Modes of the SS Soliton Presentation

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Outline I

Section 1

Introduction

Overview

My presentation is about **my recent work done in collaboration** with **Prof. Sven Bjarke Gudnason** finding number of instanton **zeromodes** allowed in static states of the **Witten-Sakai-Sugimoto** model.

The **Atiyah-Patodi-Singer index theorem**, we found the number of **zeromodes** to be $\dim \mathcal{M}_k = 6k$ where k is the number of **instantons** in the limit of infinite **'t Hooft coupling**, $\lambda \to \infty$.

Zeromodes

transformations of the gauge field that are not **gauge transformations** that leave the action invariant

Conjecture

We conjecture, $\lambda \nrightarrow \infty$, Fig. 1, where M_{kk} is geometrically a curvature scale or Kaluza-Klein mass and λ is the 'Hooft coupling.

- $2k$ modes are lifted with a scale of λM_{KK} as *heavy* modes and
- $6k - 9$ modes are lifted with a scale of M_{KK} as *light* modes with 9 zeromodes.
  - 0 modes are lifted with a scale of M_{KK} as *light* modes with 6 zeromodes when $k = 1$.

Scales

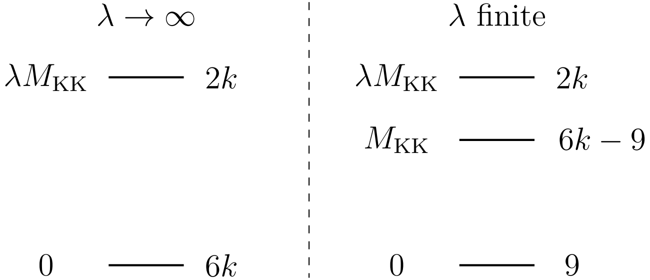

Figure 1: *The scales and number of modes at $\lambda \to \infty$ (λ finite) to the left (right).*

Section 2

Review

WSS Model

The Witten-Sakai-Sugimoto (WSS) model is given by the Yang-Mills and Chern-Simons actions.

$$S = \kappa \,\mathrm{tr} \int_{M_5} \mathcal{F} \wedge \star\mathcal{F} + \frac{9\kappa}{\lambda} \,\mathrm{tr}\, \omega_5 \tag{1}$$

$$-i\omega_5 = \mathcal{A} \wedge \mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3 \wedge \mathcal{F} - \frac{1}{10}\mathcal{A}^5 \tag{2}$$

$\mathcal{F}$ the SU(2) field tensor of $\mathcal{A}$. $\star$ is the Hodge star operator.

The manifold M_5 in endowed with a metric Eq. 3.

$$g_5 = \frac{dz^2}{H(z)} + H(z)\eta_{\mu\nu}dx^\mu dx^\nu \tag{3}$$

where $H(z)^{3/2} = 1 + z^2$ where $M_{kk} = 1$ units have been choosen.

Instantons

With an arbitrary time t_0, $\iota(z, \vec{x}) := (t_0, z, \vec{x})$

$$S = \kappa \operatorname{tr} \int_{M_4} \iota^* \mathcal{F} \wedge \star \iota^* \mathcal{F} + \frac{9\kappa}{\lambda} \operatorname{tr} \iota^* \omega_5 \tag{4}$$

where $F \equiv \iota^* \mathcal{F}$, $A \equiv \iota^* \mathcal{A}$, and $\iota(M_4) \subset M_5$.

$\mathcal{F}$ has a topological charge k.

$$c_2 = -\frac{1}{8\pi^2} \operatorname{tr} \int_{M_4} F \wedge F = k \tag{5}$$

Large Hooft Coupling

For $\lambda \to \infty$, one can neglect the Chern-Simons term and then the action Eq. 4 is approximately pure Yang-Mills.

$$S_{YM} \geq 8\pi^2 \kappa k \tag{6}$$

The BPS bound is saturated for self dual gauge fields,

$$F = \star F. \tag{7}$$

Linearization

The linearization, $A \to A + \delta A$, of the self-dual field equations Eq. 7 is Eq. 8.

$$\mathcal{P}_{-} d_A \delta A = \frac{1}{2}(1 - \star) d_A \delta A = 0 \tag{8}$$

d_A is covariant external derivative.

Fixing Gauge

Eq. 8 gives 3 equations but one can set the Lorenz gauge condition to fix the 4 δA fields.

$$d_A^\dagger \delta A \equiv -\mathcal{D}^\mu \delta A_\mu = 0 \tag{9}$$

$\mathcal{D}$ is the covariant derivative and $d_A^\dagger$ is the adjoint as of d_A with respect to the **Hodge** inner product.

$$\widetilde{\mathbb{D}}\delta A := (d_A^\dagger, \mathcal{P}_- d_A)\delta A \equiv (d_A^\dagger \delta A, \mathcal{P}_- d_A \delta A) \tag{10}$$

where $\widetilde{\mathbb{D}} : \Omega^1 \to \Omega^0 \oplus \Omega_-$ and $\dim \ker \widetilde{\mathbb{D}} = \dim \mathcal{M}_k$.

Moduli Space, $\mathcal{M}_k$

Given a **Fredholm** differential operator, $\mathbb{F}$, it's analytical index is

$$\operatorname{ind}\mathbb{F} \equiv \dim\ker\mathbb{F} - \dim\ker\mathbb{F}^\dagger. \tag{11}$$

Given an inner product $(\cdot,\cdot)$, adjoint of the operator $\mathbb{F}$ is $\mathbb{F}^\dagger$ such that $(\alpha, \mathbb{F}\beta)_{\mathbb{F}} = (\mathbb{F}^\dagger\alpha, \beta)_{\mathbb{F}^\dagger}$.

Inner Product between Covectors

$$(v, w) := \int v_\mu^* \delta^{\mu\nu} v_\nu$$

Atiyah-Patodi-Singer Index Theorem

The Atiyah-Patodi-Singer states that a analytical index is the same as a topological index for compact manifolds with boundaries. For *Dirac operators* the APS theorem can be expressed as Eq. 12.

$$\mathrm{ind}\,\mathbb{F} = \int_{\mathcal{M}} \alpha_0 - \frac{1}{2}(h + \eta) \tag{12}$$

- $\int_{\mathcal{M}} \alpha_0$ is the index neglecting boundary
- h is the dimension of the kernal of the operator projected onto the boundary, $\partial\mathbb{F}$.
- $\eta \equiv \sum_{\lambda \neq 0} \mathrm{sign}\lambda$ is the difference between the number of positive and negative eigenvalues of the operator on the boundary.

Section 3

Sketch of Method to find $6k$

Perturbation Equation of Motion

Starting with the four dimensional timeslice metric g_4 that is conformally equivalent to a locally flat metric

$$\tilde{\tilde{g}}_4 = d\xi^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \tag{13}$$

where the it's manifold has two boundaries at $\xi = \pm\xi_{\text{boundary}}$.

Yang-Mills in 4D is conformally invariant, so the moduli space should be preserved under conformal transformations.

Operator Simplification

The current form of the operator $D : \Omega^1 \to \Omega^0 \oplus \Omega_-$ is mathematically unwieldy.

Without changing the number of zeromodes, defining a new operator

$$\mathbb{D} := C \circ \widetilde{\mathbb{D}}\,. \tag{14}$$

$$C\ (\phi, \omega_-) := -d\xi \wedge \phi + 2\iota_{\partial_\xi}\omega_- \tag{15}$$

$\mathbb{D} = \eta^\mu \mathcal{D}_\mu$ where η^μ are 4D generators of $\mathfrak{su}(2)$, 't Hooft symbols.

Operator Simplification (contd.)

$$(\mathbb{D}\delta A)_\mu = \eta^\sigma{}_\mu{}^\nu \mathcal{D}_\sigma \delta A_\nu \tag{16}$$

$$(\eta^\sigma)_{\mu\nu} = \eta^\sigma{}_{\mu\nu} = \delta^\sigma_\xi \delta_{\mu\nu} + \delta^\xi_\mu \delta^\sigma_\nu - \delta^\xi_\nu \delta^\sigma_\mu - \varepsilon^{\xi\sigma}{}_{\mu\nu} \tag{17}$$

- Inner Product

$$(\alpha, \beta) := \int \mathrm{tr}_{SU(2)} \delta^{\mu\nu} \bar{\alpha}_\mu \beta_\nu \tag{18}$$

- Adjoint

$$\mathbb{D}^\dagger = \bar{\eta}^\mu \mathcal{D}_\mu = -(\eta^\intercal)^\mu \mathcal{D}_\mu \tag{19}$$

Summary Calculation with Atiyah-Patodi-Singer Theorem

$$\dim \mathcal{M}_k = \dim \ker \mathbb{D} = \dim \ker \mathbb{D}^\dagger - \int_{\mathcal{M}} \alpha_0 - \frac{1}{2}(h + \eta) \tag{20}$$

- Heat Kernal method $\Longrightarrow$ $\int_{\mathcal{M}} \alpha_0 = 8k$
- Vanishing Theorem (with a flat manifold) $\Longrightarrow$ $\dim \ker \mathbb{D}^\dagger = 0$
- The "electrical component" of $\mathcal{D}^\dagger$ vanishes on ∂M $\Longrightarrow$ $h = 4k$
- $\mathbb{Z}_2$ (left and right spinors acted on by $\mathbb{D}$ are exchangeable) $\Longrightarrow$ $\eta = 0$

Result

Therefore, according to the APS theorem Eq. 12,

$$\dim\ker\mathbb{D} = \dim\ker\mathbb{D}^\dagger - \int_{\mathcal{M}}\alpha_0 - \frac{1}{2}(h+\eta) \tag{21}$$

$$= 0 - 8k - \frac{1}{2}(4k+0)\,. \tag{22}$$

Therefore we can see that,

$$\dim\mathcal{M}_k = 6k \tag{23}$$

Section 4

Conclusion and Discussion

Conclusion

We found that the number of moduli per instanton is 6 in the $\lambda \to \infty$ limit.

This result lines up with intuitions of well separated instantons where the moduli correspond to

- 3 translations and
- 3 rotations.

We conjecture that for non-limiting case, $\lambda \nrightarrow \infty$,

- $2k$ modes are lifted with a scale of λM_{KK} as *heavy* modes and
- $6k - 9$ modes are lifted with a scale of M_{KK} as *light* modes with 9 zeromodes.
    - 0 modes are lifted with a scale of M_{KK} as *light* modes with 6 zeromodes when $k = 1$.

Outlook

With implications for holographic QCD, we propose

- further investigations of the lifted zeromodes for the λ non-limiting case, $\lambda \nrightarrow \infty$.
- Address the APS boundary conditions.

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Thank you for your attention.