Skyrmion crystals and their relatives in SU(3) chiral magnets

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Based on *PRB* **106**, L100406 (2022) and arXiv:23XX.XXXX

Collaborators：Yutaka Akagi, Sven Gudnason, Muneto Nitta, Yakov Shnir

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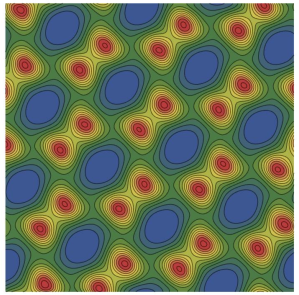
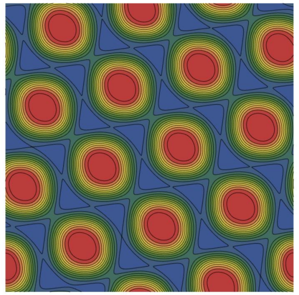
Summary

- We will discuss the ground states in an $SU(3)$ spin system obtained as an effective theory of the spin-orbit coupled spin-1 Bose-Hubbard model.
- **The $SU(3)$ spin systems host various exotic phases**:

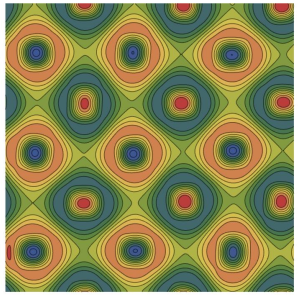
CP^2 Meron crystal



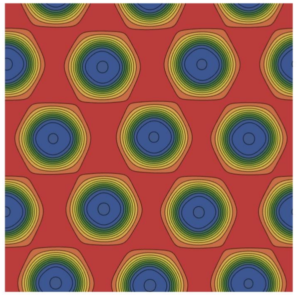
CP^2 Skyrmion crystal



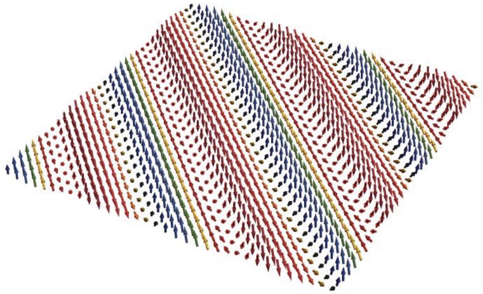
CP^2 Skyrmionium crystal



CP^2 Double-Skyrmion crystal

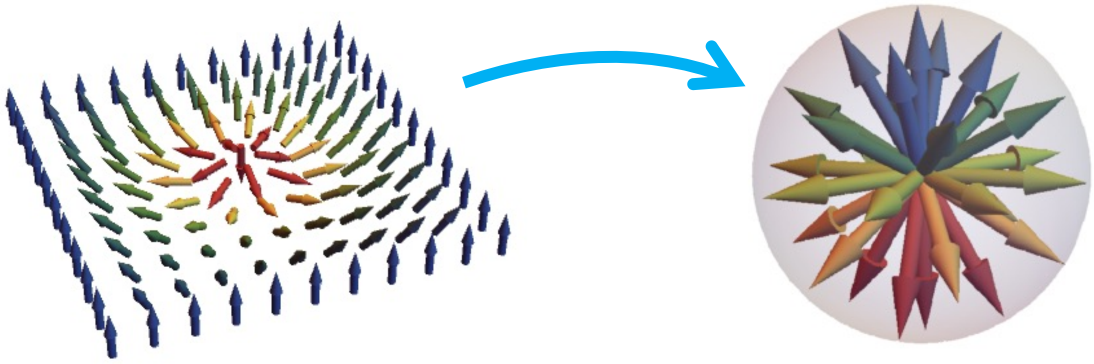


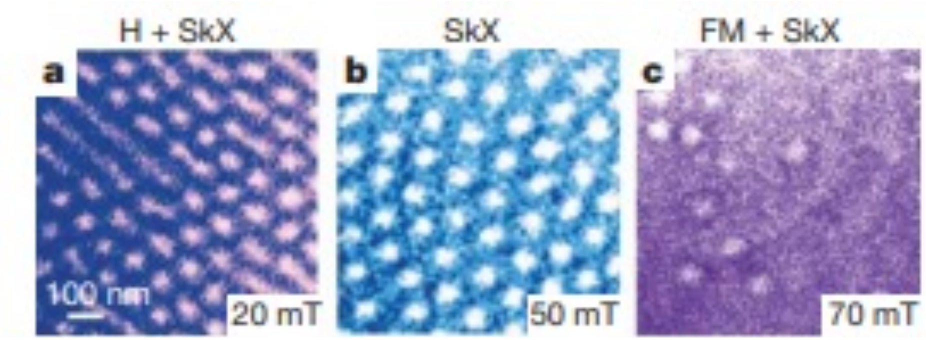
CP^2 Helix



- They possess not only *non-trivial dipole but also quadrupole moment* structures, unlike the standard magnetic Skyrmions.

Motivation – Magnetic Skyrmions





Experiments on a thin film of $Fe_{0.5}Co_{0.5}Si$
[X. Z. Yu et.al., Nature **465**, 901(2010)]

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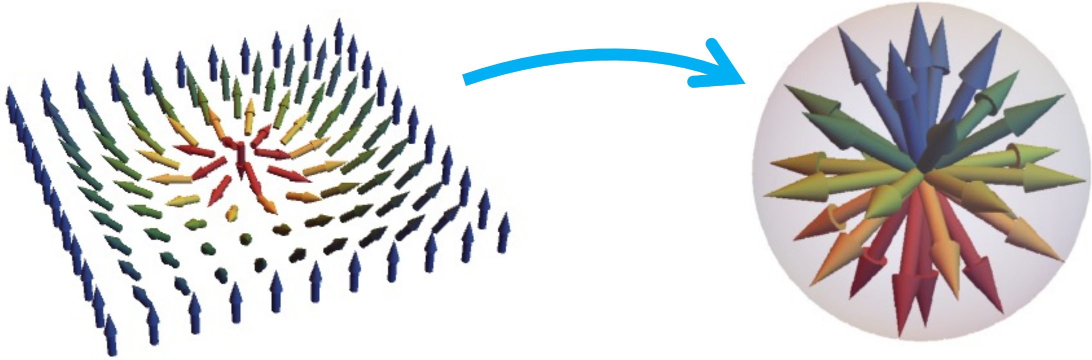
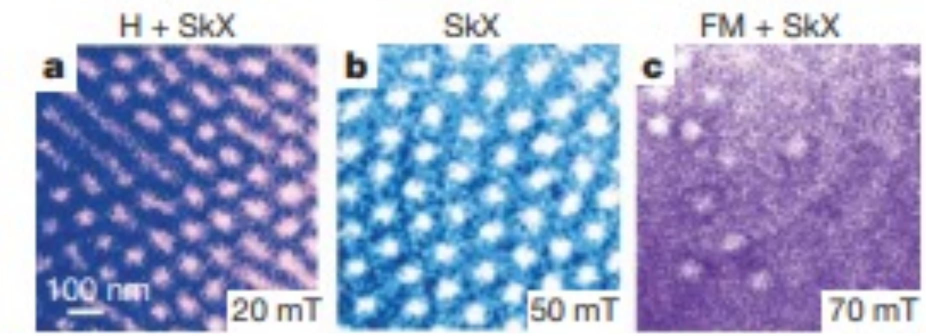




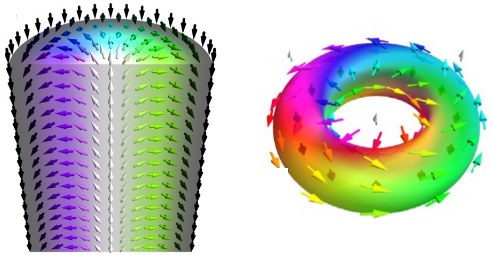
Experiments on a thin film of $\mathrm{Fe_{0.5}Co_{0.5}Si}$
[X. Z. Yu et.al., Nature **465**, 901(2010)]

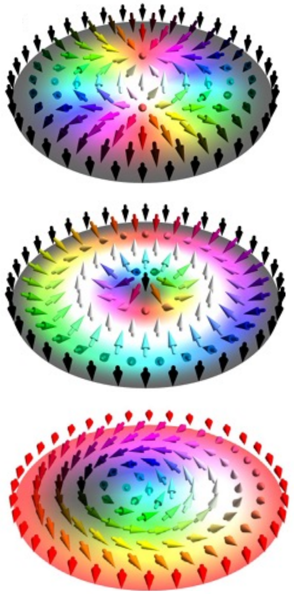
Recent trends

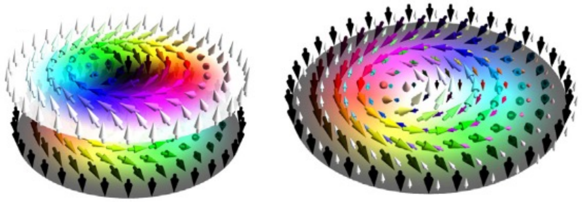
- **3D topological soliton**
  - Skyrmion string
  - Hopfion

    ⋮
- **Composite & Constituent**
  - multi-Skyrmion
  - Skyrmionium
  - fractional Skyrmion (meron)

    ⋮
- **Different surroundings**
  - anti-ferromagnets
  - ferrimagnets
  - $SU(N)$ **magnets**







These figures are taken from
[B. Göbel, I. Mertig, O. Tretiakov, Phys. Rep. **895**, 1 (2021)]

Our model

$$H = \frac{1}{2}\sum_{\langle i,j\rangle}\Big[\ \underbrace{J\,\hat{T}_i^{\alpha}\hat{T}_j^{\alpha}}_{SU(3)} + \underbrace{K\,\hat{S}_i^{a}\hat{S}_j^{a}}_{SU(2)} + \underbrace{2f_{\alpha\beta\gamma}A_{i,j}^{\alpha}\hat{T}_i^{\beta}\hat{T}_j^{\gamma}}_{Generalized\ DM}\ \Big] - h\sum_i \hat{S}_i^{z} \qquad (\ J<0\ )$$

$\hat{T}_i^{\alpha}$: $SU(3)$ spin operator, $\hat{S}_i^{a}$: Spin-1 operator

$$\begin{pmatrix}\hat{T}_i^1\\ \hat{T}_i^2\\ \hat{T}_i^3\\ \hat{T}_i^4\\ \hat{T}_i^5\\ \hat{T}_i^6\\ \hat{T}_i^7\\ \hat{T}_i^8\end{pmatrix} = \begin{pmatrix}(\hat{S}_i^x+\hat{Q}_i^4)/\sqrt{2}\\ (\hat{S}_i^y+\hat{Q}_i^6)/\sqrt{2}\\ (\hat{S}_i^z+\sqrt{3}\,\hat{Q}_i^8)/2\\ \hat{Q}_i^3\\ \hat{Q}_i^1\\ (\hat{S}_i^x-\hat{Q}_i^4)/\sqrt{2}\\ (\hat{S}_i^y-\hat{Q}_i^6)/\sqrt{2}\\ (\sqrt{3}\hat{S}_i^z-\hat{Q}_i^8)/2\end{pmatrix}$$

$$\hat{\boldsymbol{S}}_i = \left(\frac{\hat{T}_i^1+\hat{T}_i^6}{\sqrt{2}}, \frac{\hat{T}_i^2+\hat{T}_i^7}{\sqrt{2}}, \frac{\hat{T}_i^3+\sqrt{3}\hat{T}_i^8}{2}\right)$$

$\hat{Q}_i^{\alpha} = \frac{1}{2}\mathrm{Tr}(\hat{Q}_i\lambda^{\alpha})$ with quadrupole tensor $\hat{Q}_j = \hat{\boldsymbol{S}}_j \otimes \hat{\boldsymbol{S}}_j^{\mathrm{T}} - \frac{2}{3}\mathbf{1}$

Cf. Order parameter of nematic liquid crystal $Q = \boldsymbol{d}\otimes\boldsymbol{d}^{\mathrm{T}} - \frac{2}{3}\mathbf{1}$

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Spin-1 Bilinear-Biquadratic (BBQ) model

$$H_{\mathrm{BBQ}} = \sum_{\langle i,j\rangle}\left[(2J+K)\hat{\boldsymbol{S}}_i\cdot\hat{\boldsymbol{S}}_j + 2J\left(\hat{\boldsymbol{S}}_i\cdot\hat{\boldsymbol{S}}_j\right)^2\right]$$

- $NiGa_2S_4$ [Tsunetsugu, Arikawa, JPSJ **75**, 083701(2006)] [Läuchli, Mila, Penc, PRL **97**, 087205 (2006)]
- Spinor BEC [Imambekov, Lukin, Demler, PRA **68**, 063602 (2003)]

	^{7}Li	^{23}Na	^{41}K	^{87}Rb
K/J	0.912	-0.0625	0.0512	0.00925

Our model

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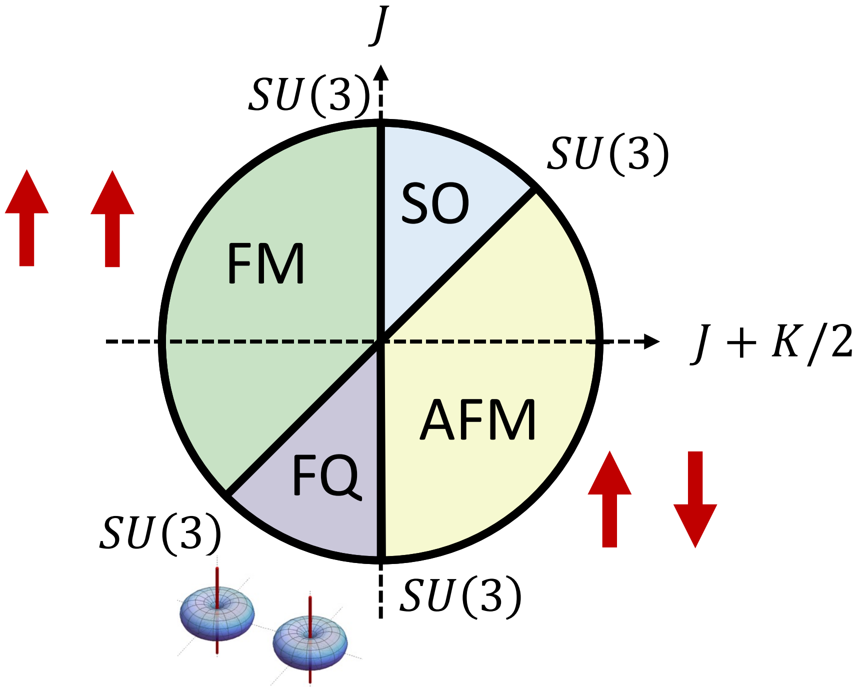
Mean-field phase diagram (square lattice)



[N. Papanicolaou, Nucl. Phys. B **305**, 367]

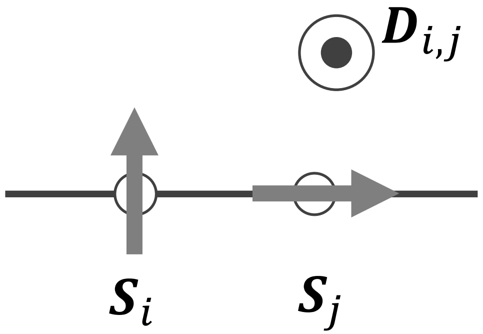
Our model

SU(3) structure constant

$$H = \frac{1}{2}\sum_{\langle i,j\rangle}\Big[\underbrace{J\,\hat{T}_i^{\alpha}\hat{T}_j^{\alpha}}_{SU(3)} + \underbrace{K\,\hat{S}_i^{a}\hat{S}_j^{a}}_{SU(2)} + \underbrace{2f_{\alpha\beta\gamma}A_{i,j}^{\alpha}\hat{T}_i^{\beta}\hat{T}_j^{\gamma}}_{\text{Generalized DM}}\Big] - h\sum_i \hat{S}_i^z \qquad (\,J<0\,)$$

Dzyaloshinskii-Moriya (DM) interaction

$$H_{\mathrm{DM}} = \sum_{\langle i,j\rangle}\boldsymbol{D}_{i,j}\cdot\left(\hat{\boldsymbol{S}}_i\times\hat{\boldsymbol{S}}_j\right) = \sum_{\langle i,j\rangle}\varepsilon^{abc}D_{i,j}^{a}\hat{S}_i^{b}\hat{S}_j^{c}$$



Our model

$SU(3)$ *structure constant*

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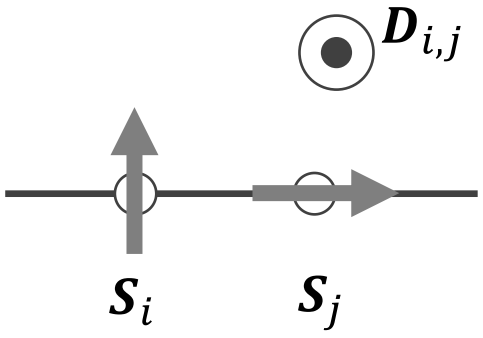

- We should introduce an interaction to evade the Derrick theorem.
- A DM-like term will induce Skyrmion crystal as the ground state.
- The generalized DM term is the $SU(3)$ extension of the standard DM term.

CP^2 Skyrmions with other type of stabilizers:

- *Long-range coulomb* (excitation, crystal) [D. L. Kovrizhin, B. Doucot, R. Moessner, PRL **110**, 186802 (2013)]
- *Skyrme term* (excitation, fractional skyrmion molecule) [Y. Akagi, **YA**, et al. JHEP **11**, 194 (2021)]
- *Frustration* (ground state, crystal) [H. Zhang, C. Batista, et al. Nat. Commun. **14**, 3626 (2023)]

Our model

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This model is an effective theory of the Spin-1 Bose-Hubbard model with spin-orbit coupling

$$H_{\mathrm{BH}} = -t\sum_{\langle ij\rangle}\left[\hat{b}_{i,\sigma}^{\dagger}\left(e^{iA_{i,j}}\right)_{\sigma\rho}\hat{b}_{j,\rho} + \mathrm{H.c.}\right] + \frac{1}{2}\sum_i\left[U_0\,\hat{n}_i(\hat{n}_i - 1) + U_2\left(\hat{\boldsymbol{S}}_i^2 + 2\hat{n}_i\right)\right] - h\sum_i \hat{S}_i^z$$

Our model

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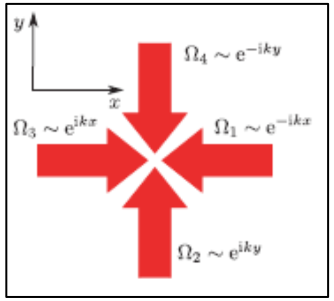
It has been theoretically proposed that the following gauge potential can be induced by applying laser beam into cold-atom gases. [Juzeliūnas, Ruseckas, Dalibard, PRA **81**, 053403 (2010)]



$$A_{i,i\pm\boldsymbol{e}_x} = \pm\frac{\kappa}{\sqrt{2}}\tau^x,\quad A_{i,i\pm\boldsymbol{e}_y} = \pm\frac{\kappa}{\sqrt{2}}\tau^y$$

τ^a: Spin-1 matrix

We work with these from now on.

Our model

$$H = \frac{1}{2}\sum_{\langle i,j\rangle}\Big[\underbrace{J\,\hat{T}_i^{\alpha}\hat{T}_j^{\alpha}}_{SU(3)} + \underbrace{K\,\hat{S}_i^{a}\hat{S}_j^{a}}_{SU(2)} + \underbrace{2f_{\alpha\beta\gamma}A_{i,j}^{\alpha}\hat{T}_i^{\beta}\hat{T}_j^{\gamma}}_{Generalized\ DM}\Big] - h\sum_i \hat{S}_i^z \qquad (\,J<0\,)$$

We study the ground state of the Hamiltonian with mean-field approximation.

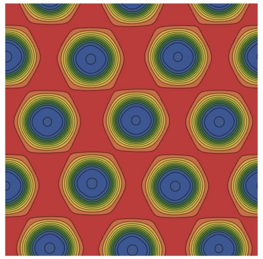
***SU*(3) spin coherent state** $|Z\rangle = \otimes_j |Z_j\rangle$ with $|Z_j\rangle = Z_j^{\sigma}|\sigma\rangle_j$ $\qquad \hat{S}_j^z|\sigma\rangle_j \equiv \sigma|\sigma\rangle_j$

- At the single site level, $|Z_j\rangle$ can describe any spin-1 state at site j.
- $\boldsymbol{Z}_j = \left(Z_j^{+1}, Z_j^{0}, Z_j^{-1}\right)^{\mathrm{T}}$ takes its value on $S^5/S^1 = SU(3)/U(2) = CP^2$.
- $\hat{T}_j^{\alpha} = (\lambda^{\alpha})_{\sigma\rho}|\sigma\rangle_j\langle\rho|_j \Rightarrow \langle\hat{T}_i^{\alpha}\rangle = \boldsymbol{Z}_j^{\dagger}\lambda^{\alpha}\boldsymbol{Z}_j$

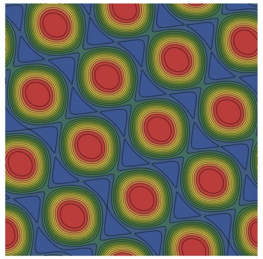
Topological charge $\quad N = -\frac{1}{64\pi}\sum_{\langle ijk\rangle} f_{\alpha\beta\gamma}\langle\hat{T}_i^{\alpha}\rangle\langle\hat{T}_j^{\beta}\rangle\langle\hat{T}_k^{\gamma}\rangle \in \pi_2(CP^2) = \mathbb{Z}$

Ground state phase diagram

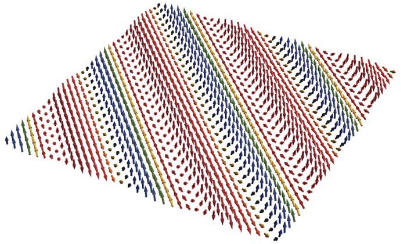
CP^2 *Double-Skyrmion crystal* (D-SkX)



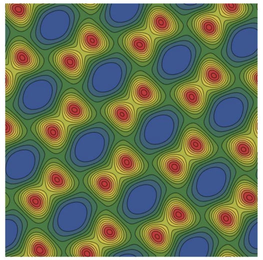
CP^2 *Skyrmion crystal* (SkX)



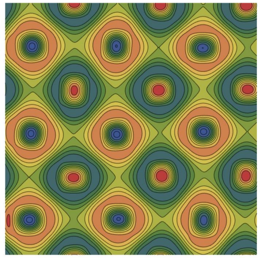
CP^2 *Helimagnetic* (HM)



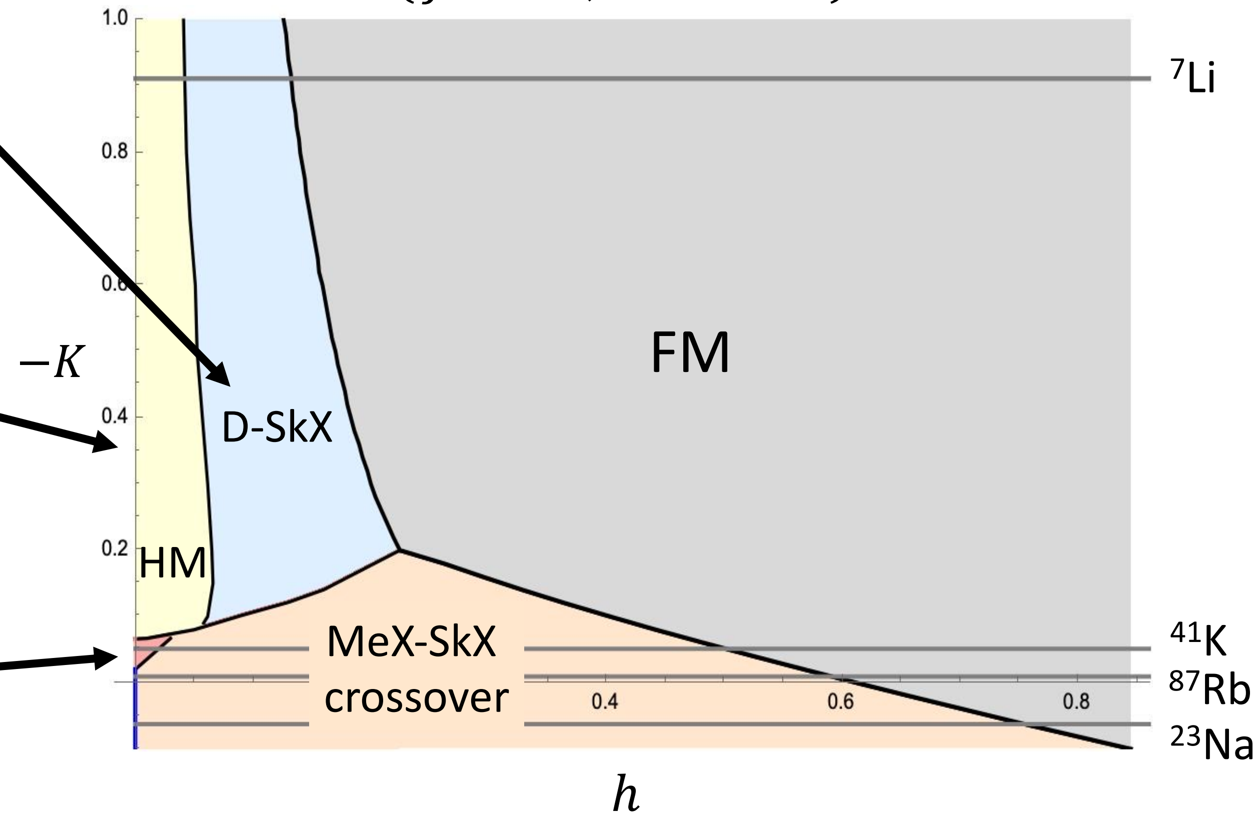
CP^2 *Meron crystal* (MeX)



CP^2 *Skyrmionium crystal*



$(J = -1, \kappa = -0.4)$



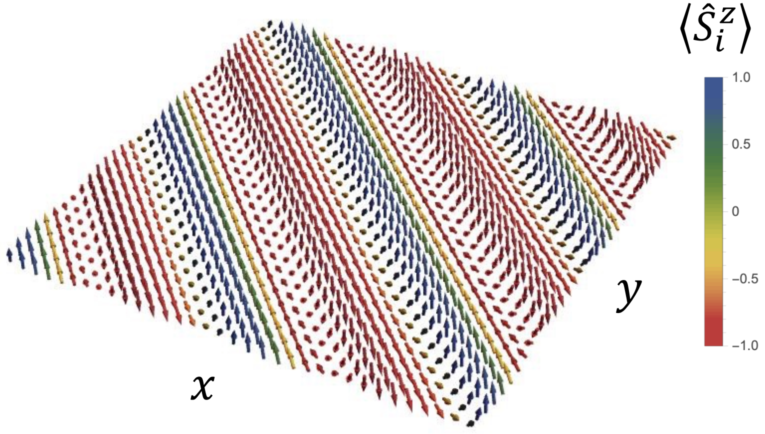
with *simulated annealing method,*
periodic boundary condition, square lattice

[**YA**, Y. Akagi, et al.,
PRB **106**, L100406 (2022)]

CP^2 Helical structure

- Helical structure with small modulation along the stripes
- Single q-state both in $\mathcal{S}(\boldsymbol{q})$ and $\mathcal{Q}(\boldsymbol{q})$

Magnetization vectors $\langle \hat{\boldsymbol{S}}_i \rangle$



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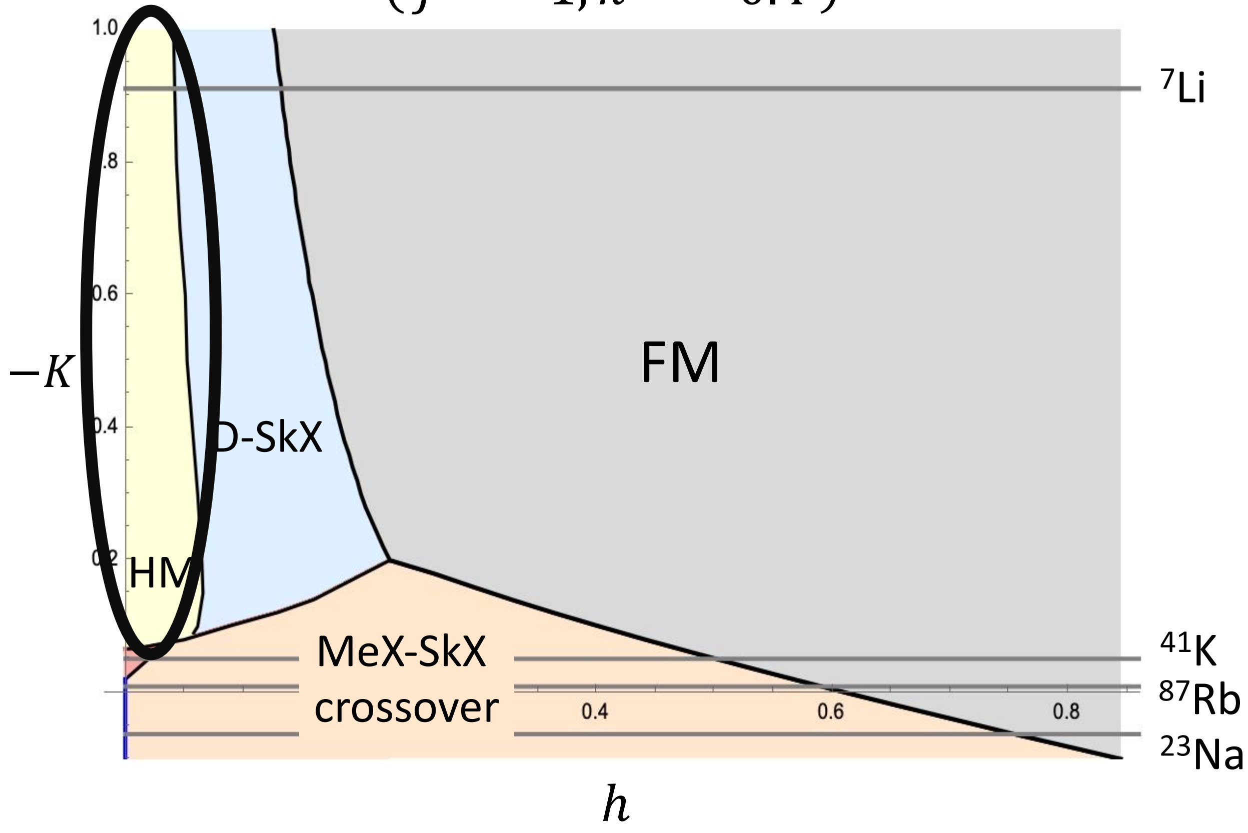

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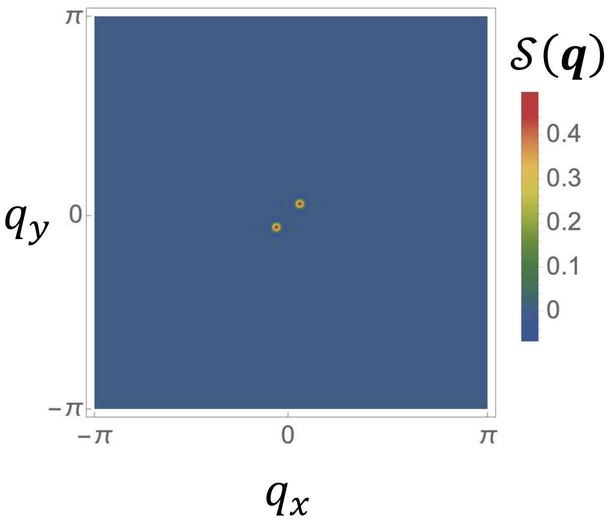
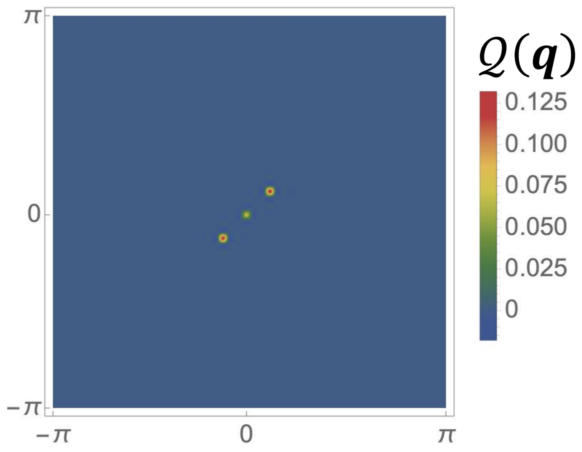
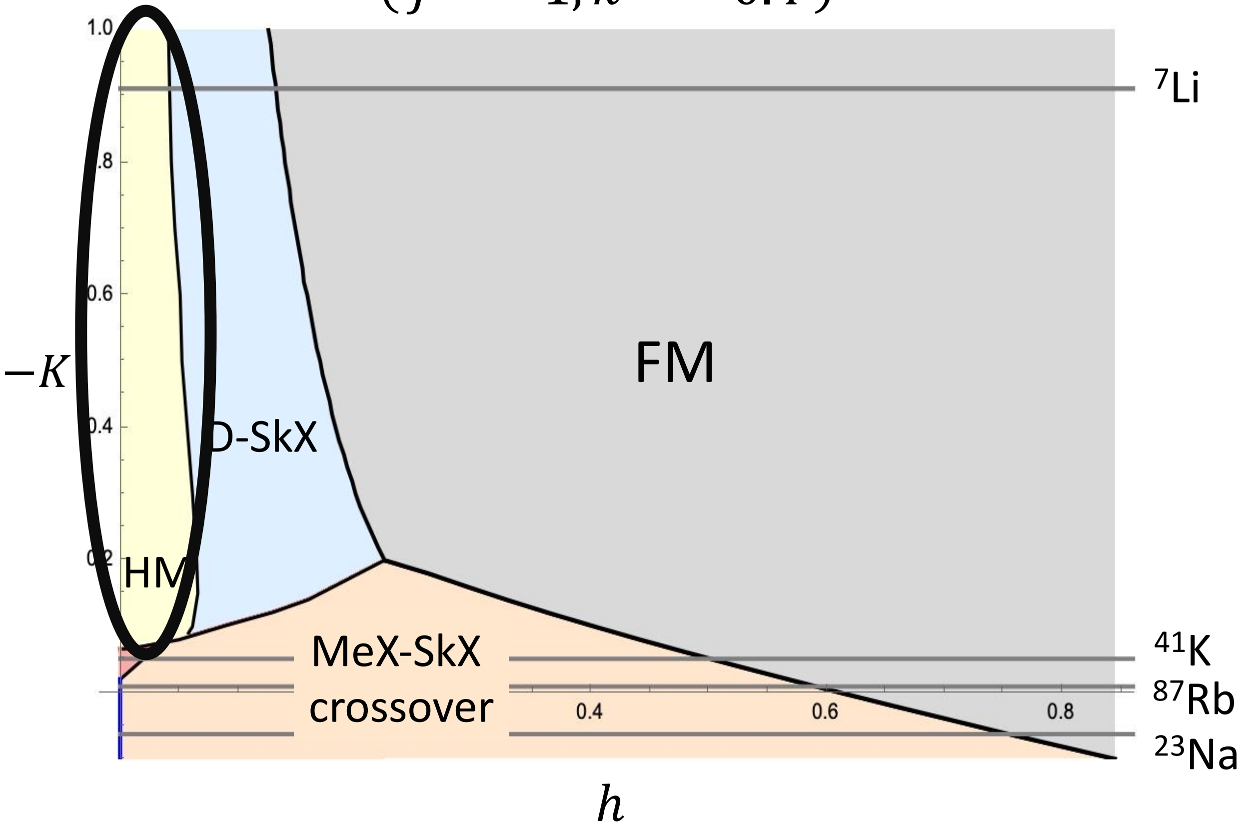
Spin Structure Factor



Quadrupole Structure Factor



$(J = -1, \kappa = -0.4)$

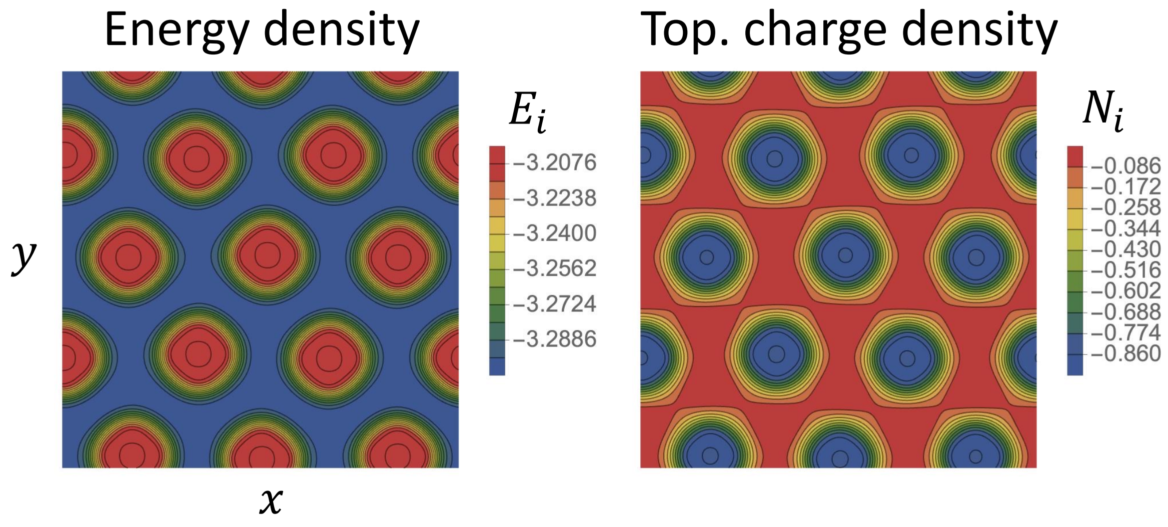


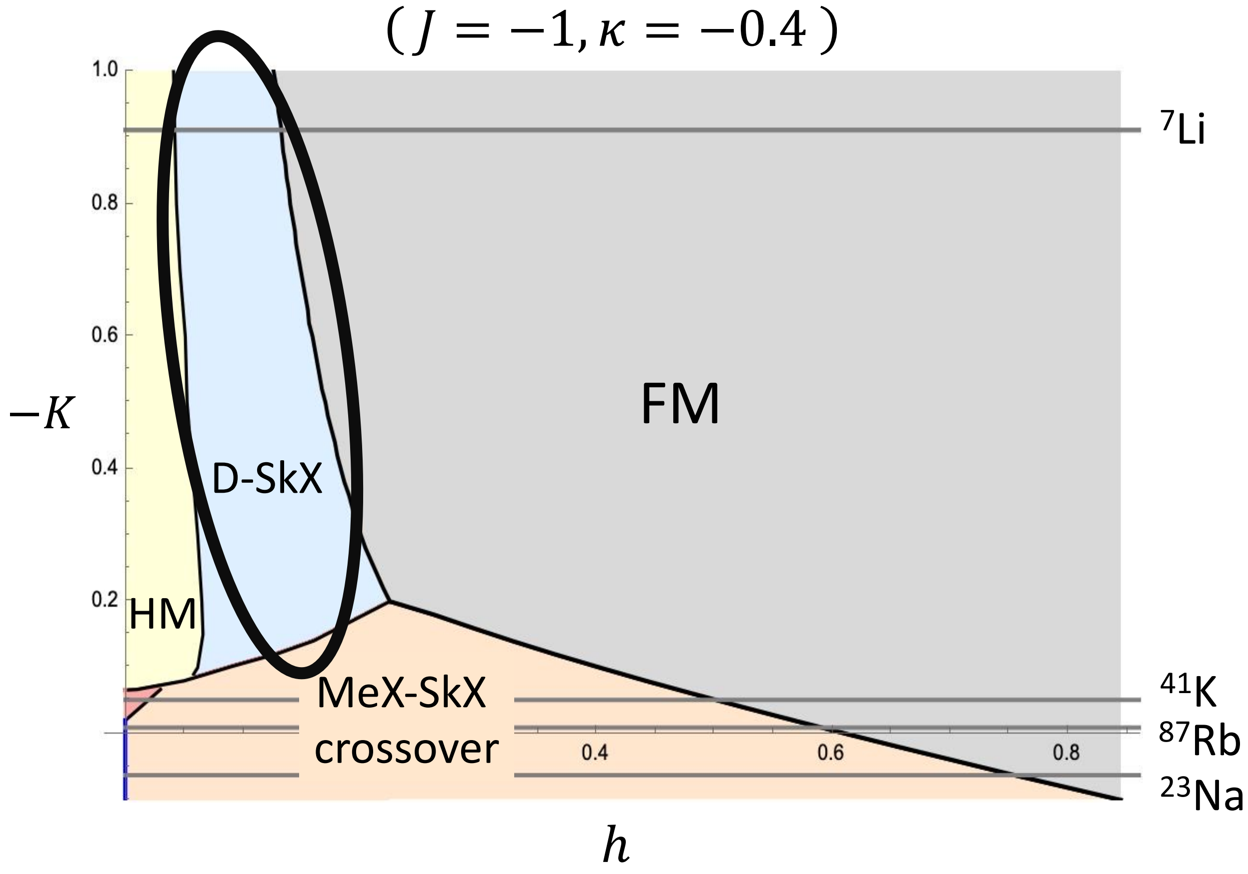
$$\mathcal{S}(\boldsymbol{q}) = \mathcal{N}^{-2} \sum_{j,k} \langle \widehat{\boldsymbol{S}}_j \rangle \cdot \langle \widehat{\boldsymbol{S}}_k \rangle \, e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_k)}$$

$$\mathcal{Q}(\boldsymbol{q}) = \mathcal{N}^{-2} \sum_{j,k} \frac{1}{2} \mathrm{Tr}\left(\langle \hat{Q}_j \rangle \langle \hat{Q}_k \rangle \right) e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_k)}$$

CP^2 Double-Skyrmion crystal

- Triangular lattice of $N = -2\ CP^2$ Skyrmions
- Magnetic Skyrmion-like magnetic structure
- But, non-trivial quandrupole structure





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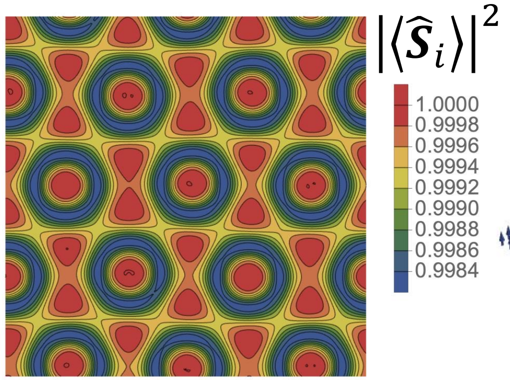
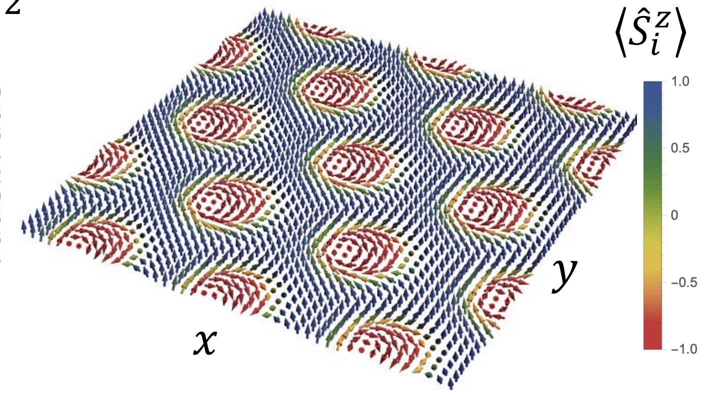
Norm of spins



Magnetization vectors



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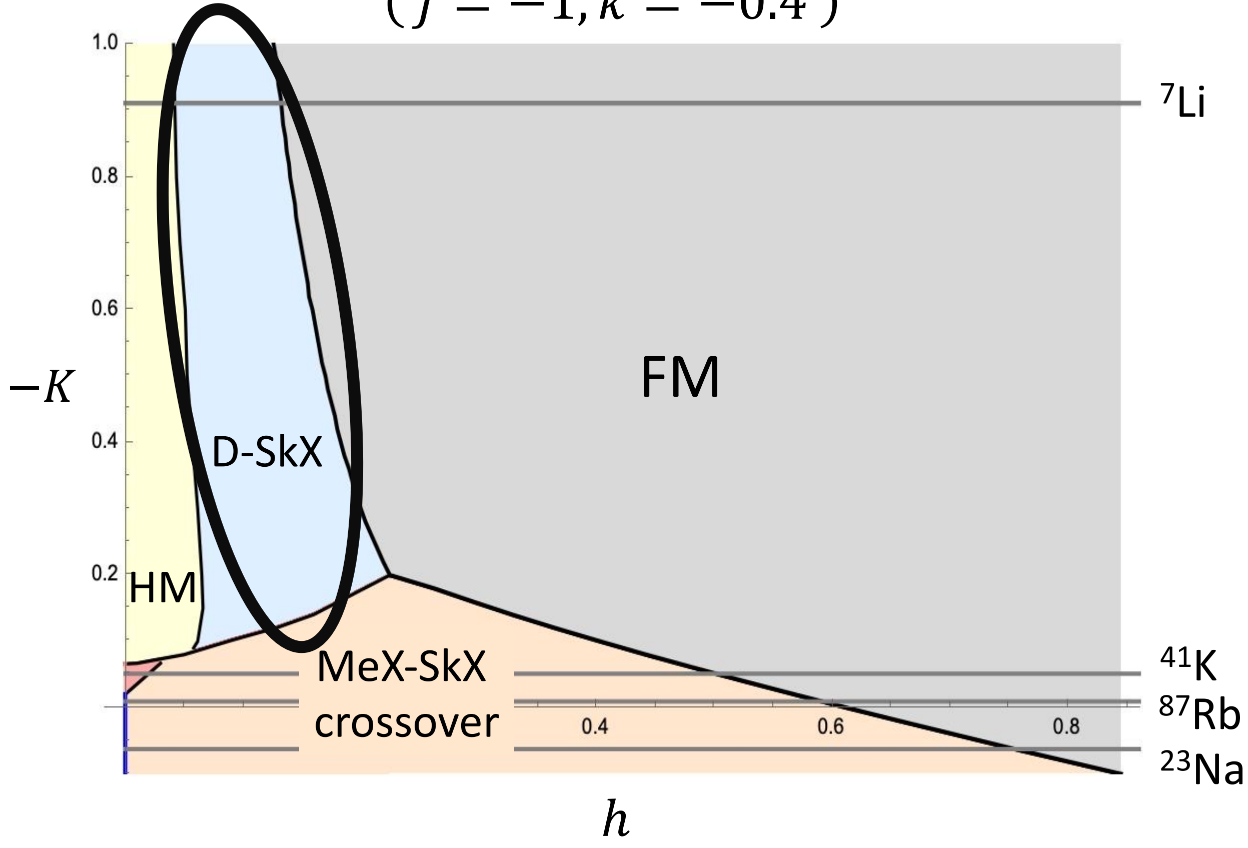

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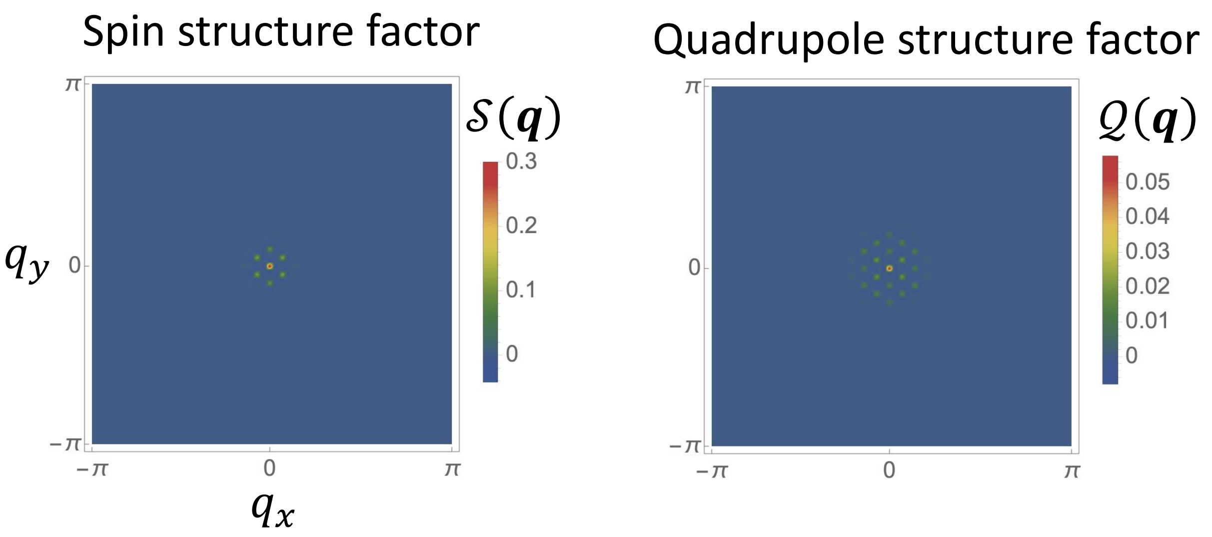
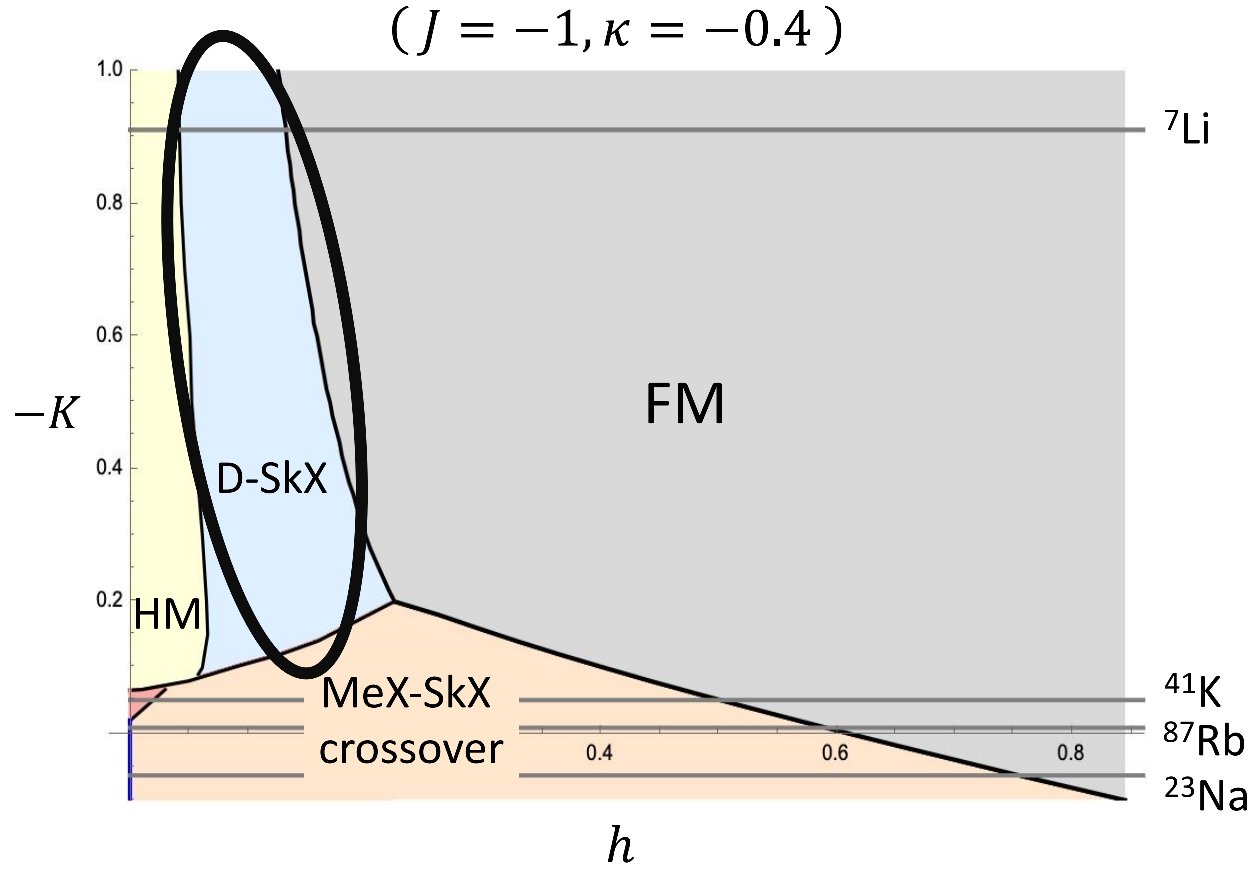


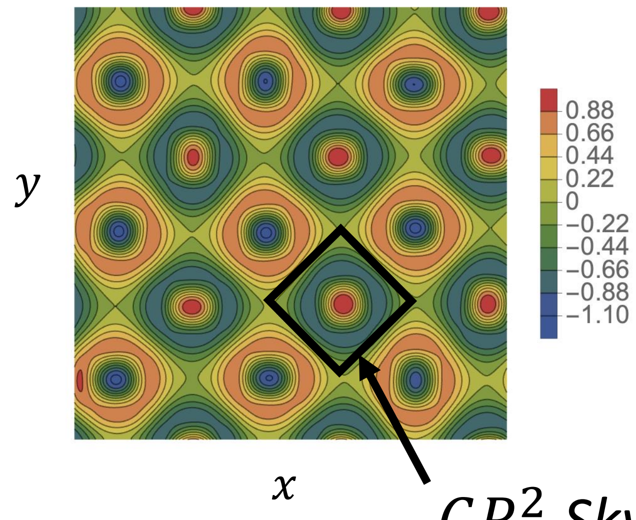


CP^2 Skyrmionium crystal

= Skyrmion surrounded by an anti-Skyrmion

- Harlequin pattern of CP^2 Skyrmioniums
- Spin nematic realize outside of Skyrmioniums
- Double q-structure in $\mathcal{S}(\boldsymbol{q})$ and $\mathcal{Q}(\boldsymbol{q})$

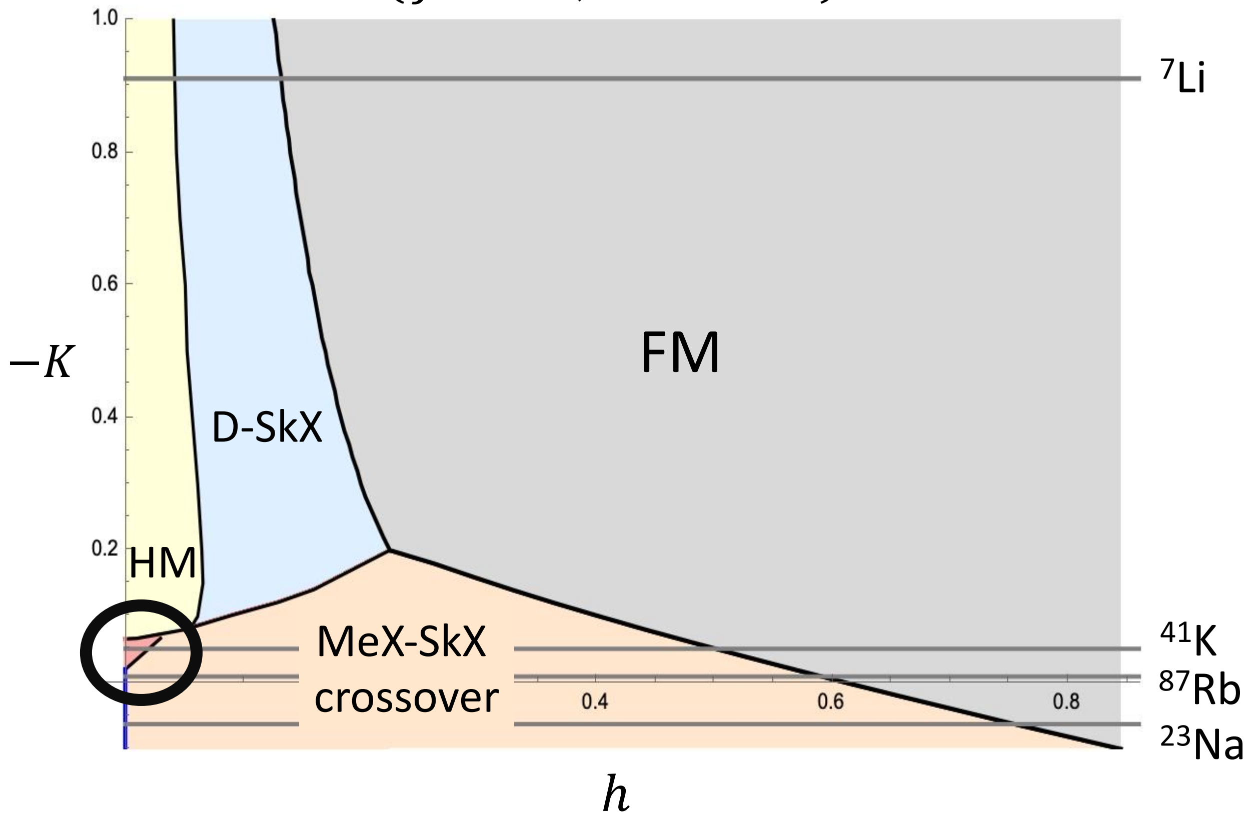
Top. charge density



CP^2 *Skyrmionium*

Cf. harlequin pattern

$(J = -1, \kappa = -0.4)$



CP^2 Skyrmionium crystal

‖ Skyrmion surrounded by an anti-Skyrmion

- Harlequin pattern of CP^2 Skyrmioniums
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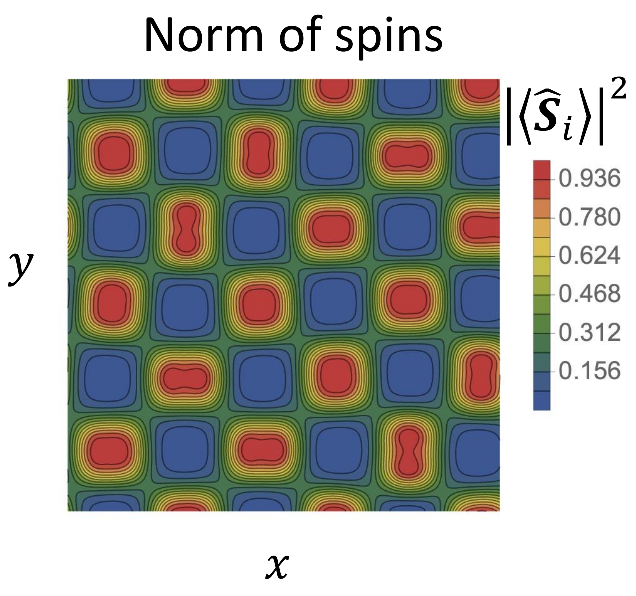

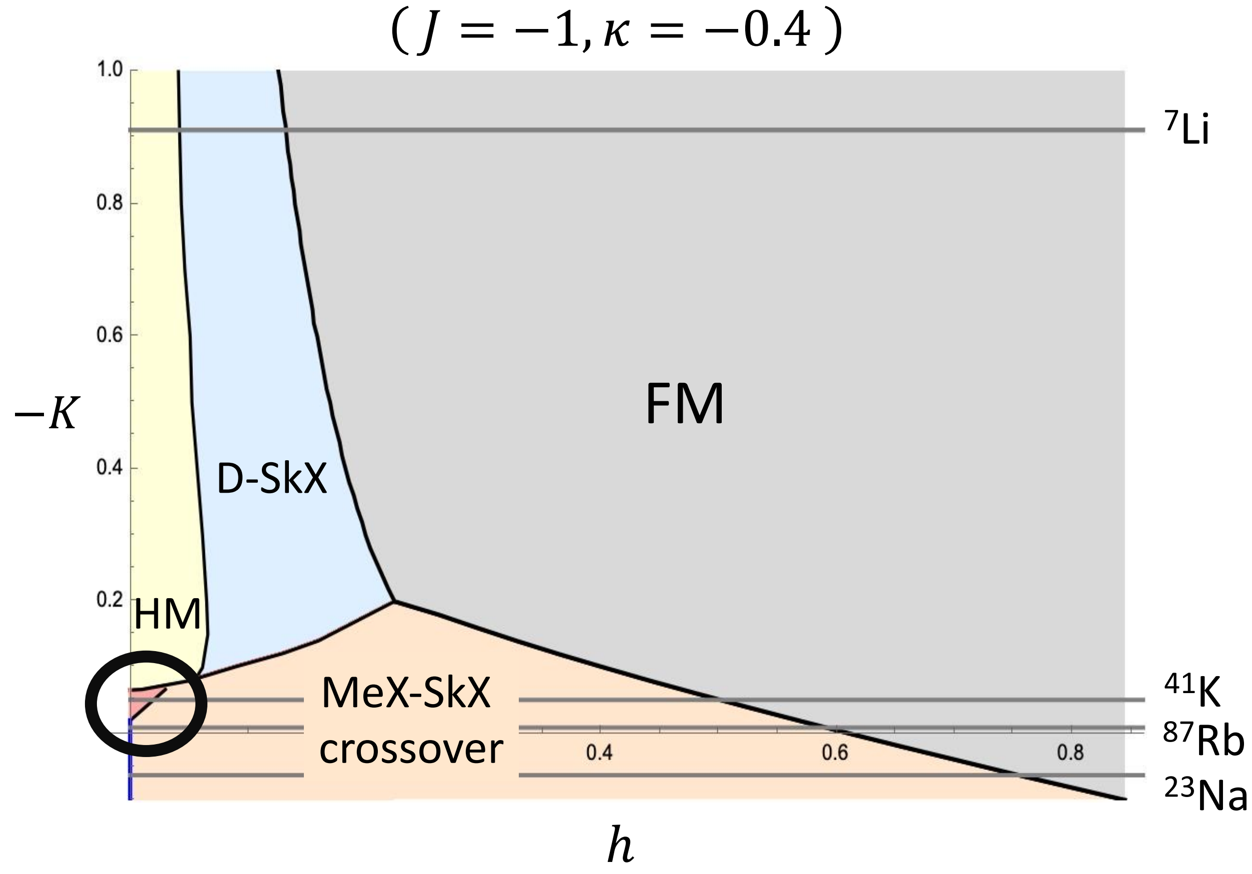

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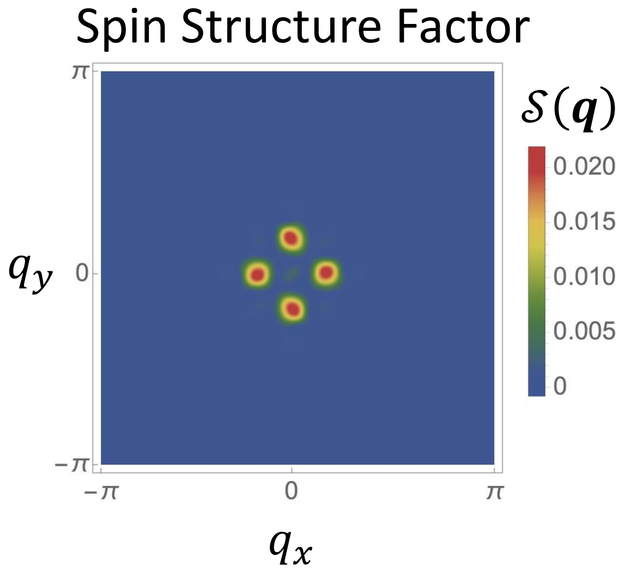
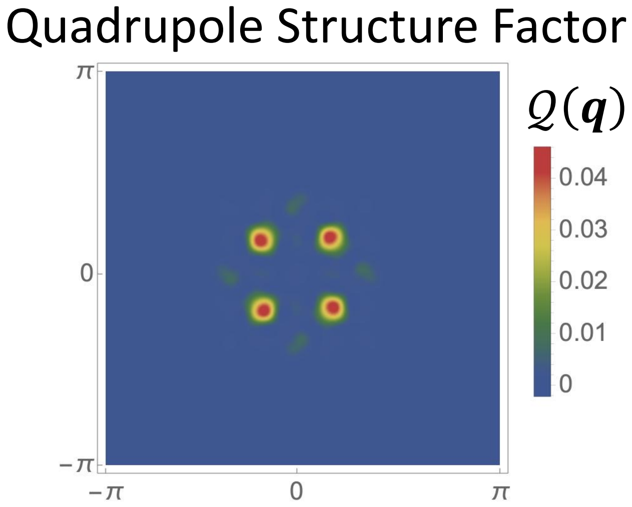
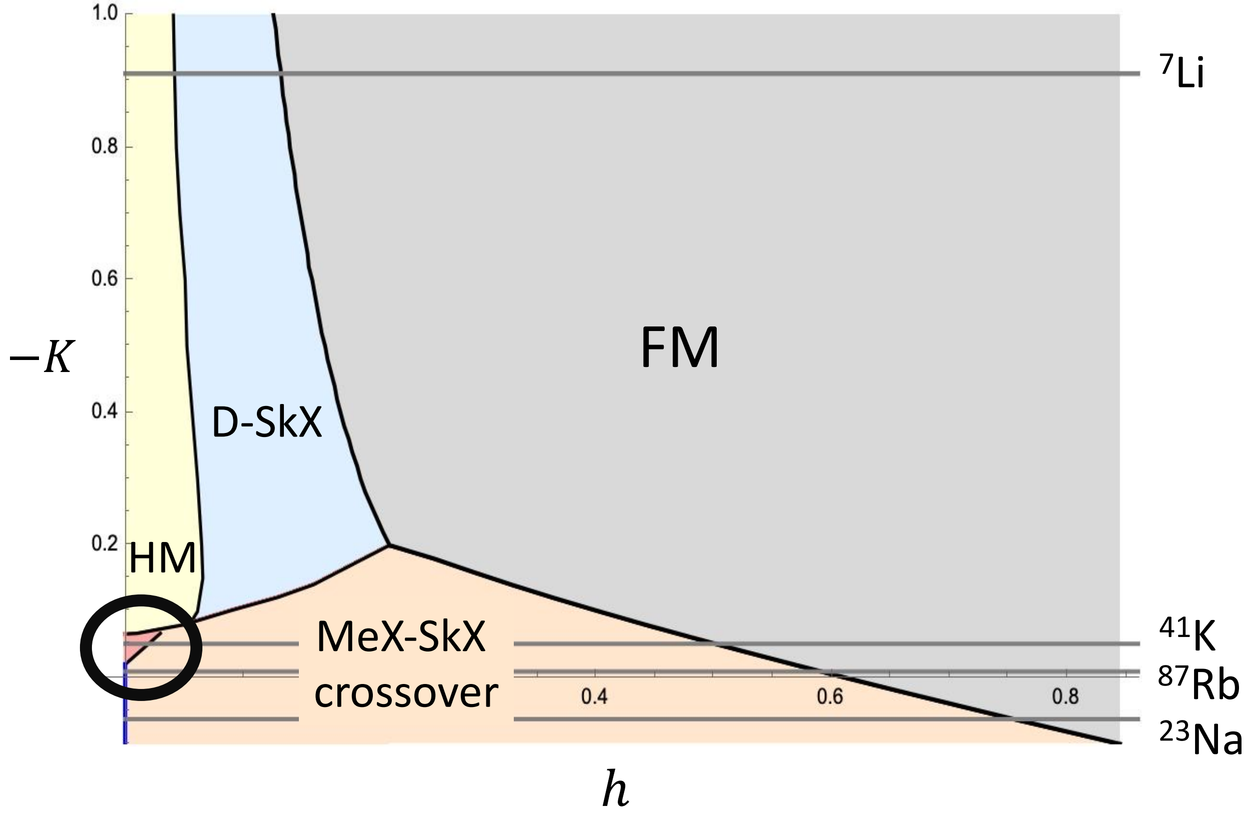


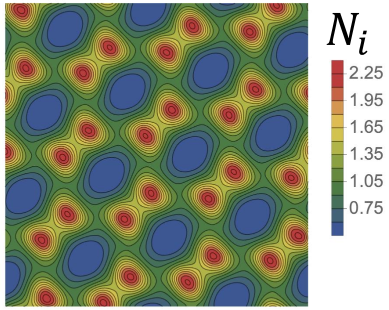




Meron crystal – Skyrmion crystal crossover

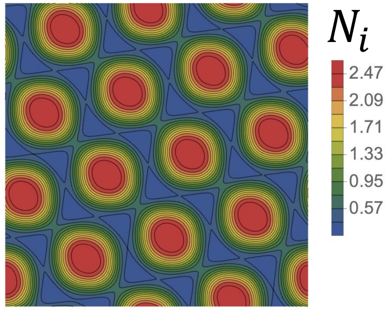
- Meron crystal = honeycomb lattice of merons
  Skyrmion crystal = triangular lattice of Skyrmions
- Spin nematic state is realized at the core of Skyrmions and outside of merons.
- Triple q-structure in $\mathcal{S}(\boldsymbol{q})$ and $\mathcal{Q}(\boldsymbol{q})$
- These state are smoothly connected.

Meron crystal



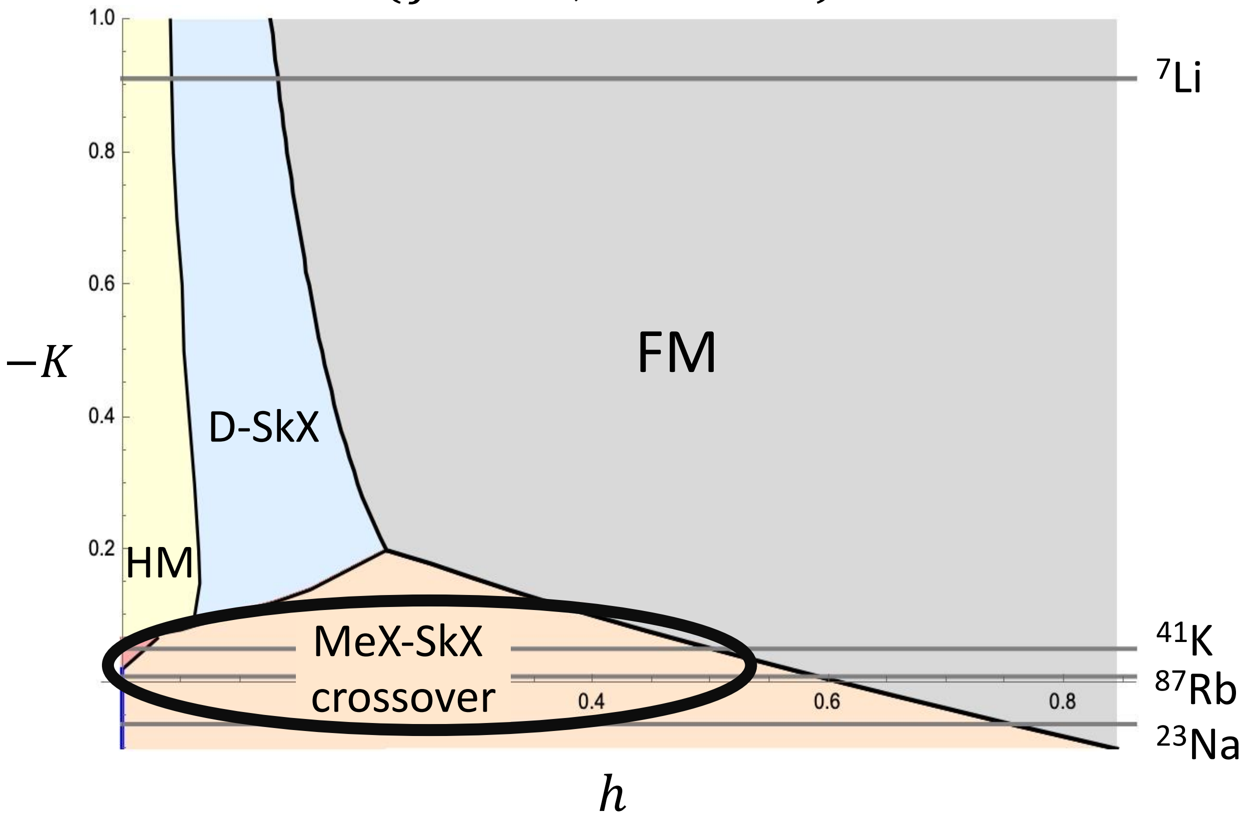
$h = 0.05$

Skyrmion crystal



$h = 0.45$

$(J = -1, \kappa = -0.4)$



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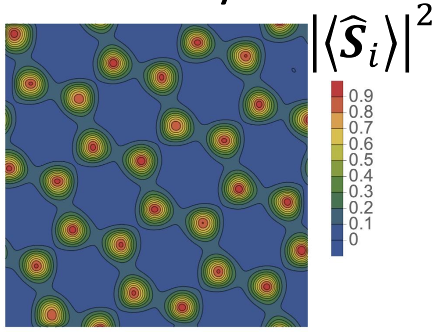

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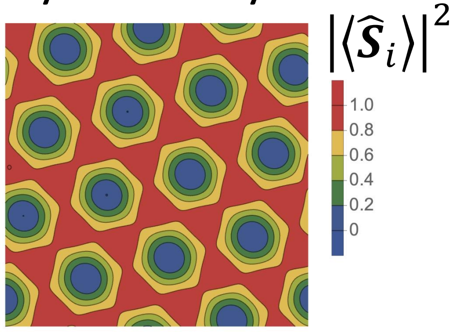

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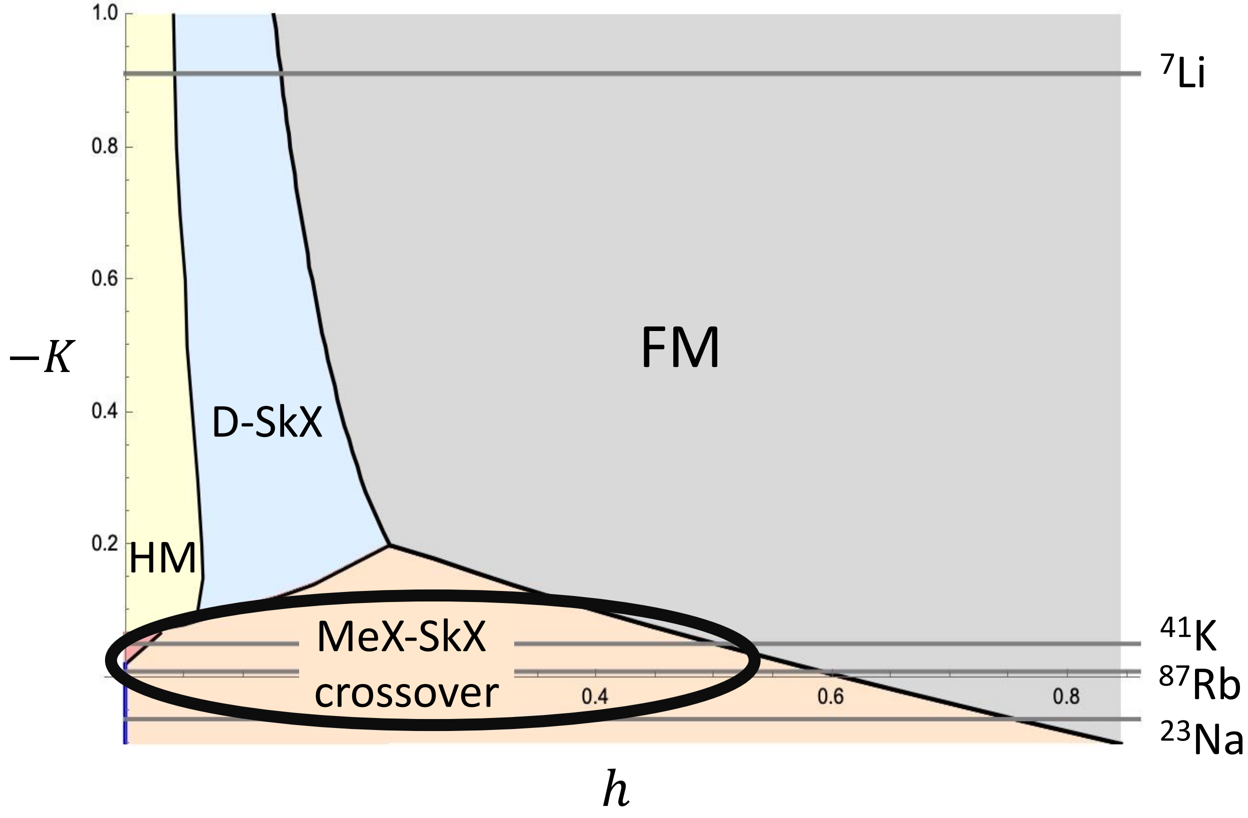

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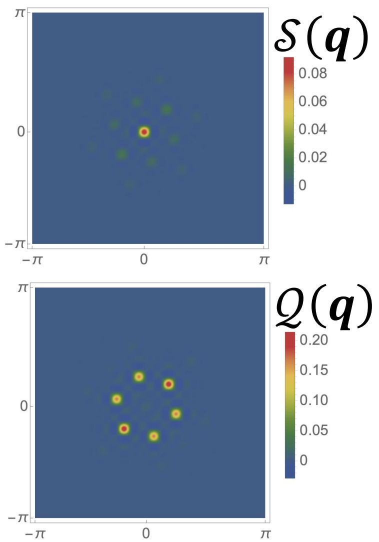
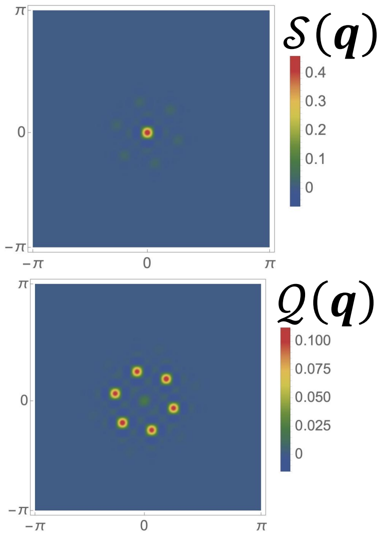
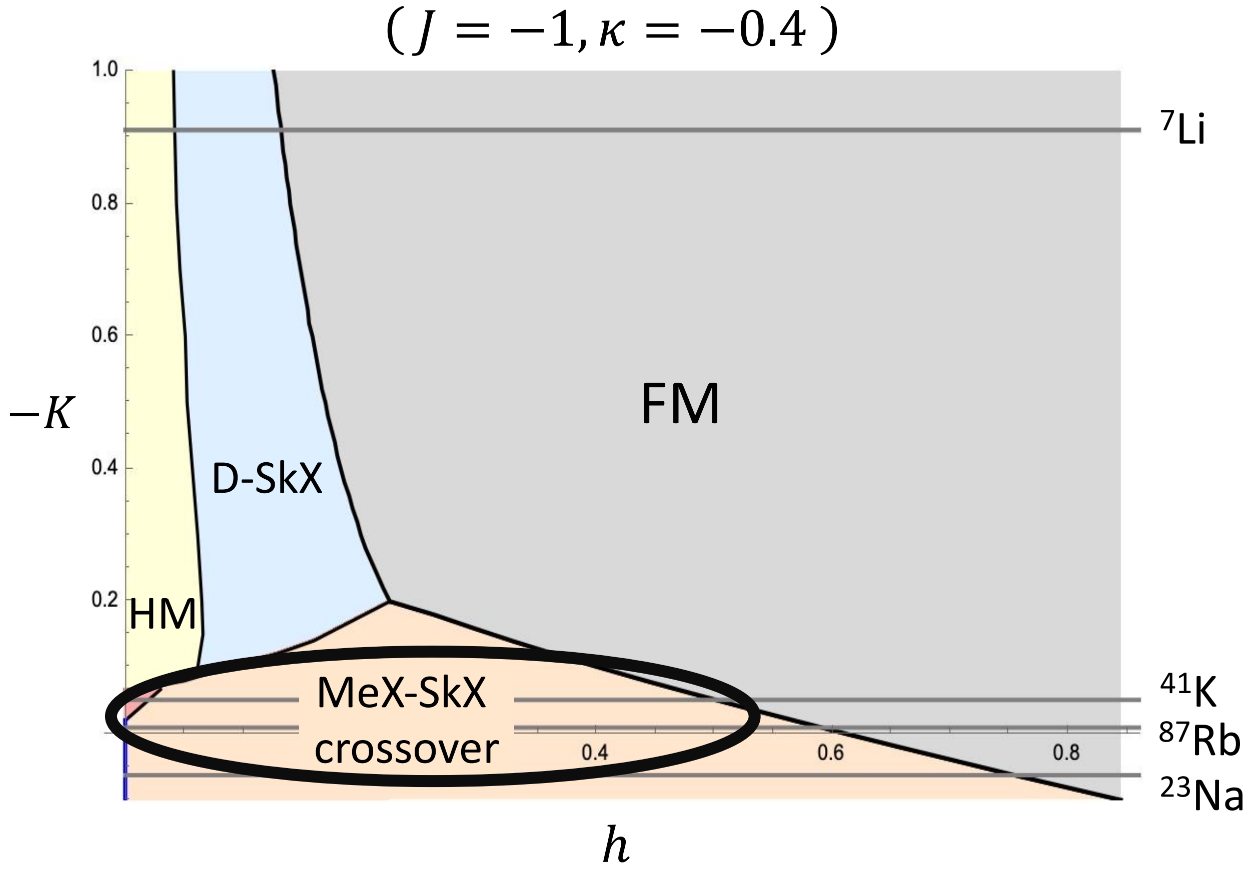


Skyrmion crystal





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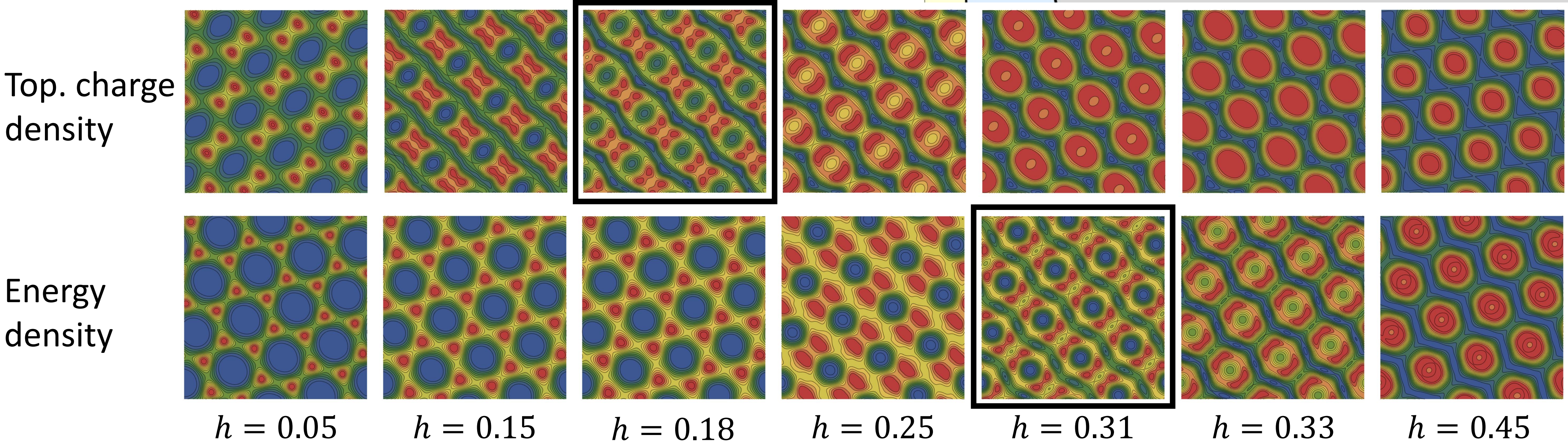
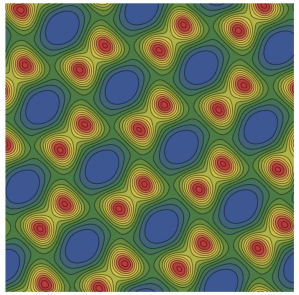


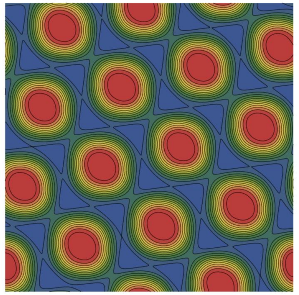
Summary

- We have studied the ground states in an $SU(3)$ spin system obtained as an effective theory of the spin-orbit coupled spin-1 Bose-Hubbard model.
- **The $SU(3)$ spin systems host various exotic phases**:

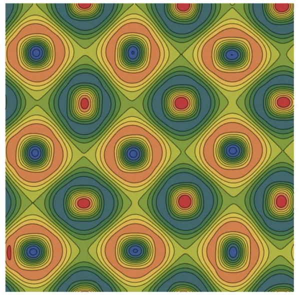
CP^2 Meron crystal



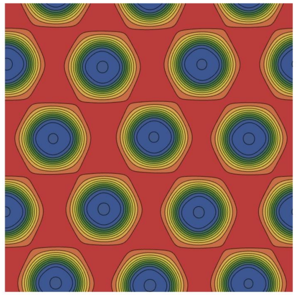
CP^2 Skyrmion crystal



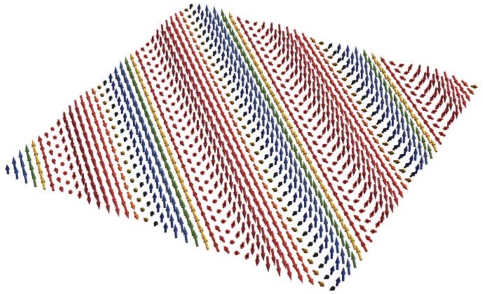
CP^2 Skyrmionium crystal



CP^2 Double-Skyrmion crystal



CP^2 Helix

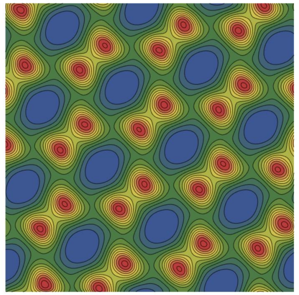


- They possess not only *non-trivial dipole but also quadrupole moment* structures, unlike the standard magnetic Skyrmions.

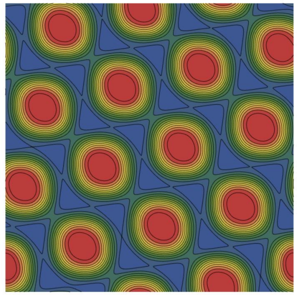
Summary

- ✓ We have studied the ground states in an $SU(3)$ spin system obtained as an effective theory of the spin-orbit coupled spin-1 Bose-Hubbard model.
- ✓ **The $SU(3)$ spin systems host various exotic phases**:

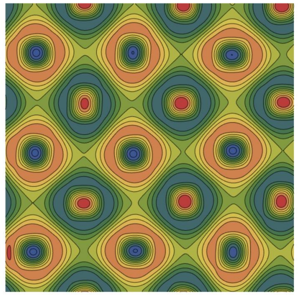
CP^2 *Meron crystal*



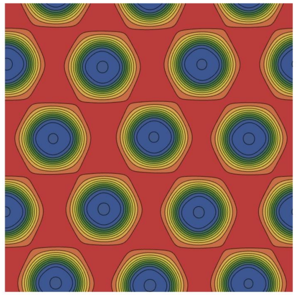
CP^2 *Skyrmion crystal*



CP^2 *Skyrmionium crystal*



CP^2 *Double-Skyrmion crystal*



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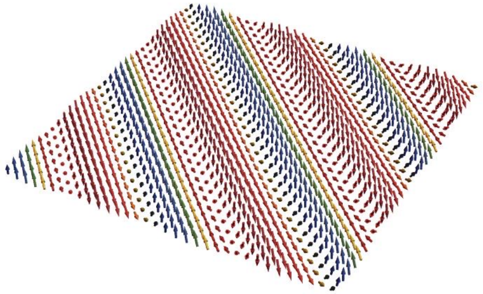

- ✓ They possess not only *non-trivial dipole but also quadrupole moment* structures, unlike the standard magnetic Skyrmions.

Thank you for your attention! Did you find your favorite phase?

Numerical method

Simulated Annealing + Conjugate Gradient → $\frac{\partial n_i^\alpha}{\partial \bar{Z}_i^\sigma}\frac{\partial E}{\partial n_i^\alpha} - \Lambda_i Z_i^\sigma = 0$

$SU(2)$ spin case : target sp. = S^2

$$\boldsymbol{m} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \rightarrow dS = \sin\theta d\theta \wedge d\phi$$

$$\boldsymbol{m} = \left(\sqrt{1-f^2}\cos\phi, \sqrt{1-f^2}\sin\phi, f\right) \rightarrow dS = df \wedge d\phi$$ more efficient

$SU(3)$ spin case : target sp. = $CP^2 \approx S^5/S^1$ $\quad (\dim(CP^2) = 4)$

$$\boldsymbol{Z} = \left(\sin f \cos g\, e^{i\phi}, \sin f \sin g\, e^{i\psi}, \cos f\right)^T$$

$$ds^2 = \frac{1}{4} dn^\alpha dn^\alpha = \boldsymbol{v}^T G \boldsymbol{v}$$ where $\boldsymbol{v} = (df, dg, d\phi, d\psi)$

➡ $$dV = \sqrt{\det G}\, df \wedge dg \wedge d\phi \wedge d\psi = \frac{1}{8} d(\sin^4 f) \wedge d(\sin^2 g) \wedge d\phi \wedge d\psi$$

$$\boldsymbol{Z} = \left(\sqrt[4]{\eta}\sqrt{1-\xi}\, e^{i\phi}, \sqrt[4]{\eta}\sqrt{\xi}\, e^{i\psi}, \sqrt{1-\sqrt{\eta}}\right)^T$$

$\eta = \sin^4 f \in [0,1]$
$\xi = \sin^2 g \in [0,1]$

See also [K. Remund, R. Pohle, Y. Akagi, J. Romhányi, & N. Shannon, *PRR 4, 033106 (2022)*]