QUANTISATION OF FREE ASSOCIATIVE DYNAMICAL SYSTEMS.

BI-QUANTUM STRUCTURE OF THE STATIONARY KDV HIERARCHY.

NON-DEFORMATION QUANTISATION OF THE VOLTERRA HIERARCHY.

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A.V. Mikhailov, *Quantisation ideals of nonabelian integrable systems*,

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V.M. Buchstaber, A.V. Mikhailov, *KdV hierarchies and quantum Novikov's equations*
arXiv:2109.06357v2 [nlin.SI] 13 Sep 2021

S.Carpentier, A.V.Mikhailov, J.P.Wang, *Quantisations of the Volterra hierarchy*

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Outline

CLASSICAL, QUANTUM MECHANICS

NEWTON'S EQUATION $\ddot{q} = F(q)$

Classical Mechanics (Newton, Hamilton): $\mathfrak{A}_0 = \mathbb{C}[p, q]$

$$[q, p] = 0, \;\; \partial_t q = p, \;\; \partial_t p = F(q), \quad H = \frac{p^2}{2} + U(q), \;\; F(q) = -U'(q)$$

$$\partial_t q = \{q, H\}, \; \partial_t p = \{p, H\}, \;\; \{q, p\} = 1, \;\; \partial_t a = \{a, H\}, \;\; a \in \mathfrak{A}_0.$$

Quantum Mechanics (Heisenberg, Dirac 1925): $\mathfrak{A}_\hbar = \mathbb{C}\langle \hat{p}, \hat{q}\rangle \diagup \langle \hat{p}\hat{q} - \hat{q}\hat{p} + i\hbar\rangle$

$$[\hat{q}, \hat{p}] = i\hbar, \;\; \partial_t \hat{q} = \hat{p}, \;\; \partial_t \hat{p} = F(\hat{q}), \quad H = \frac{\hat{p}^2}{2} + U(\hat{q}),$$

$$\partial_t \hat{q} = \frac{i}{\hbar}[H, \hat{q}], \; \partial_t \hat{p} = \frac{i}{\hbar}[H, \hat{p}], \quad [\hat{q}, \hat{p}] = i\hbar, \; \partial_t \hat{a} = \frac{i}{\hbar}[H, \hat{a}], \;\; \hat{a} \in \mathfrak{A}_\hbar.$$

Issues: (1) Consistancy, (2) Ordering.

The Fundamental Equations of Quantum Mechanics

Author(s): P. A. M. Dirac

Source: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, Dec. 1, 1925, Vol. 109, No. 752 (Dec. 1, 1925), pp. 642-653

Free Associative Mechanics:
Free associative algebra $\mathfrak{A} = \mathbb{C}\langle p, q \rangle$ with a derivation

$$\partial_t : \mathfrak{A} \mapsto \mathfrak{A}, \qquad \partial_t(a\,b) = \partial_t(a)\,b + a\,\partial_t(b), \;\; \forall\; a, b \in \mathfrak{A}.$$

Nonabelian Newton's equations:

$$\partial_t q = p, \qquad \partial_t p = F(q) \qquad p, q \in \mathfrak{A}.$$

$H_0 = [q, p] \in \mathfrak{A}$ is a constant of motion (first integral):

$$\partial_t([q, p]) = p^2 + qF(q) - F(q)q - p^2 = 0$$

of the nonabelian Newton's equations, but usual expression for the first integal of energy $H = \frac{1}{2}p^2 + U(q),\; F(q) = -U'(q)$ is not a constant of motion (if $F''(q) \neq 0$).

Example: Let $\partial_t q = p, \;\; \partial_t p = F(q) = 3q^2, \;\; U(q) = -q^3$ (*).

- $H = \frac{1}{2}p^2 - q^3$ is **not** a constant of motion

$$\partial_t(H) = \frac{1}{2}(\dot{p}p + p\dot{p}) - (\dot{q}q^2 + q\dot{q}q + q^2\dot{q})$$

$$= \frac{1}{2}(pq^2 - 2qpq + q^2p) = \frac{1}{2}[pq, q] + \frac{1}{2}[q, qp] \neq 0, \quad \partial_t(H) \in \mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}].$$

  There are infinitely many algebraically independent “first integrals” $H_k \in \mathfrak{A}/\mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}]$, such that $\partial_t(H_k) \in \mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}]$.
- Nonabelian Newton’s equation (*) has infinitely many higher symmetries. The next symmetry is:

$$\partial_{t_5} q = q^2p - 2qpq + pq^2;$$
$$\partial_{t_5} p = -qp^2 + 2pqp - p^2q;$$

$$\partial_{t_7} q = 2pq^3 - p^3 + 2q^3p - q^2pq - qpq^2;$$
$$\partial_{t_7} p = -2q^2p^2 + qpqp - 2qp^2q + pq^2p + pqpq - 2p^2q^2 + 6q^5;$$

In the commutative case H is a first integral, $H_k = H_k(H)$, and $\partial_{t_5} a(p, q) = 0, \; \partial_{t_7} a(p, q) = 2H\partial_t a(p, q)$.

In the quantum case H is a constant of motion $(H)_t = 0$, since

$$-2\hat{q}\hat{p}\hat{q} = -2\hat{p}\hat{q}^2 - 2i\hbar\hat{q}, \qquad \hat{q}^2\hat{p} = \hat{p}\hat{q}^2 + 2i\hbar\hat{q}.$$

Algebra $\mathfrak{A}$, as a $\mathbb{C}$–linear space has an additive basis of monomials

$$Mon(\mathfrak{A}) = \{p^{i_1} q^{j_1} p^{i_2} q^{j_2} \cdots p^{i_m} q^{j_m} \mid i_k, j_k \in \mathbb{Z}_{\geq 0}\}.$$

The number of monomials of a degree $n = i_1 + j_1 + \cdots + i_m + j_m$ is 2^n.

In contrast, algebras $\mathfrak{A}_0 = \mathfrak{A} / \langle qp - pq \rangle$ and $\mathfrak{A}_\hbar = \mathfrak{A} / \langle pq - qp + i\hbar \rangle$ admit additive bases of normally ordered monomials (standard)

$$Mon(\mathfrak{A}_0) = Mon(\mathfrak{A}_\hbar) = \{p^i q^j \mid i, j \in \mathbb{Z}_{\geq 0}\}$$

respectively, and the # of monomials of degree n is growing as $n + 1$.

Any element of the quotient algebra $\mathfrak{A}_0$ or $\mathfrak{A}_\hbar$ can be uniquely represented by a polynomial with normally ordered monomials.

Problem of quantisation, quantisation ideals

Fact: Any associative algebra can be represented as a quotient of a free algebra $\mathfrak{A}$ over a two sided ideal $\mathfrak{J}$.

In my opinion, the problem of quantisation of a dynamical system $\partial_t : \mathfrak{A} \mapsto \mathfrak{A}$ can be formulated as following:

We start from a dynamical system on a free associative algebra $\mathfrak{A}$.

Find such ideals $\mathfrak{J} \subset \mathfrak{A}$ that

A. $\partial_t(\mathfrak{J}) \subseteq \mathfrak{J} \quad \Leftrightarrow$ the evolutionary derivation ∂_t induces a derivation of the quotient algebra $\mathfrak{A}/\mathfrak{J}$.

B. The quotient algebra $\mathfrak{A}/\mathfrak{J}$ has an additive basis of normally ordered monomials. In other words, we know how to change the order of any two variables.

Ideals $\mathfrak{J}$ satisfying conditions A, B are called quantisation ideals and the corresponding quotient algebras $\mathfrak{A}/\mathfrak{J}$ quantised algebras.

If we start from a classical (commutative) dynamical system, then in order to apply our method we need to lift it to a free associative algebra. This step is delicate.

Let $\mathfrak{A} = \mathbb{C}\langle x_1, \ldots, x_n\rangle$ be a free associative $\mathbb{C}$ algebra with the lexicographic ordering of variables $x_i > x_j$ if $i > j$. It has the additive basis of monomials

$$Mon(\mathfrak{A}) = \left\{ x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_m}^{\alpha_m} \mid n \geq i_1, i_2, \ldots, i_m \geq 1,\ \alpha_k \geq 0 \right\}$$

Let $\mathfrak{J} = \langle f_{i,j} \mid 1 \leq i < j \leq n\rangle$ where

$$f_{i,j} = x_i x_j - \omega_{i,j} x_j x_i + g_{i,j}, \quad g_{i,j} \in \mathfrak{A},\ Lm(g_{i,j}) < x_j x_i < x_i x_j,\ \ \omega_{i,j} \neq 0.$$

Then in the quotient $\mathfrak{A} / \mathfrak{J}$ there is a monomial basis of normally ordered monomials

$$Mon(\mathfrak{A} / \mathfrak{J}) = \left\{ x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_m}^{\alpha_m} \mid n \geq i_1 > i_2 > \ldots > i_m \geq 1,\ \alpha_k \geq 0 \right\}$$

Algebra $\mathfrak{A} / \mathfrak{J}$ has Poincarée–Birkhoff–Witt basis of normally ordered monomials
$\Rightarrow \mathfrak{A} / \mathfrak{J}$ satisfies (B) condition.

PROBLEM OF QUANTISATION, QUANTISATION IDEALS

In our example of the Newton equation on $\mathfrak{A}$

$$\partial_t q = p, \qquad \partial_t p = 3q^2 \qquad p, q \in \mathfrak{A}. \tag{1}$$

a natural candidate for $\mathfrak{J}$ which implies B is

$$\mathfrak{J} = \langle J := pq - \omega qp + \alpha p + \sum_{n=1}^{M} \beta_n q^n + \gamma \rangle, \;\; \omega, \alpha, \beta, \gamma \in \mathbb{C}.$$

Then $\mathfrak{J} = \{\sum a_i J\, b_i \mid a_i, b_i \in \mathfrak{A}\}$. The condition (A): $\partial_t(\mathfrak{J}) \subseteq \mathfrak{J} \;\Leftrightarrow \partial_t(J) \in\; \mathfrak{J}$ implies that

$$\partial_t(J) = \dot{p}q + p\dot{q} - \omega\dot{q}p - \omega q\dot{p} + \alpha\dot{p} + \sum_{n=1}^{M} \beta_n \sum_{k=1}^{n} q^{k-1}\dot{q}q^{n-k}$$

$$= 3q^3 + p^2 - \omega p^2 - 3\omega q^3 + 3\alpha q^2 + \sum_{n=1}^{M} \beta_n \sum_{k=1}^{n} q^{k-1} p q^{n-k} \in \mathfrak{J}$$

$$\Rightarrow \omega = 1, \alpha = \beta_n = 0$$

and therefore $J = pq - qp + \gamma$, where $\gamma \in \mathbb{C}$.

If we add the reality arguments in the consideration we would conclude that γ is pure imaginary, i.e. $\gamma = i\hbar$ and the Heisenberg quantisation is a unique possibility for the free associative mechanics (1).

Let us consider nonabelian integrable systems: the Volterra chain (i) and the Narita-Itoh-Bogoyavlensky N–chains (ii)

$$\text{(i) } \partial_{t_2} u_n = u_{n+1}u_n - u_n u_{n-1}, \quad \text{(ii) } \partial_{t_2} u_n = \sum_{k=1}^{N}(u_{n+k}u_n - u_n u_{n-k}), \qquad n \in \mathbb{Z}. \quad (2)$$

These are infinite systems of equations for $u_k = u_k(t),\ k \in \mathbb{Z}$.

In equations (2) functions u_k are elements of a free associative algebra $\mathfrak{A} = \mathbb{C}\langle \dots u_{-1}, u, u_1, \dots \rangle$ with an infinite number of variables u_k and a natural automorphism $S : \mathfrak{A} \mapsto \mathfrak{A}$, generated by the shift operator $S(u_k) = u_{k+1},\ k \in \mathbb{Z}$, and $\partial_{t_2} S = S\partial_{t_2}$.

We begin with consideration of two-sided ideals $\mathfrak{J}_\omega \subset \mathfrak{A}$ generated by an infinite set of polynomials of the form

$$\mathfrak{J}_\omega = \langle \{ u_q u_p - \omega_{p,q} u_p u_q \,|\, p, q \in \mathbb{Z},\ p > q,\ \omega_{p,q} \in \mathbb{C}^\times \} \rangle$$

$$u_p u_q = \omega_{q,p} u_q u_p, \qquad q > p, \quad \omega_{p,q} \neq 0.$$

QUANTISATION OF THE VOLTERRA SYSTEMS

PROPOSITION

Volterra system $\partial_{t_2} u = u_1 u - u u_{-1}$ can be restricted to $\mathfrak{A}_{\mathfrak{I}_\omega}$ if and only if $\omega_{n+1,n} = \alpha,\ \omega_{n,m} = 1,\ n - m \geq 2$.

$$u_n u_{n+1} = \alpha u_{n+1} u_n, \quad u_n u_m = u_m u_n, \ \ |n - m| \geq 2. \tag{3}$$

The non-abelian Volterra system has a symmetry

$$u_{t_3} = u u_{-1} u_{-2} + u u_{-1} u_{-1} + u u u_{-1} - u_1 u u - u_1 u_1 u - u_2 u_1 u. \tag{4}$$

PROPOSITION

Equation (4) can be restricted to $\mathfrak{A}_{\mathfrak{I}_\omega}$ only in the following cases:

$$\text{(a)} \qquad u_n u_{n+1} = \alpha u_{n+1} u_n, \quad u_n u_m = u_m u_n, \quad |n - m| \geq 2 \tag{5}$$

$$\text{(b)} \qquad u_n u_{n+1} = (-1)^n \alpha u_{n+1} u_n, \quad u_n u_m = -u_m u_n, \quad |n - m| \geq 2 \tag{6}$$

PROPOSITION (S.CARPENTIER, AVM, J.P.WANG)

Every equation from the Volterra hierarchy admits quantisation (3).
Every odd degree equation from the Volterra hierarchy admits quantisation (6).

Partial quantisation (?!) The “cubic” difference ideal generated by polynomials

$$u_m u_n - u_n u_m, \ \ |m - n| > 1, \qquad u_{n+1} u_n u_{n+2} - u_n u_{n+2} u_{n+1}$$

is invariant w.r.t. the Volterra system and its symmetries.

PERIODIC VOLTERRA CHAINS

Periodic closures of the chains $u_{k+M} = u_k$ with period M result in nonabelian systems on $\mathfrak{A}^M = \mathbb{C}\langle u_1, \ldots, u_M \rangle$.

Let $M = 3$:

$$\begin{aligned} u_{1,t_2} &= u_2 u_1 - u_1 u_3, \\ u_{2,t_2} &= u_3 u_2 - u_2 u_1, \\ u_{3,t_2} &= u_1 u_3 - u_3 u_2 \end{aligned}$$

The $M = 3$ Volterra system has an obvious constant of motion $H = u_1 + u_2 + u_3$.

It has infinitely many commuting symmetries:

$$\begin{aligned} u_{1,t_3} &= u_1^2 u_3 + u_1 u_3 u_2 + u_1 u_3^2 - u_2 u_1^2 - u_2^2 u_1 - u_3 u_2 u_1, \\ u_{1,t_4} &= u_1^3 u_3 + u_1^2 u_3 u_2 + u_1^2 u_3^2 + u_1 u_2 u_1 u_3 + u_1 u_3 u_1 u_3 + u_1 u_3 u_2^2 \\ &\quad + u_1 u_3 u_2 u_3 + u_1 u_3^2 u_2 + u_1 u_3^3 - u_2 u_1^3 - u_2 u_1 u_2 u_1 - u_2 u_1 u_3 u_1 \\ &\quad - u_2^2 u_1^2 - u_2^3 u_1 - u_2 u_3 u_2 u_1 - u_3 u_2 u_1^2 - u_3 u_2^2 u_1 - u_3^2 u_2 u_1 \\ \cdots &= \cdots \end{aligned}$$

Quantisation of the periodic Volterra chains

Periodic Volterra systems with period M may admit inhomogeneous commutation relations:

$$u_q u_p = \omega_{p,q} u_p u_q + \sum_{r=1}^{M} \sigma_{p,q}^r u_r + \eta_{p,q}, \qquad 1 \leq q < p \leq M, \;\; \omega_{p,q} \neq 0.$$

Proposition

Nonabelian periodical Volterra chain with period M admits $\mathfrak{J}_M$–quantisation iff the following commutation relations

$$\begin{aligned}
M = 3: \quad & u_n u_{n+1} = \alpha u_{n+1} u_n + \beta(u + u_1 + u_2) + \eta, \;\; n \in \mathbb{Z}_3; \\
M = 4: \quad & u_1 u_2 = \alpha u_2 u_1 + \beta u_2 + \gamma u_1 - \beta\gamma, \\
& u_1 u_3 = u_3 u_1 - \beta u_2 + \beta u_4, \\
& u_4 u_1 = \alpha u_1 u_4 + \beta u_4 + \gamma u_1 - \beta\gamma, \\
& u_2 u_3 = \alpha u_3 u_2 + \beta u_2 + \gamma u_3 - \beta\gamma, \\
& u_2 u_4 = u_4 u_2 - \gamma u_3 + \gamma u_1, \\
& u_3 u_4 = \alpha u_4 u_3 + \beta u_4 + \gamma u_3 - \beta\gamma; \\
M \geq 5: \quad & u_{n+1} u_n = \alpha u_n u_{n+1}, \\
& u_n u_m = u_m u_n, \;\; |n - m| > 1, \; n, m \in \mathbb{Z}_M.
\end{aligned}$$

take place. The constants $\alpha, \beta, \gamma, \eta \in \mathbb{C}, \; \alpha \neq 0$ are arbitrary.

Classical commutative case:

$$
\begin{aligned}
u_{1,t} &= u_2 u_1 - u_1 u_3, & H_1 &= u_1 + u_2 + u_3,\\
u_{2,t} &= u_3 u_2 - u_2 u_1, & H_2 &= u_1 u_2 u_3,\\
u_{3,t} &= u_1 u_3 - u_3 u_2, & \{u_{n+1}, u_n\} &= \mu u_{n+1} u_n + \nu, \quad n \in \mathbb{Z}_3,
\end{aligned}
$$

$$
\begin{aligned}
&\{u_{n+1}, u_n\}_\mu = u_{n+1} u_n, && \{u_{n+1}, u_n\}_\nu = -1,\\
&u_{k,t} = \{u_k, H_1\}_\mu = \{u_k, H_2\}_\nu, && \{u_k, H_2\}_\mu = \{u_k, H_1\}_\nu = 0.
\end{aligned}
$$

Let $H_1 = 3$, if $H_2 = \gamma \neq 0$ solutions in elliptic functions (periodic for $0 < \gamma < 1$).

Quantum case:

$$
\begin{aligned}
&H_1 = u_1 + u_2 + u_3, && H_2 = u_3 u_2 u_1\\
&u_n, u_{n+1} = \alpha u_{n+1} u_n - \eta, && [H_1, H_2] = 0,\\
&[u_n, u_{n+1}]_\alpha = (\alpha - 1) u_{n+1} u_n, && [u_n, u_{n+1}]_\eta = -\eta, \quad n \in \mathbb{Z}_3,
\end{aligned}
$$

$$
u_{k,t} = \tfrac{1}{1-\alpha}[H_1, u_k]_\alpha = \tfrac{1}{\eta}[H_2, u_k]_\eta, \quad [H_2, u_k]_\alpha = [H_1, u_k]_\eta = 0.
$$

The periodical reduction of the system with the period $M = 4$ can be written in the form

$$\partial_{t_2} u_n = u_n u_n u_{n+3} + u_n u_{n+3} u_{n+2} + u_n u_{n+3} u_{n+3} - u_{n+1} u_n u_n - u_{n+1} u_{n+1} u_n - u_{n+2} u_{n+1} u_n, \quad (7)$$

where the lower index $n \in \mathbb{Z}_4$.
The quantisation (b) ideal $\mathfrak{J}_b$ is generated by:

$$\begin{array}{lll} uu_1 = \alpha u_1 u, & uu_2 = -u_2 u, & uu_3 = -u_3 u, \\ u_1 u_2 = -\alpha u_2 u_1, & u_1 u_3 = -u_3 u_1, & u_2 u_3 = \alpha u_3 u_2. \end{array}$$

The algebra $\mathfrak{A}_4 \diagup \mathfrak{J}_b$ has three central elements

$$\mathcal{H} = u_3 u_2 u_1 u, \qquad \mathcal{H}_1 = u_3^2 u_1^2, \qquad \mathcal{H}_2 = u_2^2 u^2.$$

The dynamical system on $\mathfrak{A}_4 \diagup \mathfrak{J}_b$ admits three first integrals

$$H = u + u_1 + u_2 + u_3, \qquad H_1 = u_3 u_1, \qquad H_2 = u_2 u.$$

On $\mathfrak{A}_4 \diagup \mathfrak{J}_b$ the quantum Volterra system can be written as

$$u_{k,t_3} = \frac{1}{\alpha^2 - 1}[H^2, u_k], \qquad k = 0, 1, 2, 3.$$

QUANTISATION OF THE NIB FAMILY OF SYSTEMS

PROPOSITION

Nonabelian N–chain $\partial_t u = \sum_{k=1}^{N}(u_k u - u u_{-k})$ *admits* $\mathfrak{J}_\omega = \langle\{u_q u_p - \omega_{p,q} u_p u_q \mid p, q \in \mathbb{Z},\ p > q,\ \omega_{p,q} \in \mathbb{C}^\times\}\rangle$ *quantisation only in the case*

$$\omega_{n+k,n} = \alpha,\ \ \textit{where}\ \ 1 \le k \le N,\ \alpha \neq 0,\ \textit{and}\quad \omega_{n,m} = 1,\ \ \textit{for}\ \ n - m > N.$$

$$u_n u_{n+k} = \alpha u_{n+k} u_n,\ \ 1 \le k \le N \qquad u_n u_m = u_m u_n,\ \ |n - m| > N.$$

PROPOSITION

There exists a modification ($u = v_2 v_1 v$)

$$v_t = v_2 v_1 v^2 + v_1 v v_{-1} v - v v_1 v v_{-1} - v^2 v_{-1} v_{-2}$$

of the nonabelian $N = 2$ *NIB chain. It admits* $\mathfrak{J}_\omega$*–quantisation only in the case*

$$\omega_{n+3m+1,n} = \alpha,\quad \omega_{n+3m+2,n} = \beta,\quad \omega_{n+3m+3,n} = \alpha^{-1}\beta^{-1},$$

$\alpha, \beta \in \mathbb{C}^\times,\ n \in \mathbb{Z},\ m \in \mathbb{Z}_{\geqslant 0}$

$$v_n v_{n+3m+1} = \alpha v_{n+3m+1} v_n,\ \ v_n v_{n+3m+2} = \beta v_{n+3m+2} v_n,\ \ v_n v_{n+3m+3} = \alpha^{-1}\beta^{-1} v_{n+3m+3} v_n,\ m \geqslant 0$$

$$u = v_2 v_1 v \ \ \Rightarrow\ \ u_n u_{n+1} = \alpha\beta u_{n+1} u_n,\ u_n u_{n+2} = \alpha\beta u_{n+2} u_n,\ u_n u_m = u_m u_n \text{ if } |m - n| > 2.$$

Non-abelian KdV hierarchy:

$$L = D^2 - u, \quad L_{t_{2k+1}} = -u_{t_{2k+1}} = [\left(L^{\frac{2k+1}{2}}\right)_+, L] = -2D\mathrm{res}\left(L^{\frac{2k+1}{2}}\right), \quad D = \partial_x$$

Notations: $D(u) = u' = u_1,\ D^2(u) = u'' = u_2, \dots$

$$\begin{aligned} u_{t_1} &= D(u) = u_1, \\ 4u_{t_3} &= D(u_2 - 3u^2), \\ 16u_{t_5} &= D(u_4 - 5u_2u - 5uu_2 - 5u_1^2 + 10u^3), \\ 64u_{t_7} &= D(u_6 - 7uu_4 - 14u_1u_3 - 21u_2^2 - 14u_3u_1 - 7u_4u \\ &+ 21u^2u_2 + 28uu_1^2 + 28uu_2u + 14u_1uu_1 \\ &+ 28u_1^2u + 21u_2u^2 - 35u^4), \\ 2^{2k}u_{t_{2k+1}} &= D(u_{2k} + \cdots) = D(G_{2k+2}). \end{aligned}$$

Non-abelian Novikov's equation: choose $N \in \mathbb{N}$

$$G_{2N+2} + \sum_{k=1}^{N-1} \alpha_{2(N+1-k)} G_{2k} = \alpha, \qquad \alpha, \alpha_k \in \mathbb{C}.$$

$$\begin{aligned} N = 1: &\quad u_2 - 3u^2 = \alpha. \\ N = 2: &\quad u_4 - 5u_2u - 5uu_2 - 5u_1^2 + \alpha_4 u = \alpha. \\ N = 3: &\quad u_6 - 7uu_4 - 14u_1u_3 + \cdots - 35u^4 + \alpha_6 u = \alpha, \dots \end{aligned}$$

Quantisation [V.Buchstaber, AVM]:

$$N=1: \quad [u_1,u]=\eta, \qquad H=\tfrac{1}{2}u_1^2-u^3-\alpha u,$$

$$\eta\partial_{t_1}(u)=[H,u]=\eta u_1,\ \eta\partial_{t_1}(u_1)=[H,u_1]=\eta(3u^2+\alpha).$$

$$N=2: \quad [u_1,u]=[u,u_2]=[u_1,u_3]=0,\ [u_3,u]=[u_1,u_2]=\eta,\ [u_3,u_2]=10\eta u;$$

$$\begin{aligned}\eta\partial_{t_1}(u_k)&=[H_1,u_k]=\eta u_{k+1}, \quad k=0,1,2\\ \eta\partial_{t_1}(u_3)&=[H_1,u_3]=\eta(\alpha-\alpha_4 u+5u_1^2+10u_2\cdot u-10u^3),\end{aligned}$$

$$\eta\partial_{t_3}(u_k)=[H_2,u_k]=\eta D^{k-1}(u_3-6u_1u) \quad k=0,1,2,3.$$

$$2H_1 = 2u_3\cdot u_1-u_2^2-10u_1^2\cdot u+5u^4+\alpha_4u^2-2\alpha u$$

$$\begin{aligned}2H_2 &= 8\eta u_1-6\alpha u^2+2\alpha u_2+\alpha_4u_1^2-2\alpha_4u_2\cdot u+4\alpha_4u^3\\ &- u_3^2-2u_2\cdot u_1^2+4u_2^2\cdot u+12u_3\cdot u_1\cdot u-30u_1^2\cdot u^2\\ &- 20u_2\cdot u^3+24u^5.\end{aligned}$$

$$\dagger: \quad u_k^\dagger=u_k, \quad (ab)^\dagger=b^\dagger a^\dagger, \quad \eta^\dagger=-\eta, \quad \alpha^\dagger=\alpha, \quad \alpha_4^\dagger=\alpha_4,$$

$$H_1^\dagger=H_1, \qquad H_2^\dagger=H_2, \qquad [H_1,H_2]=0.$$

Quantum Novikov's equations in Heisenberg algebra realisation:

Case $N = 2$. Let $\eta = 32i\hbar$:

$$q_1 = \tfrac{1}{2}u, \qquad p_1 = \tfrac{1}{16}(u_3 - 3uu_1 - 3u_1u),$$

$$q_2 = \tfrac{1}{8}(u_2 - 2u^2) - \alpha_4, \qquad p_2 = \tfrac{1}{4}u_1.$$

Then

$$[q_i, p_j] = i\hbar\delta_{i,j}, \qquad [p_i, p_j] = [q_i, q_j] = 0.$$

$$i\hbar(q_k)_{\tau_{2n-1}} = [q_k, H_n], \qquad i\hbar(p_k)_{\tau_{2n-1}} = [p_k, H_n], \qquad k, n \in \{1, 2\}$$

Operator representation in $\mathbb{R}^2$. Let $p_k \to -i\hbar\frac{d}{dq_k}$:

$$\begin{aligned} H_1 &= 4\hbar^2\left(q_1\partial_{q_2}^2 + 2\partial_{q_1}\partial_{q_2}\right) + 4\alpha_4 q_1^2 + 8\alpha_6 q_1 + 8\alpha_4 q_2 - q_1^4 + 8q_2q_1^2 + 4q_2^2, \\ H_2 &= -2\hbar^2\left(-q_1^2\partial_{q_2}^2 + 2q_2\partial_{q_2}^2 + 2\partial_{q_1}^2 + 2\partial_{q_2}\right) \\ &\quad +4\alpha_4 q_1^3 + 4\alpha_6 q_1^2 - 8\alpha_4 q_2 q_1 - 8\alpha_6 q_2 + 4q_2q_1^3 - 8q_2^2 q_1, \\ &\quad [H_1, H_2] = 0, \quad H_1 = H_1^\dagger,\ H_2 = H_2^\dagger. \end{aligned}$$

Case $N = 3$ we have

$$\begin{aligned} q_1 &= \tfrac{1}{2}u, \\ q_2 &= \tfrac{1}{8}(u_2 - 2u^2) - \alpha_4, \\ q_3 &= \tfrac{1}{96}(3u_4 - 15u_1^2 - 24u_2u + 16u^3) - \alpha_6, \end{aligned}$$

$$\begin{aligned} p_1 &= \tfrac{1}{64}(16\alpha_4u_1 - 20u_2u_1 - 10u_3u + 30u_1u^2 + u_5), \\ p_2 &= \tfrac{1}{16}(-3uu_1 - 3u_1u + u_3), \\ p_3 &= \tfrac{1}{4}u_1, \end{aligned}$$

$$[q_i, p_j] = i\hbar\delta_{i,j}, \qquad [p_i, p_j] = [q_i, q_j] = 0, \qquad \eta = 128i\hbar.$$

There are three polynomial self-adjoint oprtators $H_1,\ H_2,\ H_3$

$$[H_1,\ H_2] = [H_1,\ H_3] = [H_2,\ H_3] = 0,$$

$$(p_k)_{t_{2n-1}} = \frac{i}{\hbar}[H_n, p_k],\ (q_k)_{t_{2n-1}} = \frac{i}{\hbar}[H_n, q_k], \qquad k, n = 1, 2, 3.$$

Let $p_k \to -i\hbar \frac{d}{dq_k}$, then in $\mathbb{R}^3$ there are three polynomial commuting self-adjoint operators:

$$\begin{aligned}
H_1 = \quad & -6\hbar^2 \left(4q_1\partial_{q_3}\partial_{q_2} + q_1^2\partial_{q_3}^2 + 2q_2\partial_{q_3}^2 + 2\partial_{q_2}^2 + 4\partial_{q_1}\partial_{q_3}\right) \\
& -q_1^5 + 8q_2q_1^3 - 24q_3q_1^2 - 12q_2^2q_1 - 24q_2q_3 \\
& -4\alpha_4q_1^3 - 12\alpha_6q_1^2 - 24\alpha_4q_2q_1 - 24\alpha_8q_1 - 24\alpha_4q_3 - 24\alpha_6q_2, \\
H_2 = \quad & -24\hbar^2 \left(-q_1^3\partial_{q_3}^2 - 3q_1^2\partial_{q_2}\partial_{q_3} + 3q_3\partial_{q_3}^2 + 6q_2\partial_{q_2}\partial_{q_3} + 6\partial_{q_1}\partial_{q_2} + 6\partial_{q_3}\right) \\
& +q_1^6 - 6q_2q_1^4 + 48q_3q_1^3 + 108q_2^2q_1^2 - 144q_2q_3q_1 - 72q_2^3 - 72q_3^2 \\
& +12\alpha_4q_1^4 + 48\alpha_6q_1^3 + 144\alpha_4q_2q_1^2 + 72\alpha_8q_1^2 \\
& -144\alpha_4q_3q_1 - 144\alpha_4q_2^2 - 144\alpha_6q_3 - 144\alpha_8q_2, \\
H_1 = \quad & -12\hbar^2 \left(q_1^4\partial_{q_3}^2 + 2q_1^3\partial_{q_2}\partial_{q_3} - 6q_2q_1^2\partial_{q_3}^2 + 6q_3q_1\partial_{q_3}^2 - 12q_2q_1\partial_{q_2}\partial_{q_3}\right. \\
& \left. +12q_3\partial_{q_2}\partial_{q_3} + 6\partial_{q_1}^2 + 6\partial_{q_2}\right) \\
& -q_1^7 + 6q_2q_1^5 - 30q_3q_1^4 - 12q_2^2q_1^3 + 144q_2q_3q_1^2 + 72q_2^3q_1 - 144q_3^2q_1 - 72q_2^2q_3 \\
& -12\alpha_4q_1^5 - 24\alpha_6q_1^4 + 48\alpha_4q_2q_1^3 - 24\alpha_8q_1^3 - 72\alpha_4q_3q_1^2 + 144\alpha_6q_2q_1^2 + 144\alpha_4q_2^2q_1 \\
& -144\alpha_6q_3q_1 + 144\alpha_8q_2q_1 - 144\alpha_4q_2q_3 - 144\alpha_8q_3.
\end{aligned}$$

Let us consider quantisation of the stationary KdV equation:

$$4u_{t_3} = \partial_{t_1}(u_2 - 3u^2) = u_3 - 3u_1 u - 3uu_1 = 0 \;\Rightarrow\; \mathcal{I} = \langle u_3 - 3u_1 u - 3uu_1 \rangle$$

Then $\mathfrak{A}\big/\mathcal{I} = \mathbb{C}\langle u, u_1, u_2\rangle$ and

$$\partial_{t_1}(u) = u_1, \quad \partial_{t_1}(u_1) = u_2, \quad \partial_{t_1}(u_2) = 3u_1 u + 3uu_1.$$

Let us consider a general homogeneous differential ideal generated by

$$\mathfrak{I} = \langle f = [u, u_1] - \alpha u^2 - \beta u_1 - \gamma u - \delta \rangle$$

$(\partial_{t_1}(f) = [u, u_2] - \alpha(u_1 u + uu_1) - \beta u_2 - \gamma u_1 \in \mathfrak{I},\ \partial_{t_1}^2(f) = [u_1, u_2] + \cdots \in \mathfrak{I}, \ldots)$.

PROPOSITION

The ideal $\mathfrak{I} = \langle [u, u_1] - \alpha u^2 - \beta u_1 - \gamma u - \delta \rangle$ is ∂_{t_1} invariant if and only if $\alpha = \beta = 0$, and thus

$$[u, u_1] - \delta - \gamma u, \quad [u, u_2] - \gamma u_1, \quad [u_1, u_2] + \gamma(6u^2 - u_2) + 6\delta u.$$

The dynamical system (the stationary KdV equation)

$$\partial_{t_1} u = u_1, \quad \partial_{t_1} u_1 = u_2, \quad \partial_{t_1} u_2 = 3u_1 u + 3uu_1$$

has TWO compatible quantisations

$$\begin{array}{llll} (A): & [u, u_1]_\delta = \delta, & [u, u_2]_\delta = 0, & [u_2, u_1]_\delta = -6\delta u \\ (B): & [u, u_1]_\gamma = \gamma u, & [u, u_2]_\gamma = \gamma u_1, & [u_2, u_1]_\gamma = \gamma(6u^2 - u_2) \end{array}$$

It has two Hamiltonians

$$(1): \quad H_0 = u_2 - 3u^2, \qquad (2): \quad H_1 = \frac{1}{2}(u_1^2 - u_2 u - uu_2) + 2u^3$$

H_0 is central (Casimir) for (A) and H_1 is central for (B) comutation relations

$$[u_k, H_0]_\delta = 0, \qquad [u_k, H_1]_\gamma = 0, \qquad k = 0, 1, 2.$$

and the dynamical system can be written in the Heisenberg form:

$$\partial_{t_1} u_0 = -\tfrac{1}{\gamma}[H_0, u_0]_\gamma = -\tfrac{1}{\delta}[H_1, u_0]_\delta = u_1,$$

$$\partial_{t_1} u_1 = -\tfrac{1}{\gamma}[H_0, u_1]_\gamma = -\tfrac{1}{\delta}[H_1, u_1]_\delta = u_2,$$

$$\partial_{t_1} u_2 = -\tfrac{1}{\gamma}[H_0, u_2]_\gamma = -\tfrac{1}{\delta}[H_1, u_2]_\delta = 3u_1 u + 3uu_1.$$

BI-QUANTISATION OF THE $N = 1$ NOVIKOV EQUATION

$$\partial_{t_1}(u_2 - 3u^2) = 0 \;\Rightarrow u_2 = 3u^2 + \hat{\alpha}, \quad \text{where } \hat{\alpha} \text{ is a non-commuative constant.}$$

The invariant ($\partial_{t_1} u = u_1,\ \partial_{t_1} u_1 = 3u^2 + \hat{\alpha},\ \partial_{t_1}\hat{\alpha} = 0$) ideal in $\mathbb{C}\langle u, u_1, \hat{\alpha}\rangle$ is generated by

$$[u, u_1] = \gamma u + \delta, \quad [u, \hat{\alpha}] = \gamma u_1, \quad [u_1, \hat{\alpha}] = 3\gamma u^2 + \gamma\hat{\alpha}.$$

Thus there are two compatible quantisations:

$$\begin{array}{llll} (A): & [u, u_1]_\delta = \delta, & [u, \hat{\alpha}]_\delta = 0, & [u_1, \hat{\alpha}]_\delta = 0 \\ (B): & [u, u_1]_\gamma = \gamma u, & [u, \hat{\alpha}]_\gamma = \gamma u_1, & [u_1, \hat{\alpha}]_\gamma = \gamma(3u^2 + \hat{\alpha}) \end{array}$$

There are two Hamiltonians

$$H_0 = \hat{\alpha}, \qquad H_1 = \frac{1}{2}(u_1^2 - 2u^3 - u\hat{\alpha} - \hat{\alpha}u), \qquad [H_0, H_1] = 0,$$

such that $[H_0, \cdot]_\delta = 0,\ [H_1, \cdot]_\gamma = 0$ and

$$\begin{aligned} \partial_{t_1} u &= -\tfrac{1}{\gamma}[H_0, u\,]_\gamma = -\tfrac{1}{\delta}[H_1, u\,]_\delta = u_1, \\ \partial_{t_1} u_1 &= -\tfrac{1}{\gamma}[H_0, u_1]_\gamma = -\tfrac{1}{\delta}[H_1, u_1]_\delta = 3u^2 + \hat{\alpha}, \\ \partial_{t_1}\hat{\alpha} &= -\tfrac{1}{\gamma}[H_0, \hat{\alpha}\,]_\gamma = -\tfrac{1}{\delta}[H_1, \hat{\alpha}\,]_\delta = 0. \end{aligned}$$

In general, for the N-th Novikov equation as well as for stationary $2N+1$ KdV equation we have obtained a bi-quantum structure of the form:
In each case there are

- two compatible commutation rules $[\cdot,\cdot]_\delta,\ [\cdot,\cdot]_\gamma$;
- $N+1$ commuting operators $H_0, H_1, \ldots, H_N$;
- For stationary $2N+1$ KdV the finite quantum hierarchy of commuting symmetries which can be presented in the Heisenberg form

$$\begin{array}{rcl} & & [H_0, u_k]_\delta = 0, \\ u_{k,t_1} & = & \delta^{-1}[H_1, u_k]_\delta = \gamma^{-1}[H_0, u_k]_\gamma, \\ u_{k,t_3} & = & \delta^{-1}[H_2, u_k]_\delta = \gamma^{-1}[H_1, u_k]_\gamma, \\ & & \cdots \\ u_{k,t_{2N-1}} & = & \delta^{-1}[H_N, u_k]_\delta = \gamma^{-1}[H_{N-1}, u_k]_\gamma, \\ & & [H_N, u_k]_\gamma = 0. \end{array} \quad , \qquad k = 0, 1, \ldots 2N-1, 2N.$$

In algebra $\mathfrak{A} = \mathbb{K}\langle u, v\rangle$ we consider systems of two quadratic homogeneous equations

$$\begin{cases} u_t = \alpha_1 u^2 + \alpha_2 uv + \alpha_3 vu + \alpha_4 v^2, \\ v_t = \beta_1 v^2 + \beta_2 vu + \beta_3 uv + \beta_4 u^2 \end{cases} \tag{8}$$

possesing a hierarchy of symmetries.

Let us first consider equations (8) possesing a cubic symmetry

$$\begin{cases} u_\tau = \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uvu + \gamma_4 vu^2 + \gamma_5 uv^2 + \gamma_6 vuv + \gamma_7 v^2 u + \gamma_8 v^3, \\ v_\tau = \delta_1 u^3 + \delta_2 u^2 v + \delta_3 uvu + \delta_4 vu^2 + \delta_5 uv^2 + \delta_6 vuv + \delta_7 v^2 u + \delta_8 v^3 \end{cases} \tag{9}$$

In $\mathfrak{A} = \mathbb{K}\langle u, v\rangle$ let us consider quantisation ideals generated by one polynomial

$$\mathfrak{J} = \langle vu - \alpha uv - \delta u^2 - \beta u - \gamma v - \eta\rangle$$

Quantisation of quadratic systems with a cubic symmetry

Proposition

Any non-triangular system (8) possessing a non-zero cubic symmetry of the form (9) is equivalent to one of the following systems which admits a quantisation ideal $\mathfrak{J}$ generated by the comutation relation:

$$\mathrm{A}_1: \quad \begin{cases} u_t = u^2 - uv \\ v_t = v^2 + vu - uv \end{cases} \qquad uv = vu,$$

$$\mathrm{A}_2: \quad \begin{cases} u_t = uv \\ v_t = vu \end{cases} \qquad vu = \alpha uv, \qquad H = \alpha u - v,$$

$$\mathrm{A}_3: \quad \begin{cases} u_t = u^2 - uv \\ v_t = v^2 - vu \end{cases} \qquad vu = uv - \gamma u + \gamma v, \qquad H = uv - \gamma u$$

$$\mathrm{A}_4: \quad \begin{cases} u_t = -uv \\ v_t = v^2 + uv - vu \end{cases} \qquad vu = uv - \gamma v, \qquad H = uv + \gamma u$$

$$\mathrm{A}_5: \quad \begin{cases} u_t = uv - vu \\ v_t = u^2 + uv - vu \end{cases} \qquad vu = uv + \delta u^2 + \beta u + \eta, \quad ...$$

$$\mathrm{A}_6: \quad \begin{cases} u_t = v^2 \\ v_t = u^2 \end{cases} \qquad vu = uv + \eta, \qquad H = (v^3 - u^3)/3,$$

where $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{K}$ are arbitrary constants and $\alpha \neq 0$ and H is the Hamiltonian.

$$\mathrm{A}_6: u_t = \eta^{-1}[H, u] = v^2, \quad v_t = \eta^{-1}[H, v] = u^2, \quad H_t = 0.$$

QUANTISATION OF QUADRATIC SYSTEMS WITH A QUARTIC SYMMETRY

PROPOSITION

Any non-triangular system (8) possessing a symmetry of degree four, but not of a qubic one admit $\mathfrak{I}$ quantisation with the following commutation relations:

$$\mathrm{B}_1 \quad \begin{cases} u_t = -uv \\ v_t = v^2 + vu \end{cases} \qquad uv = vu + \gamma v, \quad H = 2uv + u^2 + \gamma u$$

$$\mathrm{B}_2 \quad \begin{cases} u_t = -vu \\ v_t = v^2 + vu \end{cases} \qquad uv = vu + \gamma v, \quad H = 2uv + u^2 + \gamma u + 2\gamma v$$

$$\mathrm{B}_3 \quad \begin{cases} u_t = u^2 - 2vu \\ v_t = v^2 - 2vu \end{cases} \qquad vu = uv + \eta, \quad H = u^2 v - uv^2,$$

$$\mathrm{B}_4 \quad \begin{cases} u_t = u^2 - uv - 2vu \\ v_t = v^2 - 2uv - vu \end{cases} \qquad vu = uv,$$

$$\mathrm{B}_5 \quad \begin{cases} u_t = u^2 - 2uv \\ v_t = v^2 + 4vu \end{cases} \qquad vu = uv,$$

where $\gamma, \eta \in \mathbb{K}$ are arbitrary constants.

Heisenberg equations:

$$\kappa u_t = [H, u], \qquad \kappa v_t = [H, v].$$

$$B_1 : \ \kappa = -2\gamma, \quad B_2 : \ \kappa = -2\gamma, \quad B_3 : \ \kappa = \eta.$$