Integrability for AdS_3/CFT_2

Alessandro Sfondrini

based on work in collaboration with

$2^{\{\text{R. Borsato, T. Lloyd, O. Ohlsson Sax, B. Stefański jr., A. Torrielli}\}}$

review: arXiv:1406.2971

1211.5119, 1212.0505, 1303.5995, 1306.2512,
1403.4543, 1406.0453, 1410.0866, 1411.3676.

GATIS

Plan

1. AdS_3/CFT_2 holography
2. Integrability in AdS/CFT
3. AdS_3/CFT_2 worldsheet integrability
4. AdS_3/CFT_2 spin-chain integrability
5. Conclusions and outlook

Gravity in AdS_3

- Early instance of holography, with Virasoro symmetry.

  [Brown, Henneaux '86]

- Rich black-hole physics.

  [Bañados, Teitelboim, Zanelli '92]

- Very relevant for string-theory black holes.

  [Strominger, Vafa '96]

- Higher-spin theories natural.

  [Gaberdiel, Gopakumar '10] [...]

AdS_3 from the D1-D5 system

Consider the D1-D5 system (16 supersymmetries)

	0	1	2	3	4	5	6	7	8	9
D1	•	•								
D5	•	•					•	•	•	•

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	$\mathfrak{so}(1,1)$		$\mathfrak{so}(4)_1$				$\mathfrak{so}(4)_2$			

and let us decompose

$$\mathfrak{so}(4)_1 = \mathfrak{su}(2)_{\mathrm{L}} \oplus \mathfrak{su}(2)_{\mathrm{R}}, \qquad \mathfrak{so}(4)_2 = \mathfrak{su}(2)_{\bullet} \oplus \mathfrak{su}(2)_{\circ}.$$

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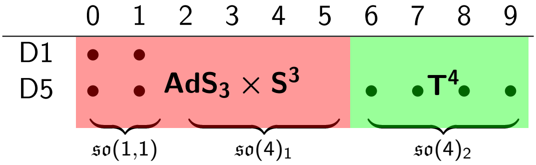

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Near-horizon geometry is

$$AdS_3 \times S^3 \times T^4 .$$

Dual gauge theory

We identify $\#\text{D1} = N_c$ and $\#\text{D5} = N_f$.

- D1-D1 strings $\longleftrightarrow \mathcal{N} = (8,8)$ $U(N_c)$ vector-multiplet;
- D1-D5 strings $\longleftrightarrow \mathcal{N} = (4,4)$ $U(N_c) \times U(N_f)$ hyper-multiplets;
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$\longrightarrow SU(N_c)$ gauge theory with vectors and hypers in the UV!

RG flow to IR sigma model on N_c-instanton moduli space in $SU(N_f)$.

Other AdS_3 superstring backgrounds

Another geometry with 16 supercharges is

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1,$$

if the curvature radii satisfy

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Both backgrounds can be supported by RR and/or NSNS fluxes.

Pure-NSNS background gives **WZW model** and is manageable.

[Maldacena, Ooguri '01]

Mixed-flux and general backgrounds are more mysterious.

Fundamental question

Can we understand general AdS_3/CFT_2 by integrability?

Plan

Integrability 101

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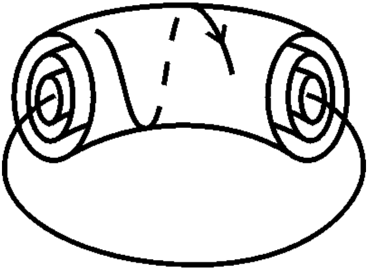
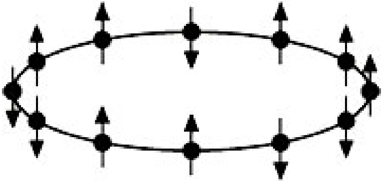
$$\{H_j, H_k\} = 0, \quad j,k = 1,\dots N \qquad \longrightarrow \qquad \text{action-angle coordinates}$$



Integrability in quantum mechanics

N-dimensional quantum system: spin chain.



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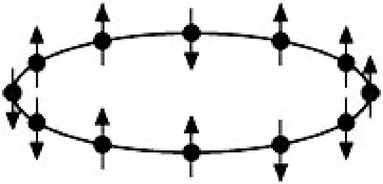

$$[H_j, H_k] = 0, \quad j,k = 1, \dots N \qquad \longrightarrow \qquad \text{simultaneous diagonalisation}$$

In practice: **Bethe Ansatz** to efficiently compute eigenvalues.

Integrability in field theory

In field theory

- ∞-many degrees of freedom (hard to exhibit conserved charges!)
- Integrability if $\{$EOM$\}\leftrightarrow\{$flatness condition$\}$. [Lax '68]

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Conserved charges follow: e.g. (Sine-Gordon) [Zamolodchikov, Zamolodchikov '79]

$$H_{2k}(p) = p^{2k}\sqrt{p^2+m^2}, \qquad H_{2k+1}(p) = p^{2k+1}.$$

Can we "quantise" this integrable structure?

Factorised scattering in two dimensions

Assume all conserved charges.

If we consider asymptotic states, $|p_1, \dots p_n\rangle_{\text{in}} \to |q_1, \dots q_{n'}\rangle_{\text{out}}$

$$\sum_{j=1}^{n} H_k(p_j) = \sum_{j=1}^{n'} H_k(q_j) \qquad \forall\, k.$$

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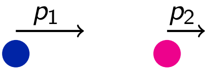

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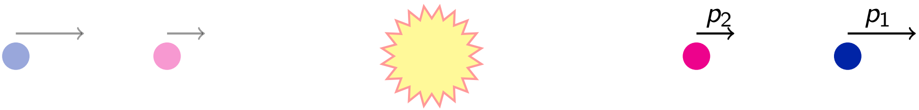

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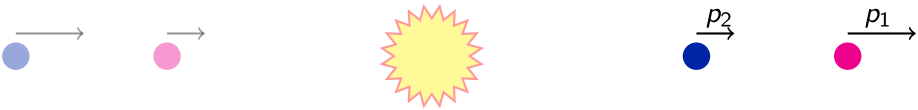

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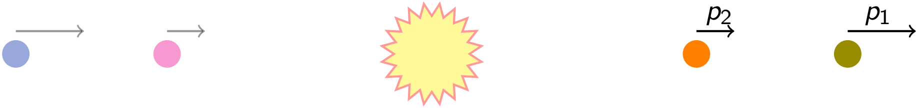

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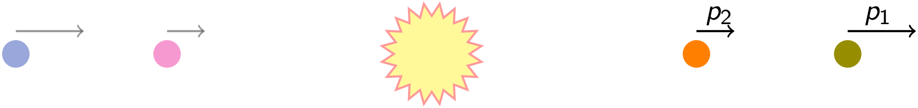
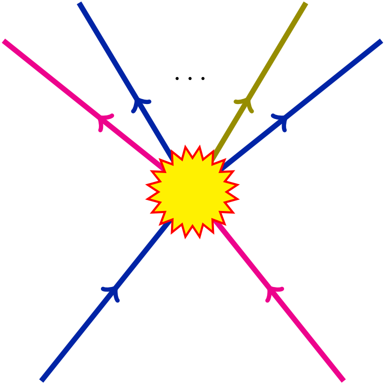


Cartoon generalises to N-particle event $\longrightarrow$ **factorisation**!

[Zamolodchikov, Zamolodchikov '79]

Consistency conditions: no particle production



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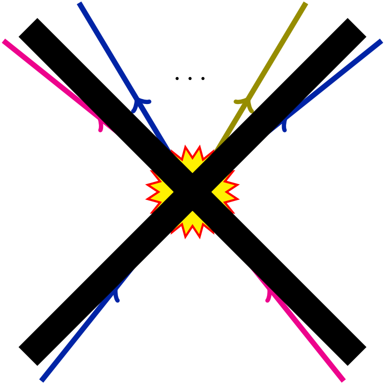
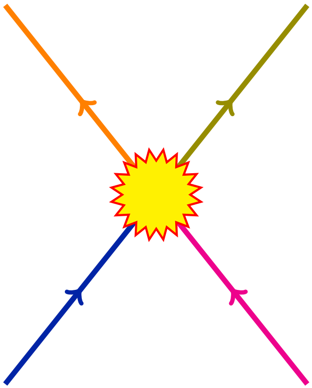
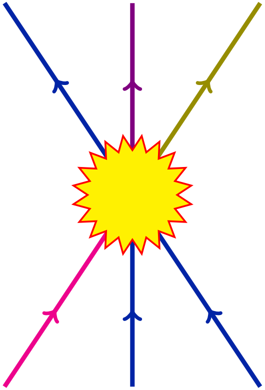


Consistency conditions: momenta transmitted



Consistency conditions: Yang-Baxter equation



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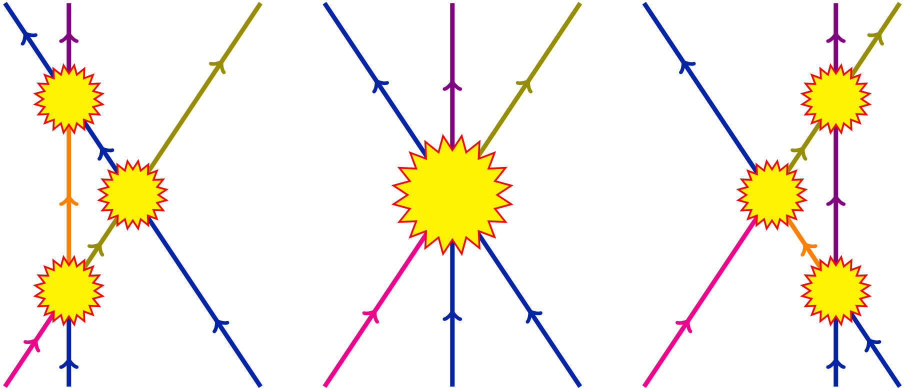

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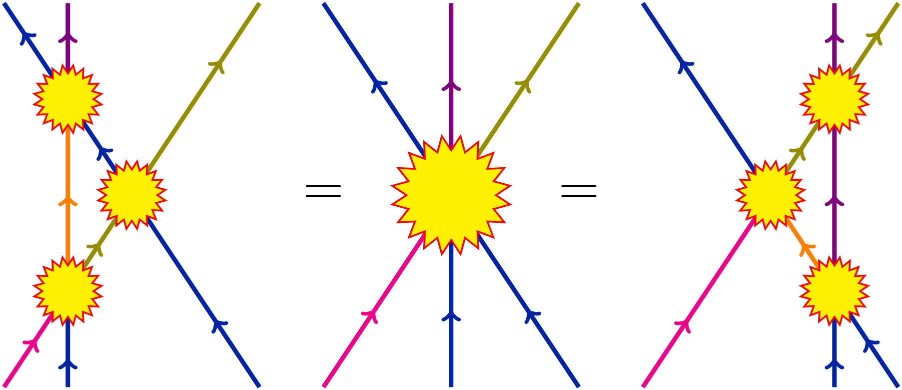

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First off, always restrict to the planar ('t Hooft) limit.

$\longrightarrow$ free strings, planar gauge theory, arbitrary λ.

Focus on the **spectral problem**:

operator (anomalous) dimensions $\longleftrightarrow$ string energy spectrum

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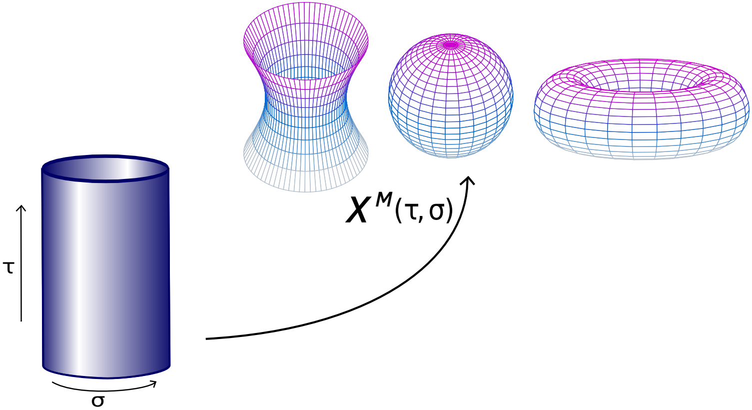
On the CFT side, consider

$$\mathbf{D}\,\mathcal{O} = \Delta\,\mathcal{O}, \qquad \mathcal{O} = \sum \mathrm{tr}\big[ZZZXZZZZXXZZZ\big]\,.$$

$\longrightarrow$ spin-chain picture! [Minahan, Zarembo '02]

Light-cone strings as a Non-Linear Sigma Model



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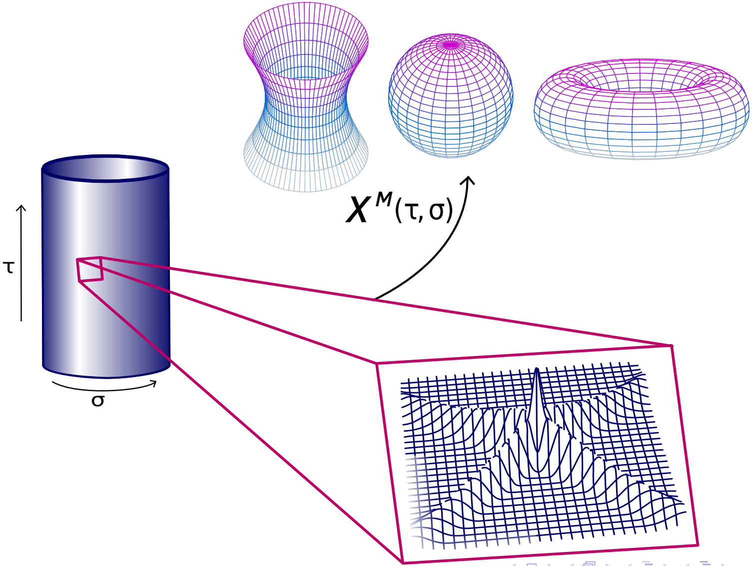

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This talk: focus on $\mathbf{AdS_3} \times \mathbf{S^3} \times \mathbf{T^4}$, pure-RR.

Plan

AdS_3/CFT_2 integrability overview

String NLSM is classically integrable for pure-RR and mixed fluxes.

[Babichenko, Stefański, Zarembo '10] [Sundin, Wulff '12] [Cagnazzo, Zarembo '12]

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Several new features and issues with respect to AdS_5/CFT_4.
Massless modes in light-cone gauge!

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Strategy:

- Explicitly work out the symmetries.
- Do not impose integrability: derive it!
- Check against perturbative calculations.

[Sundin, Wulff '12 & '14] [Beccaria, Levkovich-Maslyuk, Macorini, Tseytlin '12]
[Abbott '13] [Engelund, McKeown, Roiban '13] [Bianchi, Hoare '14]
[Roiban, Sundin, Tseytlin, Wulff '14] [Babichenko, Dekel, Ohlsson Sax '14]

The string non-linear sigma model

Start from NLSM action,

$$S = -\frac{1}{2}\int_{-r}^{+r} d^2\sigma\, \gamma^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n\, G_{mn}(X) + \text{fermions.}$$

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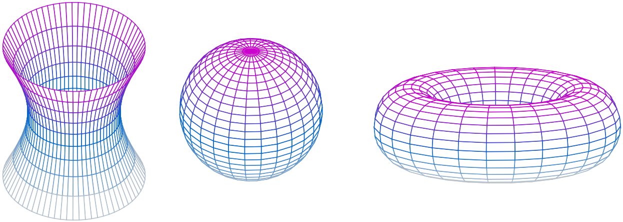
Fix **light-cone gauge**:

- Worldsheet and target-space Hamiltonian are related.
- Physical fields are manifest.
- Conformal symmetry broken on the worldsheet, r fixed
  $\longrightarrow$ we can **decompactify** the worldsheet, $r \to \infty$.
- Physical states satisfy level-matching:

$$\mathbf{P}\,|\text{physical}\rangle = 0, \qquad \mathbf{P}\,|p_1,\dots p_n\rangle = (p_1 + \dots + p_n)\,|p_1,\dots p_n\rangle.$$

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ isometries



Superisometry algebra

$$\mathfrak{psu}(1,1|2)_{\mathsf{L}} \oplus \mathfrak{psu}(1,1|2)_{\mathsf{R}} \oplus \mathfrak{u}(1)^4$$

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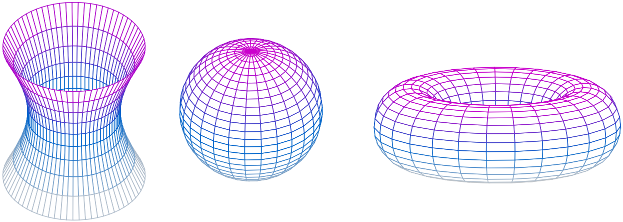

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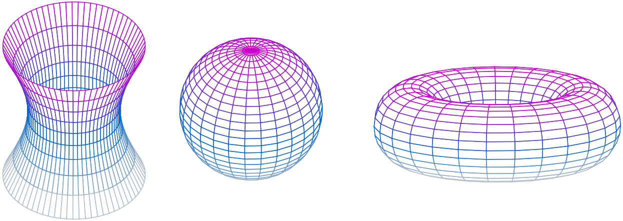

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$$\mathfrak{so}(4)_2 = \mathfrak{su}(2)_\bullet \oplus \mathfrak{su}(2)_\circ$$

Light-cone gauge symmetry algebra

Light-cone gauge breaks isometries to $\mathfrak{psu}(1|1)^4_{\text{c.e.}} \oplus \mathfrak{so}(4)_2$.

On-shell we have

$$\mathbf{P}|\text{state}\rangle = 0$$

and the algebra

$$\begin{aligned}
\{\mathbf{Q}_{\mathrm{L}\,\dot{a}}, \overline{\mathbf{Q}}_{\mathrm{L}}{}^{\dot{b}}\} &= \tfrac{1}{2}\delta_{\dot{a}}{}^{\dot{b}}(\mathbf{H}+\mathbf{M}), \qquad & \{\mathbf{Q}_{\mathrm{L}\,\dot{a}}, \mathbf{Q}_{\mathrm{R}}{}^{\dot{b}}\} &= 0,\\
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We find a non-trivial central extension given by

$$\mathbf{C} \qquad \text{and} \qquad \overline{\mathbf{C}}\,.$$

[Beisert '05] [Arutyunov, Plefka, Frolov, Zamaklar '06]

Central extension

We find the central charges to be

$$\mathbf{C} = i\,\frac{h(\lambda)}{2}\left(e^{i\,\mathbf{P}} - 1\right),$$

- Highly non-linear form
- Vanish on-shell
- Induce non-trivial co-product (Hopf algebra):

$$\mathbf{C}|p_1, p_2\rangle = \left(\ \#\,\mathbf{C} \otimes \mathbf{1} + \#\ \ \mathbf{1} \otimes \mathbf{C}\right)|p_1, p_2\rangle$$

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Representations

Perturbatively, we find 8+8 fundamental excitations:

4+4 massive particles

transverse directions of $AdS_3 \times S^3$

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flat T^4 directions

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massive

massless

Dispersion relation

From the shortening condition

$$\mathbf{H}^2 = \mathbf{M}^2 + 4\,\mathbf{C}\overline{\mathbf{C}},$$

we have the dispersion relation

$$E_p = \sqrt{m^2 + 4h^2\,\sin^2\frac{p}{2}}\,,$$

Dispersion relation

From the shortening condition

$$\mathbf{H}^2 = \mathbf{M}^2 + 4\,\mathbf{C}\overline{\mathbf{C}},$$

we have the dispersion relation

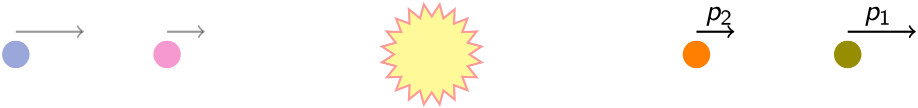
$$E_p = \sqrt{m^2 + 4h^2\,\sin^2\frac{p}{2}}\,,$$

In the case $m = 0$, we have

$$E_p = 2h\left|\sin\frac{p}{2}\right|.$$

$\longrightarrow$ non-analyticity: $m = 0$ special point, protected by (super)symmetry.

The integrable S matrix



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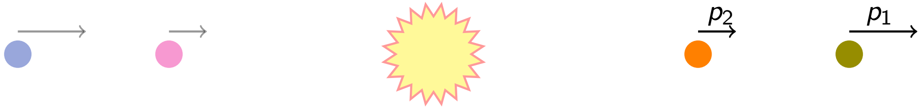

Massless scattering is usually problematic!

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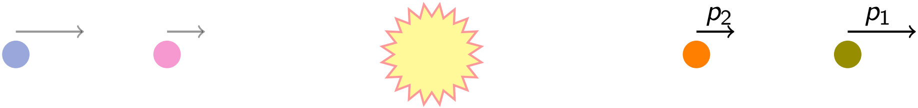

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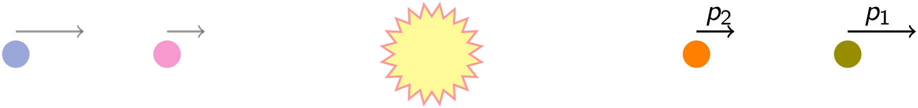

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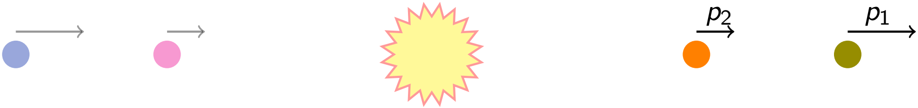

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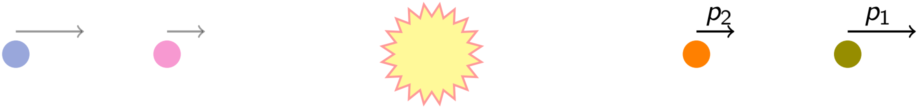

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The linear conditions leave some undetermined **“dressing” factors**.

Recap worldsheet integrability

- We used symmetries to determine **kinematics** and world-sheet **S matrix** at all loops. This is compatible with unitarity, Yang-Baxter equation, and crossing symmetry and perturbative checks.

- Five **dressing factors** are undetermined by symmetry. There is already a proposal for "massive" ones.

[Borsato, Ohlsson Sax, AS, Stefański, Torrielli '13]

- The **mixed-flux** case can be studied similarly: same symmetries, but deformed representations. $\mathbf{AdS_3 \times S^3 \times S^3 \times S^1}$ also understood.

[Lloyd, Ohlsson Sax, AS, Stefański '14]

[Borsato, Ohlsson Sax, AS, Stefański, to appear]

Plan

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2. Integrability in AdS/CFT
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4. AdS_3/CFT_2 spin-chain integrability
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Integrability on the gauge theory/CFT side?

In AdS_5/CFT_4 and AdS_4/CFT_3, "straightforward" spin-chain picture.

$$\mathbf{D}\,\mathcal{O} = \Delta\,\mathcal{O}, \qquad \mathcal{O} = \sum \mathrm{tr}\big[ZZZXZZZZXXZZZ\big]\,.$$

$\longrightarrow$ conceptual and practical tool for understanding integrability.

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In AdS_3/CFT_2:

- UV theory non-conformal $\longrightarrow$ RG flow.
- Sigma-model on instanton moduli space is complicated.
- Simpler case: symmetric-product orbifold. [Pakman, Rastelli, Razamat '10]
  - No natural "single trace" notion (but "single cycle")
  - Very large mixing at tree level
  - Even worse at one-loop

Back to the D1-D5 system

In the UV, two branches:

- Coulomb branch: D1 separated from D5;
- Higgs branch: D1 on top of D5 $\longrightarrow$ AdS_3/CFT_2. [Maldacena '98]

We want to describe the **origin of the Higgs branch**.

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We want to describe the **origin of the Higgs branch**.

In the near-horizon limit

- D1-D1 strings $\longleftrightarrow$ $\mathcal{N} = (8,8)$ $U(N_c)$ vector-multiplet:

  $\mathcal{N} = (4,4)$ vector and $\mathcal{N} = (4,4)$ adjoint hyper

- D1-D5 strings $\longleftrightarrow$ $\mathcal{N} = (4,4)$ fundamental hyper

UV action

The action is schematically

$$\begin{aligned} \text{vector} \quad & \mathcal{L}_V(\Phi) = \frac{1}{g^2}\mathrm{tr}\Big[F_{\mu\nu}F^{\mu\nu} + |\nabla\Phi|^2 + |D|^2\Big], \\ \text{adjoint hyper} \quad & \mathcal{L}_T(T,\Phi) = \mathrm{tr}\Big[|\nabla T|^2 + [T,\Phi]^2 + T[D,T]\Big], \\ \text{fundamental hyper} \quad & \mathcal{L}_H(H,\Phi) = |\nabla H|^2 + H^\dagger D H + H^\dagger \Phi\Phi H\,, \end{aligned}$$

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Flowing to the IR [Witten '96]

- $g \to \infty$ and $\mathcal{L}_V$ suppressed.
- Conformal invariance requires non-standard dimensions:

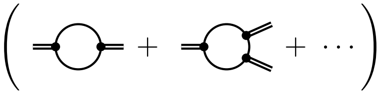
$$[A = 1, \qquad [\Phi] = 1, \qquad [D] = 2, \qquad \ldots .$$

Effective IR action

Consider the IR path-integral

$$\int [\mathrm{d}\Phi][\mathrm{d}T][\mathrm{d}H]\; e^{-S_H(H,\Phi)-S_T(T,\Phi)} = \int [\mathrm{d}\Phi][\mathrm{d}T]\; e^{-N_f\, S_{\mathrm{eff}}(\Phi)-S_T(T,\Phi)},$$

where $N_f S_{\mathrm{eff}}$ comes from integrating out the fundamental hypers:

$N_f\ S_{\mathrm{eff}} =$  N_f times

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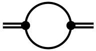
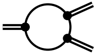
$N_f\ S_{\text{eff}} = \Bigg($  $+$  $+ \cdots \Bigg)$ N_f times

The remaining fields fit in representation-theory spin chain:

- Vector multiplet $\longleftrightarrow$ massive sector.

  [Borsato, Ohlsson Sax, AS, Stefański, Torrielli '13]

- Adjoint hyper $\longleftrightarrow$ (speculated) massless sector.

  [Ohlsson Sax, Stefański, Torrielli '13]

Large-N action and perturbation theory

Natural large-N and perturbative expansion

$$N_f \gg 1, \qquad N_c \gg 1\,, \qquad \lambda = \frac{N_c}{N_f} \ll 1\,.$$

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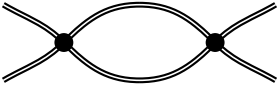
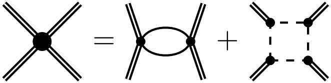
Example: we can compute the Φ-bubble



by using the propagator fixed by conformal invariance,

$$\langle \Phi_0 \Phi_x \rangle = \frac{1}{|x|^2}, \qquad \text{and}$$



[Brézin 1960s] [Coleman 1960s] [...]

The dilatation operator [Minahan, Zarembo '02]

One-loop dilatation operator $\mathbf{D} \approx \mathbf{H}$ can be found from mixing matrix.

Only neighbouring fields in $\mathcal{O}$ contribute, i.e.

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$\longrightarrow$ **integrable spin chain!**

[Ohlsson Sax, AS, Stefański, '14]

Recap spin-chain integrability

We have considered the CFT_2 at the origin of the Higgs branch.

- Nice "large-N" expansion.
- Field content fits algebraic spin-chain construction.
- One-loop integrability in a sub-sector.

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Back to our question

Can we understand general AdS_3/CFT_2 by integrability?

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Can we understand general AdS_3/CFT_2 by integrability?

It seems so.

Natural questions

- Fix the remaining dressing factors.

  [Borsato, Ohlsson Sax, AS, Stefański, Torrielli, in progress]

- Study the energy spectrum qualitatively and compare it with semiclassical integrability information.

  [Lloyd, Stefański '13] [Abbott, Aniceto, in progress]

- Spin-chain integrability beyond $\mathfrak{so}(4)$? Massless modes?

  [Ohlsson Sax, AS, Stefański, in progress]

- Understand the CFT side of $AdS_3 \times S^3 \times S^3 \times S^1$? [Tong '14]

The bigger picture

- Investigate $\mathcal{N} = (4, 4)$ symmetry and relations to integrability. Relation with symmetric-product orbifold?

  [Pakman, Rastelli, Razamat '10]

- Investigate the relation between integrability and CFT approach using mixed-flux and extrapolate towards WZW point.

  [Zamolodchikov, Fendley, Saleur '90s ??] [Bazhanov, Lukyanov, Zamolodchikov '90s ??]

- Gain new insight in higher-spin theories on AdS_3.

  [Gaberdiel, Gopakumar '14] [...]

- Study black-hole backgrounds by integrability. [David, Sadhukhan '11]