{N} \int_{S^1\times\{\sigma_j^-\}\times\{z_-\}} \mathrm{Tr}\left(\Lambda \mathcal{G}_{j(-)}^{-1} D_+ \mathcal{G}_{j(-)}\right) \, d\sigma^+ . \tag{1.90}$$

where $\mathcal{G}_{j(-)}$ and $\mathcal{G}_{j(-)}^{-1}$ are used to respectively denote $\mathcal{G}_{(-)}|_{\sigma_j^-}$ and $\mathcal{G}_{(-)}^{-1}|_{\sigma_j^-}$.

Next, we re-express the path integral measure for each surface defect field as a product of path integral measures for the values of the field at each point along the direction it was discretized. For the measure of $\mathcal{G}_{(+)}$, we set

$$\int D\mathcal{G}_{(+)} = \prod_{i=1}^{N} \int D\mathcal{G}_{i(+)}, \tag{1.93}$$

and for each field denoted $\mathcal{G}_{(-)}$, we write

$$\int D\mathcal{G}_{(-)} = \prod_{j=1}^{N} \int D\mathcal{G}_{j(-)}. \tag{1.94}$$

Given that the defect actions are exponentiated in the path integral, the partition function now takes the form

$$Z_{\mathrm{lat}}^{\mathrm{FR}} = \int DA \exp\left(\frac{i}{2\pi\hbar}\int_{T^2\times C} dz \wedge \mathrm{CS}(A)\right)$$
$$\times \prod_{j=1}^{N} D\mathcal{G}_{j(-)} \exp\left(i\int_{S^1\times\{\sigma_j^-\}\times\{z_-\}} \mathrm{Tr}(\Lambda\mathcal{G}_{j(-)}^{-1}D_+\mathcal{G}_{j(-)})\, d\sigma^+\right)$$
$$\times \prod_{i=1}^{N} D\mathcal{G}_{i(+)} \exp\left(i\int_{\{\sigma_i^+\}\times S^1\times\{z_+\}} \mathrm{Tr}(\Lambda\mathcal{G}_{i(+)}^{-1}D_-\mathcal{G}_{i(+)})\, d\sigma^-\right). \tag{1.95}$$

With an appropriate reality condition, the path integral of the coupled 4d-2d system of 4d CS theory and the discretized coadjoint orbit surface operators can be expressed as

$$\begin{aligned} Z_{\mathrm{lat}}^{\mathrm{FR}} = \int DA \exp\left(\frac{i}{2\pi\hbar}\int_{T^2\times C} dz \wedge \mathrm{CS}(A)\right) \\ \times \prod_{i=1}^{N} \mathrm{Tr}_\Lambda \mathcal{P}\exp\left(i\int_{\{\sigma_i^+\}\times S^1\times\{z_+\}} d\sigma^- A_-|_{\sigma_i^+,z_+}\right) \\ \times \prod_{j=1}^{N} \mathrm{Tr}_\Lambda \mathcal{P}\exp\left(i\int_{S^1\times\{\sigma_j^-\}\times\{z_-\}} d\sigma^+ A_+|_{\sigma_j^-,z_-}\right). \end{aligned} \tag{1.96}$$

in terms of the regularization where the number of Wilson lines in both lightcone directions, N, is taken to be finite. We have thus reproduced the discretization procedure of Faddeev and Reshetikhin from 4d Chern-Simons theory.

Free Fermion Surface Operators

Let $G = SU(N)$. For surface operators with free fermion degrees of freedom, we can also discretize to obtain 1d free fermion actions. These are known to be equivalent to Wilson lines in the direct sum of antisymmetric representations of $SU(N)$

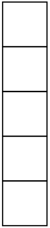
$$\int D\psi e^{\int dt i\psi^\dagger D_t \psi} = \mathrm{Tr}_\Lambda \mathrm{P}\, \exp(\oint A). \tag{1.120}$$

with the imposition of the additional constraint $\psi^\dagger \psi = \kappa$

To obtain a Wilson loop in an irreducible representation, we impose the following constraints:

$$\psi^\dagger \psi = k \,. \tag{1.133}$$

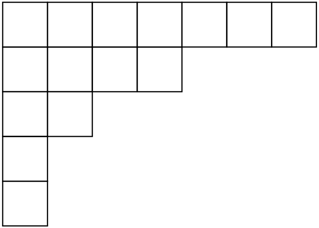
That is, we focus only on k fermionic excitations. The restricted Hilbert space then carries the k-th antisymmetric representation of $SU(N)$.



k

General irreducible representations can be realized via discretization of multiflavor chiral fermion surface defects, with appropriate constraints :

$$S_f = \int_\Sigma d^2\sigma \left(\sum_{I,J=1}^{K} \sum_{i,j=1}^{N} \psi_i^{I\dagger} (i\delta^{IJ}\delta^i_j \partial_+ + \delta^{IJ} A_+^a \rho(T_a)^i{}_j + \delta^{ij} (\tilde{A}_+)_{IJ}) \psi^j_J \right.$$
$$\left. + \sum_{I=1}^{K} k_I \, (\tilde{A}_+)_{II} \right) \tag{1.135}$$



$k_1 \quad k_2 \quad k_3 \; \dots \qquad \dots \; k_K$

Free $\beta - \gamma$ Systems

In order to realize σ-models on spheres from 4d Chern-Simons theory, one couples free $\beta - \gamma$ systems to $GL(N, \mathbb{C})$ 4d Chern-Simons theory.

$$S_{4\text{ d}-2\text{ d}} = \frac{1}{\hbar}\left[\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{C}} \mathrm{d}z\,\mathrm{CS}(A) + \int_{\mathbb{R}^2\times z_0} \beta^i \bar{\partial}_A \gamma_i + \int_{\mathbb{R}^2\times z_1} \bar{\beta}_i \bar{\partial}_A \bar{\gamma}^i\right].$$

Integrating out the gauge field A leaves us with the two-dimensional theory with the classical action.

$$S^{\text{eff}}_{2\text{ d}} = \frac{1}{\hbar}\left[\int_{\mathbb{R}^2\times z_0} \beta^i \bar{\partial}_A \gamma_i + \int_{\mathbb{R}^2\times z_1} \bar{\beta}_i \bar{\partial}_A \bar{\gamma}^i + \frac{1}{z_0 - z_1}\int \beta^i \gamma_j \bar{\beta}_i \bar{\gamma}^j\right].$$

We see that this construction has engineered the σ-model on S^{2n-1}, together with a single free boson. Discretizing the free β-γ systems leads us to quantum mechanical systems that can be straightforwardly quantized, i.e.,

$$S = \frac{1}{2\pi} \int dt \, (\beta_i \partial_t \gamma^i + A^a \beta_i \rho(t_a)^i{}_j \gamma^j). \tag{1.136}$$

Using the commutation relations, $[\gamma^i, \beta_j] = 1$, it is straightforward to show that

$$\mu_a = \beta_i \rho(t_a)^i{}_j \gamma^j \,. \tag{1.137}$$

satisfies $[\mu_a, \mu_b] = f^c_{ab} \mu_c$.

We may pick an alternate contour where the β and γ fields are not real and independent, but rather complex conjugate of each other, i.e., $\beta_i = z_i^*$ and $\gamma^i = z^i$. The associated integrable field theory, for general gauge group G, can be described as a bosonic version of the massless Thirring model.

To obtain a Wilson loop in an irreducible representation of $G = SU(N)$, we impose the following constraints:

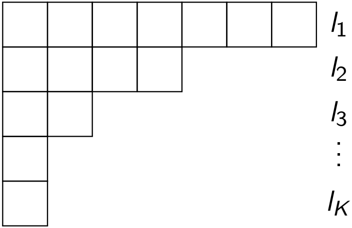
$$\sum_{i}^{N} z_i^* z^i = l\,. \tag{1.138}$$

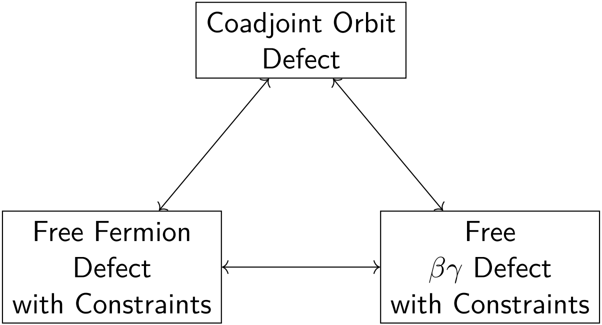
That is, we focus only on l bosonic excitations. The restricted Hilbert space then carries the l-th symmetric representation of $SU(N)$.


l

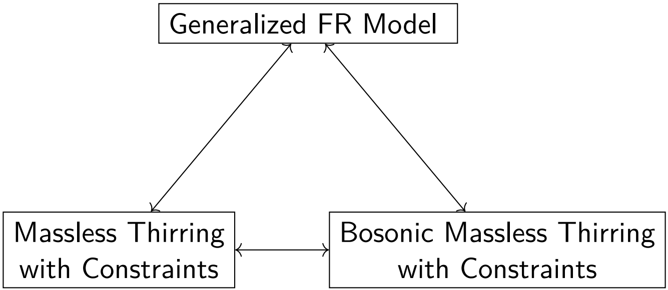
General irreducible representations can be realized via discretization of multiflavor analogues of these surface defects, with appropriate constraints :

$$S_f = \int_\Sigma d^2\sigma \left(\sum_{\alpha,\beta=1}^{K} \sum_{i,j=1}^{N} z_i[\alpha]^* (i\delta^{\alpha\beta}\delta^i_j\partial_+ + \delta^{\alpha\beta}A^a_+\rho(T_a)^i{}_j + \delta^{ij}(\tilde{A}_+)_{\alpha\beta}) z_j[\beta] + \sum_{\alpha=1}^{K} l_\alpha\, (\tilde{A}_+)_{\alpha\alpha} \right) \tag{1.142}$$





Seed triality among 1d defects.



Triality among 2d IFTs.

This implies a triality between the massless Thirring model with constraints, the bosonic massless Thirring model with constraints, as well a generalized Faddeev-Reshetikhin model.

Discretization of Curved $\beta\gamma$ Systems

Curved $\beta - \gamma$ systems are associated with a complex manifold, X, which enjoys a holomorphic G-action.

The defect is defined in terms of fields $\gamma : \mathbb{C} \to X$ and $\beta \in \Omega^{1,0}(\mathbb{C}, \gamma^* T^* X\ )$ (here, T^*X is the holomorphic cotangent bundle of X).

The coupling to 4d CS gauge fields is via $\rho : \mathfrak{g} \to \mathrm{Vect}(X)$ which is the Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of holomorphic vector fields on X. These holomorphic vector fields generate the infinitesimal G-action on X.

The local form of the chiral surface defect action is

$$\int_{\mathcal{M}\times\{z_0\}} \beta^i \bar{D}\,\gamma_i\,\mathrm{d}w\mathrm{d}\bar{w} \equiv \int_{\mathcal{M}\times\{z_0\}} \left(\beta^i\partial_{\bar{w}}\gamma_i + A_{a,\bar{w}}\beta^i\rho_{a,i}(\gamma)\right)\mathrm{d}w\mathrm{d}\bar{w}. \tag{1.150}$$

The anti-chiral $\bar{\beta}\bar{\gamma}$ system is defined such that $\bar{\gamma}$ is a map to $\bar{X}$ (defined to be X with the opposite complex structure) and $\bar{\beta} \in \Omega^{0,1}\left(\mathbb{C}, \bar{\gamma}^* T^*\bar{X}\right)$. The explicit surface defect action is

$$\int_{\mathcal{M}\times\{z_1\}} \bar{\beta}^i D\,\bar{\gamma}_i\,\mathrm{d}w\mathrm{d}\bar{w} \equiv \int_{\mathcal{M}\times\{z_1\}} \left(\bar{\beta}^i\partial_{w}\bar{\gamma}_i + A_{a,w}\bar{\beta}^i\bar{\rho}_{a,i}(\bar{\gamma})\right)\mathrm{d}w\mathrm{d}\bar{w}. \tag{1.151}$$

The discretization of these surface defects gives rise to line operators in infinite-dimensional representations.

Chiral Anomaly

For the case of a chiral free fermion order surface defect, it is a standard result that under infinitesimal gauge transformations the nontrivial transformation of the fermion measure is the exponential of

$$-\frac{1}{4\pi}\int_{\Sigma} \mathrm{Tr}\,(d\epsilon \wedge A)$$

On the other hand the 4d CS action is transformed as

$$S[A^u] = S[A] - \frac{i}{4\pi} \int_{\Sigma \times \mathbb{C}P^1} d\omega \wedge \ \mathrm{Tr} \left(d\epsilon \wedge A \right) \, . \tag{1.182}$$

In order to cancel the anomaly, we require the shift[16]

$$\omega \mapsto \omega_{\text{eff}} = \omega - \frac{\hbar}{2\pi i} \left(\frac{1}{z - z_L} - \frac{1}{z - z_R} \right) \, . \tag{1.183}$$

[16] K. Costello, M. Yamazaki (unpublished)

All the order surface defects support affine Kac-Moody algebras at some level.
In general, given a surface defect supporting an affine Kac-Moody current algebra with coupling to the gauge fields of the form

$$\frac{1}{2\pi}\int d^2 w A_{a\bar{w}} J^a, \tag{1.184}$$

we can detect a gauge anomaly by performing an infinitesimal gauge transformation of the effective as a functional of the gauge field, by using the OPE

$$J^a(w)J^b(w') \sim \frac{i f_c^{ab} J^c(w')}{w-w'} + \frac{k\delta^{ab}}{(w-w')^2}, \tag{1.185}$$

This gives the Q_{BRST}-invariant gauge anomaly for the chiral defect as

$$\delta W_{+} = \frac{k_{+}}{4\pi} \int_{\Sigma} \mathrm{Tr}(d\epsilon \wedge A)\,. \tag{1.186}$$

Similarly, the Q_{BRST}-invariant gauge anomaly for an anti-chiral defect, with Kac-Moody symmetry with level k_{-}, is given by

$$\delta W_{-} = -\frac{k_{-}}{4\pi} \int_{\Sigma} \mathrm{Tr}(d\epsilon \wedge A)\,. \tag{1.187}$$

Under the infinitesimal gauge transformation $\delta A = -d\epsilon - [A, \epsilon]$, the 4d CS action transforms as

$$\delta S_{\mathrm{CS}}[A] = \frac{1}{2\pi\hbar} \int_{\mathcal{M}\times\mathbb{CP}^1} d\omega \wedge \ \mathrm{Tr}\left(d\epsilon \wedge A\right) \ , \tag{1.188}$$

The transformation implies that the chiral anomalies can be canceled out by considering the shift in ω ,

$$\omega \mapsto \omega_{\text{eff}} = \omega - \frac{\hbar}{4\pi i} \left(\sum_{\alpha=1}^{n_+} \frac{k_{+,\alpha}}{z - z_{+,\alpha}} - \sum_{\beta=1}^{n_-} \frac{k_{-,\beta}}{z - z_{-,\beta}} \right) dz \, . \tag{1.189}$$

Indeed, the variation (1.188) is localized at the simple poles $z = z_{+,\alpha}\,, z_{-,\beta}$ of ω_{eff} as

$$
\begin{aligned}
&-\frac{1}{4\pi}\sum_{\alpha=1}^{n_+} k_{+,\alpha}\int_{\mathcal{M}\times\{z_{+,\alpha}\}} \mathrm{Tr}\,(d\epsilon\wedge A) \\
&+\frac{1}{4\pi}\sum_{\beta=1}^{n_-} k_{-,\beta}\int_{\mathcal{M}\times\{z_{-,\beta}\}} \mathrm{Tr}\,(d\epsilon\wedge A)\ ,
\end{aligned}
\tag{1.190}
$$

and these terms precisely cancel out the gauge-dependent terms of the chiral anomalies which take the form given in (1.186) and (1.187).

Emergence of Disorder Defects

For a chiral and anti-chiral defect, the one-loop corrected one-form is given by

$$\omega_{\text{eff}} = \omega - \frac{\hbar}{4\pi i}\left(\frac{k}{z - z_+} - \frac{k}{z - z_-}\right) dz\,, \qquad (1.192)$$

By rewriting the one-form as

$$\omega_{\text{eff}} = \frac{(z - \zeta_+)(z - \zeta_-)}{(z - z_+)(z - z_-)} dz\,, \qquad (1.193)$$

we find that ω_{eff} has the following poles and zeros :

$$\mathfrak{p} = \{z_\pm, \infty\}\,, \qquad \mathfrak{z} = \{\zeta_\pm\}\,, \qquad (1.194)$$

where $z = \infty$ is a double pole, $z = z_\pm$ are simple poles, and $z = \zeta_\pm$ are simple zeros

$$\zeta_\pm = \frac{1}{2}\left(z_+ + z_- \pm \sqrt{(z_+ - z_-)\left(z_+ - z_- - \frac{i\hbar k}{\pi}\right)}\right) . \quad (1.195)$$

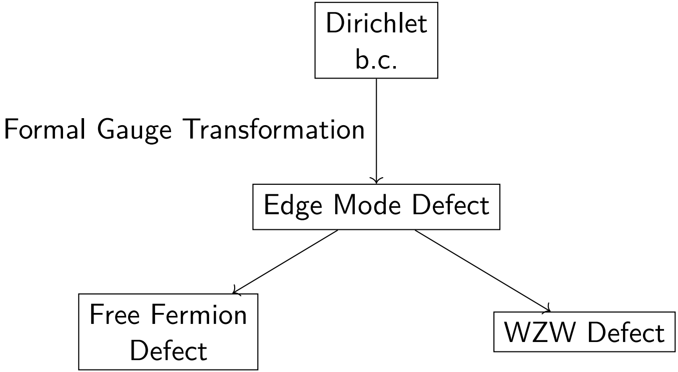
Note that in the semiclassical limit $\hbar \to 0$ the poles $z_\pm$ and zeros $\zeta_\pm$ will cancel each other. In other words, the quantum correction "pair-creates" a pair of poles and zeroes of the one-form.

Quartet of Defects and Nonabelian Bosonizations

We have thus far studied various order defects, their discretizations, and how their gauge anomalies lead to the emergence of quantum corrections to ω that can be interpreted as disorder defects.

Free fermion surface defects can be shown to have equivalent descriptions in terms of "edge mode" defects that arise from formal gauge transformations of chiral Dirichlet boundary conditions, as well as WZW defects.

These equivalances lead to two instances of bosonizations, which are themselves related via boson-boson duality.



The relationship between Dirichlet boundary conditions, edge mode defects, free fermion defects, and WZW defects.

Simply put, an edge mode defect action is just the effective action of a free fermion defect.[17]

$$\int \mathcal{D}\psi_L \mathcal{D}\psi_L^* \exp\left(\frac{i}{\hbar_{\mathrm{2d}}}\int_{\mathcal{M}\times\{z_+\}} d^2\sigma \sum_{I=1}^{k_+} \psi_{L,I}^* i \not{D}_L \psi_{L,I}\right) = \exp\left(-iN_F S_+^{\mathrm{edge}}[g_{(+)}, A]\right) \quad (1.203)$$

$$\int \mathcal{D}\psi_R \mathcal{D}\psi_R^* \exp\left(\frac{i}{\hbar_{\mathrm{2d}}}\int_{\mathcal{M}\times\{z_-\}} d^2\sigma \sum_{I=1}^{k_-} \psi_{R,I}^* i \not{D}_R \psi_{R,I}\right) = \exp\left(-iN_F S_-^{\mathrm{edge}}[g_{(-)}, A]\right) \quad (1.204)$$

By multiplying the edge mode defects by the constant factor, $\int \mathcal{D}\mathcal{G} e^{iS[\mathcal{G}]}$, we can use the Polyakov-Wiegmann identity to equate it with the WZW defect.

[17] Goldstone Fields in Two Dimensions with Multivalued Actions - Polyakov, Wiegmann

- The Lagrangians of the chiral and anti-chiral edge mode defects are given by

$$S_+^{\mathrm{edge}}[g,A] = S_{\mathrm{WZW}}[g] + \frac{1}{2\pi}\int_{\mathcal{M}\times\{z_+\}} d^2\sigma\, \mathrm{Tr}(A_+A_-) \tag{1.205}$$

with the constraint

$$A_-|_{z_+} = g^{-1}\partial_- g\,, \tag{1.206}$$

and

$$S_-^{\mathrm{edge}}[g,A] = \tilde{S}_{\mathrm{WZW}}[g] + \frac{1}{2\pi}\int_{\mathcal{M}\times\{z_-\}} d^2\sigma\, \mathrm{Tr}(A_+A_-) \tag{1.207}$$

where $\tilde{S}_{\mathrm{WZW}}[g] = S_{\mathrm{WZW}}[g^{-1}]$, with the constraint

$$A_+|_{z_-} = g^{-1}\partial_+ g\,, \tag{1.208}$$

where $S_{\mathrm{WZW}}[g]$ is the WZW action

$$S_{\mathrm{WZW}}[g] = -\frac{1}{2\pi}\int_\Sigma d^2\sigma\, \mathrm{Tr}\left(g^{-1}\partial_+ g g^{-1}\partial_- g\right) + S_{\mathrm{WZ}}[g]\,. \tag{1.209}$$

Note that here g is not a separate dynamical degrees of freedom, but rather an "edge mode" of the bulk gauge field A.

The Lagrangians for the WZW defects are given by

$$\begin{aligned} S^b_+[A,\mathcal{G}_{(+)};N_F] &= N_F S_{\mathrm{WZW}}[\mathcal{G}_{(+)}] - \int_{\mathcal{M}\times\{z_+\}} d^2\sigma \ \mathrm{Tr}(A_- \mathcal{J}_+) \\ &\quad - \frac{N_F}{2\pi}\int_{\mathcal{M}\times\{z_+\}} d^2\sigma \ \mathrm{Tr}(A_+ A_-)\,, \\ S^b_-[A,\mathcal{G}_{(-)};N_F] &= N_F S_{\mathrm{WZW}}[\mathcal{G}^{-1}_{(-)}] + \int_{\mathcal{M}\times\{z_-\}} d^2\sigma \ \mathrm{Tr}(A_+ \mathcal{J}_-) \\ &\quad - \frac{N_F}{2\pi}\int_{\mathcal{M}\times\{z_-\}} d^2\sigma \ \mathrm{Tr}(A_+ A_-)\,, \end{aligned} \tag{1.210}$$

where

$$\mathcal{J}_+ = \frac{N_F}{\pi}\partial_+\mathcal{G}_{(+)}\mathcal{G}_{(+)}^{-1}\,, \tag{1.211}$$

$$\mathcal{J}_- = \frac{N_F}{\pi}\partial_-\mathcal{G}_{(-)}\mathcal{G}_{(-)}^{-1}\,. \tag{1.212}$$

We emphasize that the fields $\mathcal{G}_{(+)}$ and $\mathcal{G}_{(-)}$ are dynamical degrees of freedom, which are integrated in the path integral.[18]

The defects have the same gauge transformation as the chiral anomaly of the free fermion systems, rendering the entire 4d-2d system gauge invariant.

[18]The WZW defects play a crucial role in showing the holomorphic factorization of the WZW model path integral [Witten '91].

If we consider these defects in the context of 4d CS with the meromorphic one-form given by

$$\omega_{eff} = \left(1 - \frac{\hbar}{4\pi i}\left(\sum_{\alpha=1}^{n_+} \frac{k_{+,\alpha}}{z - z_{+,\alpha}} - \sum_{\beta=1}^{n_-} \frac{k_{-,\beta}}{z - z_{-,\beta}}\right)\right) dz\,, \quad (1.213)$$

then there is a further equivalence to the Dirichlet boundary conditions for chiral / anti-chiral components of the gauge fields

$$\begin{aligned} A_-|_{z_{+,\alpha}} &= 0\,, \\ A_+|_{z_{-,\alpha}} &= 0\,. \end{aligned} \quad (1.214)$$

Let us explore how the Dirichlet defect relates to the edge mode defect via the formal gauge transformation

$$A = -d\hat{g}\hat{g}^{-1} + \hat{g}\mathcal{L}\hat{g}^{-1}. \tag{1.215}$$

By using this gauge transformation, one can set $\mathcal{L}_{\bar{z}} = 0$
The 4d CS action is transformed into

$$S_{\mathrm{CS}}[A] = \frac{1}{2\pi\hbar}\int_{\Sigma\times C}\omega_{\mathrm{eff}}\wedge\left(\mathrm{CS}(\mathcal{L}) + d\left(\mathrm{Tr}(\hat{g}^{-1}d\hat{g}\wedge\mathcal{L})\right) + I_{\mathrm{WZ}}[\hat{g}]\right), \tag{1.216}$$

where $I_{\mathrm{WZ}}[\hat{g}]$ is the Wess-Zumino three-form defined as

$$I_{\mathrm{WZ}}[\hat{g}] = \frac{1}{3}\mathrm{Tr}\left(\hat{g}^{-1}d\hat{g}\wedge\hat{g}^{-1}d\hat{g}\wedge\hat{g}^{-1}d\hat{g}\right), \tag{1.217}$$

while the boundary conditions are transformed to

$$\begin{aligned} \mathcal{L}_{-}|_{z_{+,\alpha}} &= g_{+,\alpha}^{-1}\partial_{-}g_{+,\alpha}|_{z_{+,\alpha}}, \\ \mathcal{L}_{+}|_{z_{-,\beta}} &= g_{-,\beta}^{-1}\partial_{+}g_{-,\beta}|_{z_{-,\beta}}, \end{aligned} \tag{1.218}$$

where we have denoted the value of $\hat{g}$ at $z_{+,\alpha}$ and $z_{-,\beta}$ to be $g_{+,\alpha}$ and $g_{-,\beta}$, respectively.

The second and third terms in (1.216) can be shown, using the Cauchy-Pompeiu integral formula, to take the form

$$-\sum_{\alpha=1}^{n_+}\frac{k_{+,\alpha}}{4\pi}\bigg(\int_{\Sigma\times\{z_{+,\alpha}\}} d\sigma^+ d\sigma^- \mathrm{Tr}(g_{+,\alpha}^{-1}\partial_+ g_{+,\alpha}\mathcal{L}_- - g_{+,\alpha}^{-1}\partial_- g_{+,\alpha}\mathcal{L}_+) + \int_{\Sigma\times\mathbb{R}_+\times\{z_{+,\alpha}\}} I_{\mathrm{WZ}}[g_{+,\alpha}]\bigg)$$
$$+\sum_{\beta=1}^{n_-}\frac{k_{-,\beta}}{4\pi}\bigg(\int_{\Sigma\times\{z_{-,\beta}\}} d\sigma^+ d\sigma^- \mathrm{Tr}(g_{-,\beta}^{-1}\partial_+ g_{-,\beta}\mathcal{L}_- - g_{-,\beta}^{-1}\partial_- g_{-,\beta}\mathcal{L}_+) + \int_{\Sigma\times\mathbb{R}_+\times\{z_{-,\beta}\}} I_{\mathrm{WZ}}[g_{-,\beta}]\bigg)\,. \tag{1.219}$$

To relate to edge mode defects, we use the boundary constraints (1.218), to rewrite (1.219) into

$$
\begin{aligned}
&-\sum_{\alpha=1}^{n_+}\frac{k_{+,\alpha}}{4\pi}\bigg(\int_{\Sigma\times\{z_{+,\alpha}\}} d\sigma^+ d\sigma^- \mathrm{Tr}(g_{+,\alpha}^{-1}\partial_+ g_{+,\alpha} g_{+,\alpha}^{-1}\partial_- g_{+,\alpha} - \mathcal{L}_+\mathcal{L}_-)\\
&\qquad\qquad + \int_{\Sigma\times\mathbb{R}_+\times\{z_{+,\alpha}\}} I_{\mathrm{WZ}}[g_{+,\alpha}]\bigg)\\
&-\sum_{\beta=1}^{n_-}\frac{k_{-,\beta}}{4\pi}\bigg(\int_{\Sigma\times\{z_{-,\beta}\}} d\sigma^+ d\sigma^- \mathrm{Tr}(g_{-,\beta}^{-1}\partial_+ g_{-,\beta} g_{-,\beta}^{-1}\partial_- g_{-,\beta} - \mathcal{L}_+\mathcal{L}_-)\\
&\qquad\qquad - \int_{\Sigma\times\mathbb{R}_+\times\{z_{-,\beta}\}} I_{\mathrm{WZ}}[g_{-,\beta}]\bigg)\,.
\end{aligned}
\tag{1.220}
$$

Since we know that edge mode defects are equivalent to free-fermion defects, the field theory describing the effective dynamics of these edge modes obtained from 4d CS will be a **bosonization of a Massless Thirring model**!

At this point, one may go on to derive an integrable field theory which takes the form of integrable coupled WZW models following the techniques of Costello-Yamazaki[19] as expounded by Delduc, Lacroix, Magro, Vicedo[20], that is by solving the 4d CS equation of motion $\omega \wedge \partial_{\bar{z}} \mathcal{L} = 0$ with the given boundary conditions.

A convenient method to obtain the IFT is to note that the 4d CS action in this case has

$$\begin{aligned} \omega_{\text{eff}} &= \prod_{\alpha=1}^{N} \frac{(z - \zeta_{-,\alpha})(z - \zeta_{+,\alpha})}{(z - z_{-,\alpha})(z - z_{+,\alpha})} dz \\ &= \tilde{\varphi}_{+}(z) \tilde{\varphi}_{-}(z) \,, \end{aligned} \tag{1.221}$$

for some $\hbar$-dependent zeroes $\zeta_{+,\alpha}$ and $\zeta_{-,\alpha}$,

[19]K. Costello, M. Yamazaki, "Gauge Theory and Integrability, III"
[20]F. Delduc, S. Lacroix, M. Magro, and B. Vicedo, “A unifying 2D action for integrable -models from 4D Chern–Simons theory"

where

$$\tilde{\varphi}_+(z) = \prod_{\alpha=1}^{N} \frac{(z - \zeta_{-,\alpha})}{(z - z_{-,\alpha})}\,, \qquad \tilde{\varphi}_-(z) = \prod_{\alpha=1}^{N} \frac{(z - \zeta_{+,\alpha})}{(z - z_{+,\alpha})}\,. \tag{1.222}$$

It turns out that the resulting model is a special limit of the coupled WZW models studied by Delduc et al.[21] and Costello-Yamazaki. The twist function of the latter corresponds to the meromorphic one-form

$$\omega = \frac{\prod_{i=1}^{2N}\left(z - q_i^+\right)\prod_{j=1}^{2N}\left(z - q_j^-\right)}{\prod_{k=1}^{2N}(z - p_k)^2}\,\mathrm{d}z\,, \tag{1.223}$$

where non-chiral Dirichlet boundary conditions

$$A_\pm|_{z=p_k} = 0 \tag{1.224}$$

are imposed at the double poles at p_k, while at the zeroes q_i^+ and q_i^-, A_+ and A_- are respectively allowed to have poles.

[21] F. Delduc, S. Lacroix, M. Magro, and B. Vicedo, "Integrable Coupled Sigma Models"

The resulting integrable field theory consists of coupled WZW models with negative levels

$$\begin{aligned}&-\sum_{\alpha=1}^{N}\Big[k_{+,\alpha}S_{\mathrm{WZW}}[g_{+,\alpha}]+k_{-,\alpha}S_{\mathrm{WZW}}[g_{-,\alpha}^{-1}]\\&+\sum_{\beta=1}^{N}\rho_{\alpha\beta}\int_{\Sigma}d^2\sigma\ \mathrm{Tr}\left(j_+^{(\alpha)}j_-^{(\beta)}\right)\Big],\end{aligned} \tag{1.225}$$

where

$$\rho_{\alpha\beta}=\frac{2i}{\hbar}\frac{\tilde{\varphi}_{+,\alpha}(z_{-,\alpha})\tilde{\varphi}_{-,\beta}(z_{+,\beta})+\tilde{\varphi}_{-,\beta}(z_{+,\beta})\tilde{\varphi}_{+,\alpha}(z_{-,\alpha})}{z_{-,\alpha}-z_{+,\beta}}, \tag{1.226}$$

and

$$\tilde{\varphi}_{\pm,\alpha}(z)=(z-z_{\mp,\alpha})\tilde{\varphi}_{\pm}(z), \qquad j_{\pm}^{(\alpha)}(\tau,\sigma)=g_{\mp,\alpha}^{-1}\partial_{\pm}g_{\mp,\alpha}. \tag{1.227}$$

The edge mode defects can be identified with multiple chiral free fermion surface operators at $z_{+,\alpha}$ and multiple anti-chiral free fermion surface operators at $z_{-,\alpha}$. Taking $\hbar$ to be small and integrating out the gauge fields gives us an integrable field theory with the action

$$\frac{1}{\hbar_{2d}}\int_{\mathcal{M}} d^2\sigma\Big(\sum_{\alpha=1}^{N}\sum_{I=1}^{k_{+,\alpha}}\bar{\Psi}^I_\alpha iP_+\not{\partial}\Psi^I_\alpha+\sum_{\beta=1}^{N}\sum_{J=1}^{k_{-,\beta}}\bar{\Psi}^J_\beta iP_-\not{\partial}\Psi^J_\beta$$
$$+\frac{\hbar_{2d}}{4}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{i\hbar}{z_{+,\alpha}-z_{-,\beta}}\Big(\sum_{I=1}^{k_{+,\alpha}}\bar{\Psi}^I_\alpha i\rho(t^a)P_+\Psi^I_\alpha\Big)\Big(\sum_{J=1}^{k_{-,\beta}}\bar{\Psi}^J_\beta i\rho(t_a)P_-\Psi^J_\beta\Big)\Big), \tag{1.228}$$

which is a generalized multiflavour massless Thirring model We have thus derived a bosonization duality of this massless Thirring model from 4d CS!

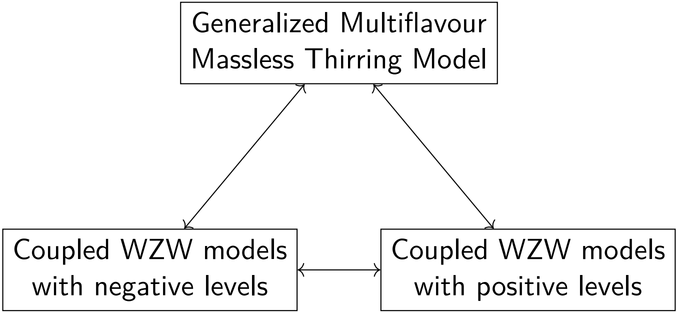
The edge mode defect can alternatively be dualized to WZW defects, via the Polyakov-Wiegmann identity.
The resulting coupled 4d-2d system can be shown to be equal (for a chiral and anti-chiral defect) to

$$\begin{aligned} S_{\mathrm{2d}} =& S^{(k)}_{\mathrm{WZW}}[g_{(+)}] + S^{(k)}_{\mathrm{WZW}}[g_{(-)}] \\ & - \frac{8i}{\hbar}\rho_{+-}\int_{\mathcal{M}} d^2\sigma \ \mathrm{Tr}(\partial_+ g_{(+)} g^{-1}_{(+)} g^{-1}_{(-)} \partial_- g_{(-)}) \,, \end{aligned} \tag{1.229}$$

where ρ_{+-} is given by

$$\rho_{+-} = -\frac{(z_+ - \zeta_+)(z_- - \zeta_-)}{2(z_+ - z_-)} = \frac{(z_+ - z_- - (\zeta_+ - \zeta_-))^2}{8(z_+ - z_-)} \,. \tag{1.230}$$

Here, the levels of the WZW models are positive.
Thus, we find another bosonization duality for the multiflavour massless Thirring model.

- 

Triality among 2d IFTs.

Thermodynamic Limit of Wilson Lines from Polarization of D-branes

In this section, we shall furnish a string theoretic interpretation of the process whereby the a large number of Wilson lines gives rise to an order surface operator in 4d CS theory.
We shall utilize the embedding of 4d CS theory with gauge group $GL(N, \mathbb{C})$ in type IIB string theory derived by Costello and Yagi, utilizing a stack of N D5-branes in an Ω-background.

Let us briefly recall this type IIB string theory configuration. The 10d (Euclidean) spacetime is specified to take the form

$$ds^2 = ds^2_{T^*\Sigma} + ds^2_C + \mathrm{d}s^2_{\mathrm{TN}} \tag{1.231}$$

where $\mathrm{d}s^2_{\mathrm{TN}}$ refers to the following Taub-NUT background,

$$\mathrm{d}s^2_{\mathrm{TN}} = U\ \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} + \frac{1}{U}(\ \mathrm{d}\theta + \vec{\omega}\cdot\mathrm{d}\vec{x})^2. \tag{1.232}$$

Here, $\vec{x}$ is a coordinate of $\mathbb{R}^3$ and θ parametrizes a circle of radius r, which is referred to as the Taub-NUT circle. In addition, $U = \frac{1}{r} + \frac{1}{\lambda^2}$, while $\vec{\omega}$ is a vector on $\mathbb{R}^3$ that satisfies $\mathrm{d}U = \star_{\mathbb{R}^3}(\vec{\omega}\cdot\mathrm{d}\vec{x})$. Parametrizing $\mathbb{R}^3$ by a radial coordinate, $\rho = \sqrt{\vec{x}\cdot\vec{x}}$, and two angular coordinates, one finds that a 2d surface at fixed values of these angular coordinates is a cigar with coordinates r and θ.

The supergravity background of interest preserves a supercharge, Q, that induces an Ω-deformation of the worlvolume theory of any D-brane that wraps a cigar in the Taub-NUT geometry, such that the theory can be described as an Ω-deformed B-model whose target space is a space of maps.

This supercharge squares to a Lie derivative generating a rotation of the Taub-NUT circle. In addition, the type IIB string theory background of interest includes a nontrivial dilaton and RR 2-form. To realize $GL(N, \mathbb{C})$ 4d CS theory, one needs to place a stack of N D5-branes along a cigar in the Taub-NUT geometry, and $\Sigma \subset T^*\Sigma$.[22]

[22] The twist of the normal bundle to $\Sigma \subset T^*\Sigma$ realizes the R-symmetry twist that ensures topological invariance along Σ.

We shall now utilize this type IIB string theory background with additional D3-branes, which will realize Wilson lines within 4d CS theory, as shown in the figure below :

	Σ		$\mathbb{R}^2_{\hbar}$		C		$N\Sigma\subset T^*\Sigma$		$\mathbb{R}^2_{-\hbar}$	
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
D5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
$\mathbf{D3}_b^+$	$\times$		$\times$	$\times$				$\times$		
$\mathbf{D3}_b^-$		$\times$	$\times$	$\times$			$\times$			
$\mathbf{D3}_f^+$	$\times$							$\times$	$\times$	$\times$
$\mathbf{D3}_f^-$		$\times$					$\times$		$\times$	$\times$

Here, we shall take the 0 and 1 directions to be lightcone coordinates. In addition, we have denoted the Taub-NUT geometry as $\mathbb{R}^2_{\hbar} \times \mathbb{R}^2_{-\hbar}$, where the two planes can be indentified with two antipodal cigars in the Taub-NUT geometry. The opposite signs of the deformation parameters indicate that the supercharge Q squares to a symmetry that rotates the two cigars in opposite directions.

The $D3_b - D5$ system shares a 3d worldvolume with topology $\mathbb{R} \times \mathbb{R}^2_\hbar$, where strings stretched between them give rise to a 3d $\mathcal{N} = 4$ hypermultiplet, which further localizes to a 1d theory of bosons, with the action

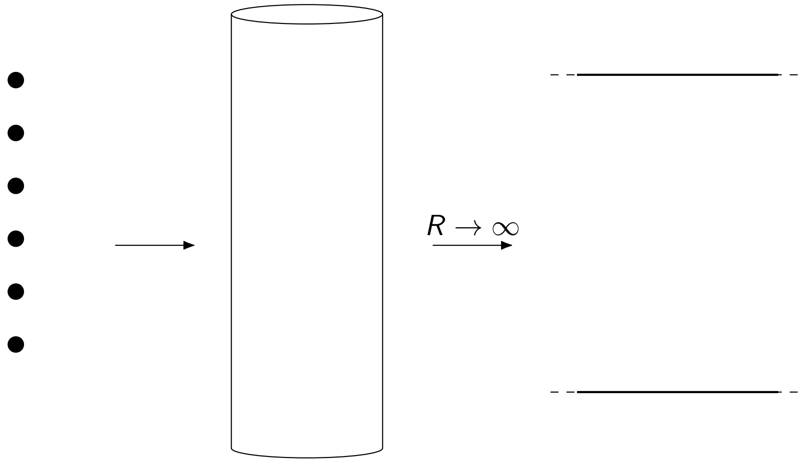
$$\frac{1}{\hbar} \int_{\mathbb{R}} \mathrm{Tr}_{\mathbb{C}^N} \left(\varphi \, \mathrm{d}^A \tilde{\varphi} \right), \tag{1.233}$$

where φ and $\tilde{\varphi}$ are bosonic fields in the fundamental representation of $U(N)$, since we are considering a single $D3_b$ brane. Note that if we were to consider a stack of k $D3_b$ branes instead, we would obtain bosonic fields in bifundamental representations of $U(N) \times U(k)$; in deriving (1.233) we have frozen the D3-brane center-of-mass degree of freedom. This is precisely the line operator that we obtain when discretizing free $\beta\gamma$ systems.

In addition, the $D3_f - D5$ strings are described by the dimensional reduction of the $D4 - D6$ I-brane system given by N chiral free fermions (if we were to consider a stack of k $D3_f$ branes instead, we would obtain bosonic fields in bifundamental representations of $U(N) \times U(k)$). with the action

$$\frac{1}{\hbar}\int_{\mathbb{R}} \mathrm{Tr}_{\mathbb{C}^N}\left(\psi \; \mathrm{d}^A \tilde{\psi}\right). \tag{1.234}$$

This is the analytic continuation of the line operator that we obtain via discretization of chiral free fermion surface operators.



The polarization of a large number of D3-branes (depicted as points) to a D5-brane that realizes the thermodynamic limit of Wilson lines in the 4d CS theory. For example, for $\mathrm{D3}_b^+$, the vertical direction is '1' while the horizontal direction is '6'.

The resulting configuration is shown in the following figure:

	Σ		$\mathbb{R}^2_\hbar$		C		$N\Sigma \subset T^*\Sigma$		$\mathbb{R}^2_{-\hbar}$	
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
D5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
$\mathbf{D5}_b^+$	$\times$	$\times$	$\times$	$\times$			$\times$	$\times$		
$\mathbf{D5}_b^-$	$\times$	$\times$	$\times$	$\times$			$\times$	$\times$		
$\mathbf{D5}_f^+$	$\times$	$\times$					$\times$	$\times$	$\times$	$\times$
$\mathbf{D5}_f^-$	$\times$	$\times$					$\times$	$\times$	$\times$	$\times$

Here, the $D5_b - D5$ system shares a four-dimensional worldvolume with topology $\mathbb{R}^2 \times \mathbb{R}^2_\hbar$, where strings stretched between them give rise to a 4d $\mathcal{N} = 2$ hypermultiplet subject to the Ω-deformation of Kapustin's topological-holomorphic twist, which further localizes to the 2d theory of bosons which can be identified with a free $\beta - \gamma$ surface operator in 4d CS:

$$\frac{1}{\hbar} \int_\Sigma \mathrm{Tr}_{\mathbb{C}^N} \left(\varphi \, \partial^A_{\bar{w}} \tilde{\varphi} \right) . \tag{1.235}$$

The $D5_f - D5$ brane system is T-dual to the well-known D4-D6 I-brane system that can be described by a surface operator supporting N chiral free fermions with the action

$$\frac{1}{\hbar}\int_{\Sigma} \mathrm{Tr}_{\mathbb{C}^N}\left(\psi\ \partial^{A}_{\bar{w}}\tilde{\psi}\right). \tag{1.236}$$

This is a free-fermion surface operator in 4d CS.

Summary

- We have described how order surface operators in 4d Chern-Simons theory can be discretized to form lightcone lattices of Wilson lines.
- Different integrable field theories can sometimes have equivalent discretizations, implying equivalences/dualities between such theories, including an instance of bosonization.
- We have shown how gauge anomalies of order defects are cancelled via the emergence of disorder defects.
- We have employed these disorder defects to derive bosonization dualities for a generalized multi-flavor Massless Thirring model and a boson-boson duality.
- It is hoped that the bosonization procedure together with discretization would help with the quantization of non-ultralocal integrable field theories.

Future Work

1. Trigonometric and elliptic bosonizations from 4d CS (work in progress).
2. Discretization/quantization of non-ultralocal integrable field theories (work in progress).

Thank you!