Infinitely many new (3+1)-dimensional integrable systems from contact geometry

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Why do we need integrability?

While studying an ordinary or partial differential system it is often helpful to know exact formulas for at least some of its solutions: at the very least, they provide useful benchmarks for numerical methods applied to the system under study, and, quite obviously, the more of such exact solutions is known, the better off we are.

Of particular interest is the case when the partial differential system under study can, at least in some sense, be solved exactly, meaning that either that a general solution is available in some reasonable form or that sufficiently many exact solutions are known.

Integrability for systems of ODEs

We have **integrability by quadratures**, when a general solution of the system under study can be written down using quadratures.

Example. A single 2nd order ODE

$$d^2x/dt^2 = F(x)$$

is integrable by quadratures: its general solution is

$$t = \int_{x_0}^{x} (2(E - V(z)))^{1/2} dz$$

where E and x_0 are arbitrary constants and $V(x)$ satisfies $V' = -F$. Here V and F are assumed smooth on an interval $[x_0, x_1]$.

Liouville integrability theorem

Theorem *Let* $\mathscr{M}$ *be an open domain in* $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$ *with coords* q^i, p_i, $i = 1, \dots, n$, *and functionally independent* $I_j \in C^\infty(\mathscr{M})$, $j = 1, \dots, n$, *satisfy*

$$\{I_k, I_\ell\} = 0, \quad k, \ell = 1, \dots, n,$$

where $\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right)$ *is the Poisson bracket*

Then for any $H = F(I_1, \dots, I_n)$ *with smooth* F *the system*

$$dq^i/dt = \partial H/\partial p_i, \quad dp_i/dt = -\partial H/\partial q_i, \quad i = 1, \dots, n.$$

is integrable by quadratures.

Integrability for *nonlinear* PDEs via linearizability

There are nonlinear PDEs whose general solution can be written down in some reasonable form but they are very scarce, and in a sense not too interesting as they are often transformable to linear PDEs.

E.g. a general solution $u(x,t)$ for the Burgers equation

$$u_t = u_{xx} + uu_x$$

can be written, via the Cole–Hopf transformation, as

$$u = 2v_x/v$$

where $v(x,t)$ is a general solution for the linear heat eqn

$$v_t = v_{xx}$$

Beyond linearizability

For a long time it appeared that the only sensible approach to defining exact solvability for nonlinear PDEs is mimicking the linear case by requiring a general solution in a tractable form but this is not too satisfying for many reasons.

This has changed about 60 years ago with the discovery of multisoliton solutions for the Korteweg–de Vries equation

$$u_t + u_{xxx} + 6uu_x = 0$$

describing inter alia shallow water waves.

Solitons and their interactions

M. Kruskal & N. Zabusky (USA, ca. 1965) have discovered through numerical simulation solutions for the KdV eqn that describe two solitary waves that recover their shape after collision, which is very unusual for nonlinear systems. Now these are known as two-soliton solutions

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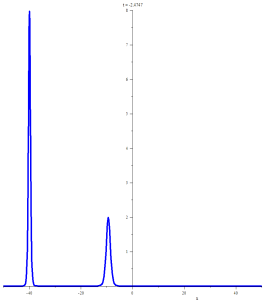

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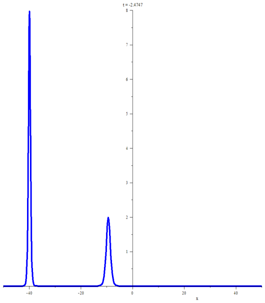

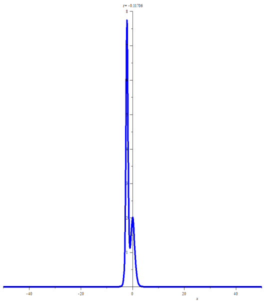

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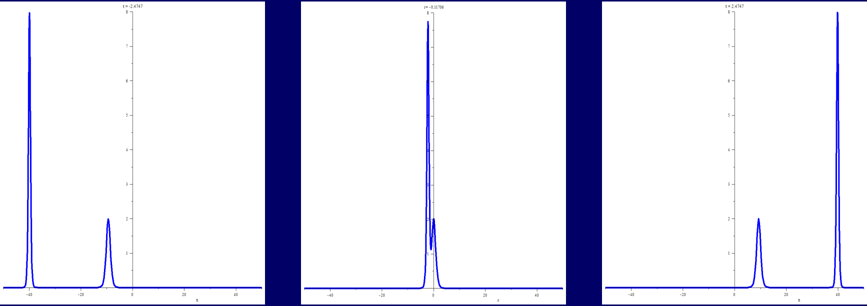

Integrable systems as exactly solvable models

C. Greene, J. Gardner, M. Kruskal & R. Miura (USA) have found explicit exact formulas for multisoliton KdV solutions

P. Lax (USA) has shown that this happens since KdV, albeit nonlinear, can be written as a compatibility condition of an (overdetermined) *linear* system, and thus KdV is an *integrable system*. Later many more integrable systems were found, e.g. the nonlinear Schrödinger (NLS) equation.

As many properties and features of such systems can be studied *exactly*, without having to resort to numerical simulation or approximations, integrable systems, even though their general solutions cannot be written down, were eventually included among exactly solvable models

Integrable *nonlinear* systems from *linear* Lax pairs

A nonlinear partial differential system $\mathcal{S}$ is *(Lax) integrable* if $\mathcal{S} \Leftrightarrow [L, M] = 0$ for a pair of 'nice' linear partial differential operators L and M.

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Then L and M are called *Lax operators*,
$L\psi = 0$, $M\psi = 0$ a *Lax pair*,
and $[L, M] = 0$ a *Lax-type representation* for $\mathcal{S}$

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Under further technical assumptions such an $\mathcal{S}$ has

- infinitely many conservation laws & symmetries
- infinitely many exact solutions

KdV equation: the prototypic integrable system

Let nD indicate n independent variables a.k.a. n dimensions: 2D or $(1+1)$D for $n = 2$ etc.

The 2D Korteweg–de Vries equation for $u = u(x, t)$,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

has a Lax-type representation $[L, M] = 0$ with

$$L = -\partial_x^2 - u - \lambda, \quad M = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x.$$

$[L, M] = 0 \Rightarrow$ compatibility of Lax pair for $\psi(x, t, \lambda)$:

$$Q\psi = \lambda\psi, \qquad M\psi = 0, \tag{2}$$

where $Q = -\partial_x^2 - u$ and λ is the *spectral parameter*

Multisoliton solutions for the KdV equation

The function $u = 2\dfrac{\partial^2 \ln\tau}{\partial x^2}$, where

$$\tau = \sum_{\alpha_1=0}^{1} \cdots \sum_{\alpha_N=0}^{1} \exp\left(2\sum_{i=1}^{N} \alpha_i \Theta_i + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \ln A_{ij}\right),$$

and $\Theta_i = a_i x - 4a_i^3 t + \delta_i, \quad A_{ij} = \left(\frac{a_i - a_j}{a_i + a_j}\right)^2$, satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Conservation laws for the KdV related to the Lax pair

$$w_0 := u, \quad w_1 := u_x,$$

$$w_j := \partial_x w_{j-1} - \sum_{k=0}^{j-2} w_k w_{j-2-k}, \quad j = 2, 3, 4, \ldots$$

Then we have nontrivial conservation laws for KdV

$$\partial_t w_{2s} + \partial_x \sigma_{2s} = 0, \quad s = 0, 1, 2, \ldots$$

where σ_j are certain polynomials in u and its x-derivatives;

$$u_{t_j} = \partial_x \sum_{k=0}^{2j} (-\partial_x)^k \partial w_{2j}/\partial u_k, \quad u_k \equiv \partial^k u/\partial x^k$$

commute $(u_{t_i})_{t_j} = (u_{t_j})_{t_i}$, $i, j = 1, 2, \ldots$ and define symmetries of the original KdV: $(u_t)_{t_j} = (u_{t_j})_t$.

Nonisospectral Lax pairs: An example

Lax operators may contain derivatives w.r.t. variables not present in the associated nonlinear system

Example. The dKP eqn $(u_t + uu_x)_x + u_{yy} = 0$ is known to admit a Lax-type rep with the Lax operators

$$L = \partial_y + p\partial_x - u_x\partial_p, \quad M = \partial_t + (p^2 + u)\partial_x + (u_y - pu_x)\partial_p$$

containing derivatives w.r.t. p, so they, as well as the associated Lax pair $L\chi = 0, M\chi = 0$ for $\chi = \chi(x, y, t, p)$, are *nonisospectral* and p is the *variable spectral parameter*.

The isomonodromic representations for the Painlevé equations are apparently the first known examples of nonisospectral Lax pairs.

Integrable systems in two independent variables

A huge number of examples with the Lax operators

$$L = \partial_x - A, \quad M = \partial_t - B,$$

where A and B are square matrices of the same dimension that depend on the unknown (vector) function and its finitely many derivatives and the spectral parameter λ.

Example. Nonlinear Schrödinger equation

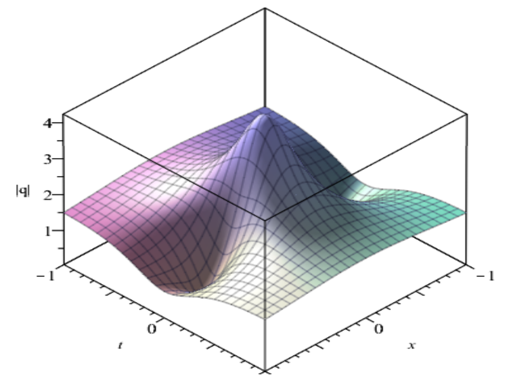
$$iq_t + q_{xx}/2 \mp |q|^2 q = 0$$

has ($f = -i(\lambda^2 \pm |q|^2/2)$)

$$A = \begin{pmatrix} i\lambda & q \\ \pm\bar{q} & -i\lambda \end{pmatrix}, \; B = \begin{pmatrix} f & \lambda q + iq_x/2 \\ \pm\lambda\bar{q} \mp i\bar{q}_x/2 & -f \end{pmatrix}.$$

Peregrine soliton for NLS $iq_t + q_{xx}/2 + |q|^2 q = 0$

$$q = \sqrt{2}\exp(2it)(1 - 4(1 + 4it)/(1 + 8x^2 + 16t^2))$$



Integrable systems in three independent variables

Many examples with the Lax operators

$$L = \partial_y - \mathscr{A}, \quad M = \partial_t - \mathscr{B},$$

where $\mathscr{A}$ and $\mathscr{B}$ are diff. operators of the general form

$$\mathscr{A} = \sum_{j=0}^{n} u_j \partial_x^j, \quad \mathscr{B} = \sum_{k=0}^{m} v_k \partial_x^k$$

Example. For the KP system

$$u_t + 6uu_x + u_{xxx} + 3\sigma^2 v_x = 0, \quad v_x - u_y = 0, \quad \sigma^2 = \pm 1$$

$$L = \partial_y + \sigma^{-1}\left(\partial_x^2 + u\right), \quad M = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x - 3\sigma v$$

Integrable systems in three independent variables II

Many integrable systems for $\boldsymbol{u} = \boldsymbol{u}(x, y, t)$ of general form

$$A_1(\boldsymbol{u})\boldsymbol{u}_x + A_2(\boldsymbol{u})\boldsymbol{u}_y + A_0(\boldsymbol{u})\boldsymbol{u}_t = 0$$

arise from Lax pairs with the Lax operators of the form

$$L = \partial_y - \mathcal{X}_f, \quad M = \partial_t - \mathcal{X}_g \qquad (*)$$

where $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$ are the *Lax functions*; $\mathcal{X}_h = h_p\partial_x - h_x\partial_p$ formally looks like a Hamiltonian vector field in one d.o.f. with the Hamiltonian $h(p, \boldsymbol{u})$.

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Lax operators $(*)$ involve $\partial_p \Rightarrow$ are *nonisospectral*, so p is called the *variable spectral parameter* (recall that $\boldsymbol{u}_p \equiv 0$).

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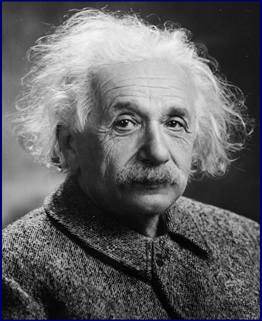
Example. For $f = u - p^2$, $g = 4p^3 - 6up + 3v$ we get the dispersionless KP system $u_y = v_x$, $u_t = 3v_y - 6uu_x$

Integrability in dimension four



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Einstein's GR $\Rightarrow$ our spacetime is 4D, so 4D partial differential systems are of particular relevance for applications

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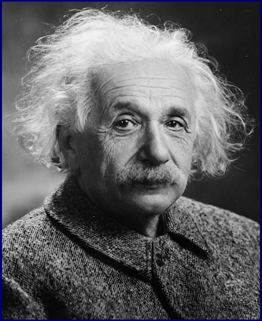

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Einstein's GR $\Rightarrow$ our spacetime is 4D, so 4D partial differential systems are of particular relevance for applications

For a long time it appeared that, unlike 2D and 3D, integrable 4D systems are scarce, and there is no effective construction for them.

Integrable systems in 4D: What was known so far

The most important ones are (anti-)self-dual vacuum Einstein equations and (anti-)self-dual Yang–Mills equations; some other examples are related to them, e.g. the Przanowski equation or the general heavenly equation.

There also is a number of other examples, e.g. the $4\mathrm{D}$ Martínez Alonso–Shabat equation and its modified version, the Dunajski equation etc.

The overwhelming majority of the known integrable $4\mathrm{D}$ systems can be written in dispersionless form, i.e., as quasi-linear homogeneous first-order partial differential systems.

Self-dual Yang–Mills eqs on a matrix Lie group

They boil down to a single equation for the Yang matrix J:

$$(J_{y^-}J^{-1})_{y^+} + (J_{z^-}J^{-1})_{z^+} = 0,$$

and can be rewritten in dispersionless form as

$$J_{z^-}J^{-1} - W_{y^+} = 0, \quad J_{y^-}J^{-1} + W_{z^+} = 0$$

The associated Lax pair reads

$$(\partial_{y^+} + \lambda(\partial_{z^-} - A_{z^-}))\psi = 0, \ (\partial_{z^+} - \lambda(\partial_{y^-} - A_{y^-}))\psi = 0,$$

where $A_{y^-} = J_{y^-}J^{-1}$ and $A_{z^-} = J_{z^-}J^{-1}$.

How it appeared

3D effective constructions
(central extension,
Hamiltonian vec. fields)
+ sporadic examples

4D sporadic examples

Integrable systems: 3D vs 4D

	How it appeared	**How it really is**
3D	effective constructions (central extension, Hamiltonian vec. fields) + sporadic examples	effective constructions (central extension, Hamiltonian vec. fields) + sporadic examples
4D	sporadic examples	*effective construction* (*contact vec. fields*) + sporadic examples

New kind of Lax pairs for 4D systems

Let $L = \partial_y - X_f$ and $M = \partial_t - X_g$, where

- $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$ are the *Lax functions*;
- $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$ is the vector of unknown functions for the associated nonlinear system
- p is the *variable spectral parameter* ($\boldsymbol{u}_p \equiv 0$)
- $X_h = h_p \partial_x + (p h_z - h_x) \partial_p + (h - p h_p) \partial_z$ formally looks exactly like the 3D contact vector field w.r.t. $dz + p dx$ with the contact Hamiltonian h

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The Lax pair $L\chi = 0$, $M\chi = 0$ can be rewritten as

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

where $\chi = \chi(x, y, z, t, p)$.

Infinitely many new integrable 4D systems

Theorem *For all natural* m *and* n *and all* (f,g) *given by*

i) $f = p^{n+1} + \sum_{i=0}^{n} u_i p^i, \; g = p^{m+1} + \frac{m}{n} u_n p^m + \sum_{j=0}^{m-1} v_j p^j$

with $\boldsymbol{u} = (u_0, \dots, u_n, v_0, \dots, v_{m-1})^{\mathrm{T}}$*, and*

ii) $f = \sum_{i=1}^{m} \frac{a_i}{(p-u_i)}, \quad g = \sum_{j=1}^{n} \frac{b_j}{(p-v_j)}$

with $\boldsymbol{u} = (a_1, \dots, a_m, u_1, \dots, u_m, b_1, \dots, b_n, v_1, \dots, v_n)^{\mathrm{T}}$;

Lax pairs $\chi_y = X_f(\chi)$*,* $\chi_t = X_g(\chi)$ *for* $\chi = \chi(x,y,z,t,p)$ *with* $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$ *yield* $4\mathrm{D}$ *integrable systems for* $\boldsymbol{u} = \boldsymbol{u}(x,y,z,t)$ *transformable into Cauchy–Kowalevski form.*

A simple example

Let $f = p^2 + wp + u$, $g = p^3 + 2wp^2 + rp + v$,
i.e. $m = 2$, $n = 1$, $u_0 \equiv u, u_1 \equiv w, v_0 \equiv v, v_1 \equiv r$,
in class i) of the above thm.

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in class i) of the above thm.

The Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ then reads

$$\begin{aligned}
\chi_y &= (2p + w)\chi_x + (-p^2 + u)\chi_z \\
&\quad + (w_z p^2 + (u_z - w_x)p - u_x)\chi_p, \\
\chi_t &= (r + 4wp + 3p^2)\chi_x + (v - 2wp^2 - 2p^3)\chi_z \\
&\quad + (2w_z p^3 + (r_z - 2w_x)p^2 + (v_z - r_x)p - v_x)\chi_p.
\end{aligned}$$

Recap : $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$

A simple example II

For $f = p^2 + wp + u$ and $g = p^3 + 2wp^2 + rp + v$ the above Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ yields a system

$$\begin{aligned}
&u_t - vu_z - ru_x + uv_z + wv_x - v_y = 0,\\
&2u_z + w_x + 2ww_z - r_z = 0,\\
&2r_x - 3u_x - 2w_y + 2wu_z - v_z - 2ww_x + 2uw_z = 0,\\
&w_t - r_y + 2v_x - 4wu_x + wr_x - rw_x - vw_z + ur_z = 0.
\end{aligned} \tag{3}$$

A simple example II

For $f = p^2 + wp + u$ and $g = p^3 + 2wp^2 + rp + v$ the above Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ yields a system

$$
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\end{aligned}
\tag{3}
$$

Proposition *System (3) is, up to a simple change of variables, an* integrable *generalization to the case of* four *independent variables for the well-known dKP equation*

$$(u_t + 6uu_x)_x - 3u_{yy} = 0.$$

Compatibility condition for the Lax pairs

Proposition *For* $L = \partial_y - X_f$ *and* $M = \partial_t - X_g$ *the condition* $[L, M] = 0$ *holds iff*

$$f_t - g_y + \{f, g\} = 0,$$

where $\{,\}$ *is the* contact bracket

$$\{f, g\} \stackrel{\mathrm{df}}{=} f_p g_x - g_p f_x - p\,(f_p g_z - g_p f_z) + f g_z - g f_z.$$

In turn, $[L, M] = 0$ implies compatibility of the Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi)$$

Reminder: $X_h = h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$

Lax pairs: dynamical systems interpretation

The function χ in the Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi)$$

has a straightforward interpetation: it is a joint integral of motion for the following pair of contact dynamical systems

$$\begin{aligned} dx/dy &= -f_p, & dx/dt &= -g_p, \\ dz/dy &= pf_p - f, & dz/dt &= pg_p - g, \\ dp/dy &= f_x - pf_z, & dp/dt &= g_x - pg_z, \end{aligned}$$

which are compatible if we substitute there a sufficiently smooth solution $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$ of the associated nonlinear system

Reminder: $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$

Relation to previously known 3D construction

Consider an integrable nonlinear $4\mathrm{D}$ system with a Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi) \qquad (*)$$

and impose a reduction $\boldsymbol{u}_z = 0$ and $\chi_z = 0$.

Then $(*)$ boils down to a $3\mathrm{D}$ Lax pair of a well-known type,

$$\chi_y = \mathcal{X}_f(\chi), \ \chi_t = \mathcal{X}_g(\chi),$$

where $\mathcal{X}_h = h_p\partial_x - h_x\partial_p$ formally looks like a Hamiltonian vector field with one degree of freedom (recall that $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$).

Lax functions polynomial in p

Let m and n be arbitrary natural numbers,

$\boldsymbol{u} = (u_0, \ldots, u_n, v_0, \ldots, v_{m-1})^{\mathrm{T}}$,

$$f = p^{n+1} + \sum_{i=0}^{n} u_i p^i, \quad g = p^{m+1} + \frac{m}{n} u_n p^m + \sum_{j=0}^{m-1} v_j p^j.$$

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where $X_h = h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$,
yields a system shown at the next slide.

Lax functions polynomial in p: Part II

$$
\begin{aligned}
&(u_k)_t - (v_k)_y + m\,(u_{k-m-1})_z - n\,(v_{k-n-1})_z \\
&+(n+1)\,(v_{k-n})_x - (m+1)\,(u_{k-m})_x \\
&+\sum_{i=0}^{n}\Big\{(k-i-1)v_{k-i}\,(u_i)_z - (i-1)u_i\,(v_{k-i})_z \\
&-(k+1-i)v_{k+1-i}\,(u_i)_x + iu_i\,(v_{k+1-i})_x\Big\} = 0.
\end{aligned}
$$

Here $k = 0, \dots, n+m$, $u_i \stackrel{\text{def}}{=} 0$ for $i > n$ and $i < 0$, $v_j \stackrel{\text{def}}{=} 0$ for $j > m$ and $j < 0$; $v_m \stackrel{\text{def}}{=} (m/n)u_n$.

This is an evolution system in disguise: it can be solved w.r.t. the z-derivatives $(u_i)_z$ and $(v_j)_z$ for all i and j.

Lax functions rational in p

$$\forall m, n \in \mathbb{N} \ \text{ let } \ f = \sum_{i=1}^{m} \frac{a_i}{(p - u_i)}, \quad g = \sum_{j=1}^{n} \frac{b_j}{(p - v_j)},$$

$\boldsymbol{u} = (a_1, \ldots, a_m, u_1, \ldots, u_m, b_1, \ldots, b_n, v_1, \ldots, v_n)^{\mathrm{T}}.$

The associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

where, as before, $X_h = h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$, yields a system for $\boldsymbol{u}$ shown at the next slide that can be brought into Cauchy–Kowalevski form e.g. by passing from t to $T = y + t$ with all other variables intact

Lax functions rational in p: Part II

$$(u_i)_t + \sum_{j=1}^{n} \left\{ \left(\frac{b_j}{v_j - u_i} \right)_x - \left(\frac{b_j u_i}{v_j - u_i} \right)_z - \frac{2 b_j (u_i)_z}{v_j - u_i} \right\} = 0, \quad i = 1, \ldots, m,$$

$$(v_j)_y + \sum_{i=1}^{m} \left\{ - \left(\frac{a_i}{v_j - u_i} \right)_x + \left(\frac{a_i v_j}{v_j - u_i} \right)_z + \frac{2 a_i (v_j)_z}{v_j - u_i} \right\} = 0, \quad j = 1, \ldots, n,$$

$$(a_i)_t + \sum_{j=1}^{n} \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z + \frac{3 a_i (b_j)_z}{v_j - u_i} + \frac{3 a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad i = 1, \ldots, m,$$

$$(b_j)_y + \sum_{i=1}^{m} \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z + \frac{3 a_i (b_j)_z}{v_j - u_i} + \frac{3 a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad j = 1, \ldots, n.$$

Lax functions algebraic in p: an example

Let $\boldsymbol{u} = (u, v, a, b, r, s)^{\mathrm{T}}$,

$$f = \sqrt{p^2 + 2up + 2v},$$
$$g = a + bp + (r + sp)\sqrt{p^2 + 2up + 2v}.$$

The compatibility condition for the associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

where $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$, yields a system shown at the next slide.

Lax functions algebraic in p: an example

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This is the first known example of a $4\mathrm{D}$ integrable system with a nonisospectral Lax pair whose Lax operators are algebraic in the spectral parameter p.

Lax functions algebraic in p: an example cont'd

$$\begin{aligned}
a_y &= -sv_x + ur_x + 2vr_z,\\
b_y &= -su_x + sv_z + r_x + ur_z + us_x + 2vs_z,\\
r_y &= -2wu_x - su_y - 2uwu_z + wv_z - uw_x\\
&\quad +2(v-u^2)w_z + b_x + ub_z,\\
s_y &= wu_z + w_x + uw_z,\\
u_t &= bu_x - 4uwu_x + ru_y - 2usu_y\\
&\quad +(-4u^2w + 2vw + a)u_z + 2wv_x + sv_y\\
&\quad +2uwv_z + 2vw_x - 2u^2w_x + (-4u^3 + 6uv)w_z\\
&\quad -a_x - ua_z + ub_x + (2u^2 - 2v)b_z,\\
v_t &= -4vwu_x - 2vsu_y - 4uvwu_z + bv_x + rv_y\\
&\quad +(2vw + a)v_z - 2uvw_x + 4v(v-u^2)w_z - ua_x\\
&\quad -2va_z + 2vb_x + 2uvb_z.
\end{aligned}$$

More details in AS, Appl. Math. Lett. 92 (2019), 196–200, arXiv:1812.02263

Open questions

? Find examples of Lax functions $f(p, \boldsymbol{u})$ and $g(p, \boldsymbol{u})$ *transcendental* in p such that the associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi), \qquad (*)$$

where $X_h = h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$, yields a $4\mathrm{D}$ integrable system for $\boldsymbol{u}(x, y, z, t)$

Open questions

- Find examples of Lax functions $f(p, \boldsymbol{u})$ and $g(p, \boldsymbol{u})$ *transcendental* in p such that the associated Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi), \qquad (*)$$

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- For a given natural N, where $\boldsymbol{u} = (u^1, \dots, u^N)^{\mathrm{T}}$), classify all pairs of Lax functions $f = f(p, \boldsymbol{u})$ and $g = g(p, \boldsymbol{u})$ such that $(*)$ yield $4\mathrm{D}$ integrable systems

- Can we find any such pairs (f, g) for $N < 4$?

Summary of main results

- Far more integrable $4\mathrm{D}$ systems than it appeared before: infinitely many new ones with Lax pairs of the form
$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$
where $\chi = \chi(x, y, z, t, p)$, $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$, $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$, $X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$
- The first known example of a $4\mathrm{D}$ integrable system with a nonisospectral Lax pair whose Lax operators are algebraic in the spectral parameter p

Main ref.: AS, Lett. Math. Phys. 108 (2018), 359-376 (arXiv:1401.2122)

どうもありがとうございます