

Five Lectures on Determinants

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§1 Foundation of Determinants

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad i, j \in [n] := \{1, 2, \dots, n\}$$

Def 1.1 (Determinant of A)

$$\det A := \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1, \sigma(1)} a_{2, \sigma(2)} \dots a_{n, \sigma(n)}$$

$\left(\begin{matrix} \text{''' } \\ |A| \end{matrix} \right)$ — permutation gp.

Thm 1.2 (Cramer's formula)

Consider $A \vec{x} = \vec{b}$... (*) w/ $|A| \neq 0$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Solution to (*) is given by

$$x_i = \frac{1}{|A|} \begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{ni} & \dots & b_n & \dots & a_{nn} \end{vmatrix}$$

$\leftarrow i$ -th column is replaced with \vec{b}

Thm 1.3 (Laplace) $|A| \neq 0$, $B := A^{-1}$

$$b_{ij} = (A^{-1})_{ij} = (-1)^{i+j} \frac{|A^{ji}|}{|A|}$$

\leftarrow submatrix of A deleting j -th row & i -th column

\leftarrow Quiz 1. Prove it for $n=3$

Cor 1.4 $\tilde{a}_{ij} := (-1)^{i+j} |A^{ji}|$, $\tilde{A} := (\tilde{a}_{ji}) \Rightarrow A \tilde{A} = \tilde{A} A = |A| I$

Relation to exterior algebra

$\Lambda^n(V)$: exterior algebra of V

$$v_i, v_j \quad v_i \wedge v_j = -v_j \wedge v_i \quad (v_i \wedge v_i = 0)$$

Prop 1.5 $w_i = \sum_{j=1}^n a_{ij} v_j$

$$w_1 \wedge \dots \wedge w_n = |A| v_1 \wedge \dots \wedge v_n$$

(Ex) $n=2$) $w_1 \wedge w_2 = (a_{11} v_1 + a_{12} v_2) \wedge (a_{21} v_1 + a_{22} v_2)$ - $v_1 \wedge v_2$

$$= a_{11} a_{22} v_1 \wedge v_2 + a_{12} a_{21} \overbrace{v_2 \wedge v_1}^{\text{green}}$$

$$= \underbrace{(a_{11} a_{22} - a_{12} a_{21})}_{\sum_{\sigma \in S_2} \text{sgn}(\sigma) a_{1, \sigma(1)} a_{2, \sigma(2)}} v_1 \wedge v_2 = |A| v_1 \wedge v_2$$

Thm 1.6 (Laplace Expansion Theorem)

$$I = \{i_1, \dots, i_r \in [n] \mid i_1 < \dots < i_r\}$$

$$J = \{j_1, \dots, j_r \in [n] \mid j_1 < \dots < j_r\}$$

$$[n] \setminus I = \{i_{r+1}, \dots, i_n \mid i_{r+1} < \dots < i_n\}$$

$$[n] \setminus J = \{j_{r+1}, \dots, j_n \mid j_{r+1} < \dots < j_n\}$$

$$A_{I,J} := (a_{ip, j_q})_{1 \leq p, q \leq r} \quad A_{([n] \setminus I, [n] \setminus J)} = (a_{ip, j_q})_{r+1 \leq p, q \leq n}$$

$r \times r$ $(n-r) \times (n-r)$

$$|A| = \sum_{\substack{I \subset [n] \\ \#I=r}} (-1)^{\Sigma(I) + \Sigma(J)} |A_{I,J}| |A_{([n] \setminus I, [n] \setminus J)}| \quad \text{for a given } J$$

sum is taken over all I

$$= \sum_{\substack{J \subset [n] \\ \#J=r}} (-1)^{\Sigma(I) + \Sigma(J)} |A_{I,J}| |A_{([n] \setminus I, [n] \setminus J)}| \quad \text{for a given } I$$

$(\Sigma(I) := i_1 + \dots + i_r)$

(Proof) ($n=4, r=2$)

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$$W_1 \wedge W_2 \wedge W_3 \wedge W_4 = |A| V_1 \wedge V_2 \wedge V_3 \wedge V_4$$

|| on the other hand

$$(W_1 \wedge W_2) \wedge (W_3 \wedge W_4) \quad (\text{cf. Report Prob 1 (2)})$$

$$\begin{aligned} W_1 \wedge W_2 &= (a_{11}V_1 + a_{12}V_2 + a_{13}V_3 + a_{14}V_4) \wedge (a_{21}V_1 + a_{22}V_2 + a_{23}V_3 + a_{24}V_4) \\ &= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} V_1 \wedge V_2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} V_1 \wedge V_3 + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} V_1 \wedge V_4 \\ &\quad + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} V_2 \wedge V_3 + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} V_2 \wedge V_4 + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} V_3 \wedge V_4 \end{aligned}$$

$$W_3 \wedge W_4 = \text{Quiz 2 (Calculate this)}$$

$$\therefore (W_1 \wedge W_2) \wedge (W_3 \wedge W_4)$$

$$\begin{aligned} &= \left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \right. \\ &\quad \left. + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \right\} \\ &\quad \begin{matrix} // \text{ expanded by "2x2 det"} \\ |A| \\ 4 \times 4 \end{matrix} \quad \dots (**) \quad V_1 \wedge V_2 \wedge V_3 \wedge V_4 \end{aligned}$$

Similarly $(n\text{-form}) = (r\text{-form}) \wedge ((n-r)\text{-form})$

↓
Thm 1.6 (report)

Rmk 1.7 $r=1$ or $n-1 \Rightarrow$ ordinary expansion in a ^{row} column

Thm 1.8 (Plücker relation)

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$$\alpha_1, \dots, \alpha_{r-1} \in [n] \quad \alpha_1 < \dots < \alpha_{r-1}$$

$$\beta_1, \dots, \beta_{r+1} \in [n] \quad \beta_1 < \dots < \beta_{r+1}$$

unnecessary
 ~~$\alpha_i = \beta_j \ (i, j)$~~

$$\sum_{k=1}^{r+1} (-1)^{k-1} \begin{vmatrix} a_{1\alpha_1} & \dots & a_{1\alpha_{r-1}} & a_{1\beta_k} \\ \vdots & & \vdots & \vdots \\ a_{r\alpha_1} & \dots & a_{r\alpha_{r-1}} & a_{r\beta_k} \end{vmatrix} \begin{vmatrix} a_{1\beta_1} & \dots & \overset{\text{omit}}{a_{1\beta_k}} & \dots & a_{1\beta_{r+1}} \\ \vdots & & \vdots & & \vdots \\ a_{r\beta_1} & \dots & a_{r\beta_k} & \dots & a_{r\beta_{r+1}} \end{vmatrix} = 0$$

$r \times r$ $r \times r$

Ex 1.9 $n=4, r=2 \quad \alpha_1=1, \beta_1=2, \beta_2=3, \beta_3=4$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{23} & a_{23} \end{vmatrix} \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0$$

$k=1$ $k=2$ $k=3$ \vdots

Rmk This is a defining eq. of $Gr(2,4)$ in CP^5 (***)

☺ Laplace expansion (***) for $n=4, r=2$ with

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \end{vmatrix} = 0$$

This will be important (P.11)

Rmk 1.10 Maya diagram representation

$$\begin{aligned}
 & (1\ 2)(3\ 4) \\
 (***) \Leftrightarrow & -(1\ 3)(2\ 4) \Leftrightarrow \begin{array}{c} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \\
 & + (1, 4)(2, 3) = 0 \quad + \begin{array}{c} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} = 0
 \end{array}$$

§2 KdV Equation and Wronskian

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2.0 Introduction (\rightarrow Slide)

2.1 Definition & Properties

Def 2.1 (KdV eq.) $u = u(t, x)$: defined on (1+1)-dim space-time

$$u_t + u_{xxx} + 6u u_x = 0 \quad \dots \textcircled{1}$$

↑ partial derivatives ↑ time ↑ space
~~~~~ non-linear term (hard to solve)

Rmk 2.2 Three coefficients are arbitrary

i.e.  $\textcircled{1} \iff \tilde{u}_{\tilde{t}} + A \tilde{u}_{xxx} + B \tilde{u} \tilde{u}_x = 0$

$$\tilde{u} = \frac{6A}{B} u$$
$$\tilde{t} = \frac{1}{A} t$$

Def 2.3 For  $D(t, x), F(t, x), F(t, x \rightarrow \pm\infty) = 0$ ,

$D_t + F_x = 0$  is called conservation law.

$D$ : conserved density,  $F$ : flux

Prop 2.4  $I := \int_{-\infty}^{\infty} D dx$  is conserved (i.e.  $\frac{dI}{dt} = 0$ )

$$\textcircled{\ominus} \frac{dI}{dt} = \int_{-\infty}^{\infty} D_t dx = - \int_{-\infty}^{\infty} F_x dx = -F \Big|_{x=-\infty}^{x=+\infty} = 0 \quad \square$$

Thm 2.5 KdV eq has  $\infty$  conserved densities

c.g.  $D_1 = u, D_2 = u^2, D_3 = u^3 - \frac{1}{2} u_x^2, \dots$        $\textcircled{\ominus}$  See, e.g. [D]

## 2.2 Hirota trf. & Wronskian solution ( $\tau$ -fcn) ⑥

Def 2.6 (Hirota trf.)  $u = 2 \partial_x^2 \log \tau(t, x) \dots$  ②

Prop 2.7 KdV eq. is transformed by ② to

$$\tau \tau_{xt} - \tau_x \tau_t + 3 \tau_{xx}^2 - 4 \tau_x \tau_{xxx} + \tau \tau_{xxxx} = 0 \dots$$
 ③

(Hirota's) bilinear eq.

☺ **Quiz 1: Prove this.** ( ①  $\xrightarrow{②}$   $(\dots)_x = 0 \xrightarrow{x\text{-integ. \& } C=0}$  ③ )

Def 2.8 (Wronskian)

$$\text{Wr}(f_1, \dots, f_n) := \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix} \quad \begin{array}{l} f_i : C^\infty \text{ fcn of } x \\ f_i^{(k)} : k\text{-th } x\text{-derivative of } f_i \end{array}$$

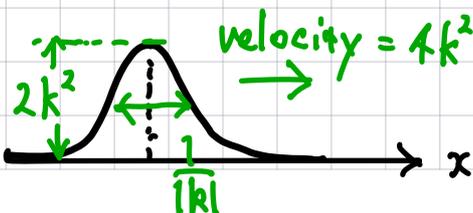
Thm 2.9 General solutions of ③ are the following Wronskians:

$$\tau = \text{Wr}(f_1, \dots, f_N) \quad f_{i,t} = -4 f_{i,xxx}$$

N-soliton solutions are given by the choice:

$$f_i = e^{\theta_i} + a_i e^{-\theta_i}, \quad \theta_i := k_i x - 4k_i^3 t \quad (a_i > 0)$$

Ex 1-soliton solution:  $u = 2 \partial_x^2 \log (e^{kx - 4k^3 t} + a e^{-kx + 4k^3 t})$



$$= 2k^2 \text{cosh}^{-2} (kx - 4k^3 t - \frac{1}{2} \log a)$$

# Asymptotic behavior of the N-soliton solution

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For simplicity, set  $a_i \equiv 1$  &  $0 < k_1 < k_2 < \dots < k_N$

Consider the comoving frame where  $X_I = x - 4k_I^2 t$  is finite.

$$\frac{\theta_i}{k_i} = x - 4k_i^2 t = \underbrace{x - 4k_I^2 t}_{\text{finite}} - 4(k_i^2 - k_I^2)t$$

$$f_i = e^{\theta_i} + e^{-\theta_i} \xrightarrow{t \rightarrow +\infty} \begin{cases} e^{\theta_i} & i < I \\ e^{\theta_I} + e^{-\theta_I} & i = I \\ e^{-\theta_i} & i > I \end{cases}$$

( $t \rightarrow -\infty$  case)  
is similar

$$\tau \xrightarrow{t \rightarrow \infty} \begin{vmatrix} e^{\theta_1} & \dots & e^{\theta_I} + e^{-\theta_I} & \dots & e^{-\theta_N} \\ k_1 e^{\theta_1} & \dots & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & \dots & (-k_N) e^{-\theta_N} \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} e^{\theta_1} & \dots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & \dots & (-k_N)^{N-1} e^{-\theta_N} \\ 1 & \dots & e^{\theta_I} + e^{-\theta_I} & \dots & 1 \\ k_1 & & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & & (-k_N) \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} & \dots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & & (-k_N)^{N-1} \end{vmatrix}$$

$\Delta(a_1, \dots, a_N)$

Vandermonde det

$\prod_{i < j} (a_i - a_j)$

$e^{\text{linear}}$

not contribute  $u$

$$= \underbrace{\Delta(k_1, \dots, k_I, (-k_{I+1}), \dots, (-k_N))}_{\Delta_{I,1}} e^{\theta_I} + \underbrace{\Delta(k_1, \dots, k_{I-1}, (-k_I), \dots, (-k_N))}_{\Delta_{I,2}} e^{-\theta_I}$$

$$u = \partial_x^2 \log \tau = 2k_I^2 \cosh^{-2} \left( k_I x - 4k_I^3 t + \frac{1}{2} \log \frac{\Delta_{I,1}}{\Delta_{I,2}} \right)$$

(I-th) 1-soliton with the position shift (phase shift)

## 2.3 Lax form & KaV hierarchy

[MJD][D]

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### Prop. 2.10 (Lax form of KaV)

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Consider the linear system:

$$\begin{cases} P\psi = 0 \\ \partial_t \psi = B\psi \end{cases} \quad \begin{array}{l} \text{Lax} \\ \text{pair} \end{array} \quad \begin{cases} P := \partial_x^2 + u \\ B := -4\partial_x^3 - 6u\partial_x - 3u_x \end{cases}$$

The compatible condition  $[\partial_t - B, P] = 0 \dots \textcircled{4}$

give rise to the KaV eq.

( $\nearrow u_t = [B, P]$ )  
evolution eq. or flow eq.

☹ **Quiz 1 Show this**  $\uparrow$   
(a little bit difficult)

$$[A, B] := AB - BA$$

of. Quantum Mechanics

$$\hat{x} = x, \hat{p} = \frac{\hbar}{i} \frac{d}{dx} \Rightarrow [\hat{x}, \hat{p}] = i\hbar$$

(for  $\forall \varphi$ )

$$\Leftrightarrow \text{essentially } \left[ \frac{d}{dx}, x \right] = 1 \Leftrightarrow \frac{d}{dx} (x\varphi) - x \left( \frac{d}{dx} \varphi \right) = 1 \cdot \varphi$$

$\swarrow$  cancel  $\searrow$

### Rmk 2.11

2 flow eqs.  $u_t = K[u]$  and  $u_s = \hat{K}[u]$  are commute

$$\text{i.e. } \partial_s K = \partial_t \hat{K} \quad (\Leftrightarrow u_{st} = u_{ts})$$

$\Rightarrow u$  can be considered as  $u = u(t, s)$

Thm 2.12 There exist  $\infty$  commuting flow eqs. for KdV 9

$$\frac{\partial u}{\partial t_{2n+1}} = [B_{2n+1}, P] \quad \text{now } u = u(x, t_3, t_5, t_7, \dots)$$

such that  $\partial_{t_{2m+1}} \partial_{t_{2n+1}} u - \partial_{t_{2n+1}} \partial_{t_{2m+1}} u = 0 \quad (\forall m, n)$   
 (commuting flows)

This is called the KdV hierarchy

$$\textcircled{5} \begin{cases} u_{t_3} = [B_3, P] \Rightarrow \text{(3rd) KdV eq} & (t_3 \equiv t, B_3 \equiv B \text{ in } \textcircled{4}) \\ u_{t_5} = [B_5, P] \Rightarrow \text{(5th) " (exist)} \\ u_{t_7} = [B_7, P] \Rightarrow \text{(7th) " (" )} \\ \vdots & \vdots \end{cases}$$

Prop 2.13

(1)  $\textcircled{5}$  have common conserved density:  $\frac{\partial}{\partial t_{2n+1}} \mathcal{D}_k = 0$  (conserved density in Thm 2.5)

(2)  $\textcircled{5}$  are transformed to bilinear eqs. by the Hirota trf.

3rd KdV  $\longrightarrow$  bilinear eq.  $\textcircled{3}$

5th KdV  $\longrightarrow$  another bilinear eq.

$$\vdots \quad u = 2 \partial_x^2 \log \tau \quad \vdots$$

(3)  $\tau(x, t_3, t_5, t_7, \dots)$  has common Wronskian solutions

$$\tau = \text{Wr}(f_1, \dots, f_N) \quad f_i = f_i(x, t_3, t_5, t_7, \dots) \quad \text{rescaled}$$

$$\text{(for soliton } f_i = e^{\theta_i} + a_i e^{-\theta_i}, \theta_i = k_i x + \sum_{n=1}^{\infty} k_i^{2n+1} t_{2n+1}) \quad \partial_{t_{2n+1}} f_i = \partial_x^{2n+1} f_i$$

# §3 KP equation & Plücker relation



## 3.1 Definition & Properties

Def 3.1 (Kadomtsev-Petviashvili (KP) eq.)  $u = u(x, y, t)$

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

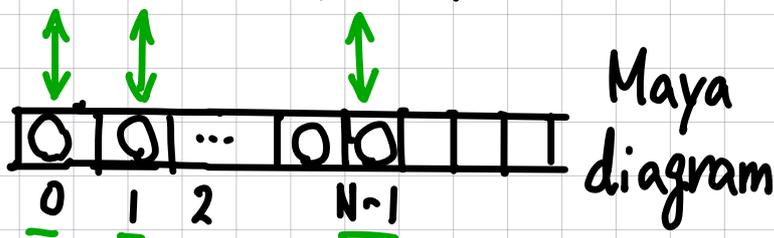
$$\downarrow u = 2 \partial_x^2 \log \tau$$

$$(\tau_{xxxx} - 4\tau_{tx} + 3\tau_{yy})\tau - 4(\tau_{xxx} - \tau_t)\tau_x + 3(\tau_{xx} - \tau_y)(\tau_{xx} + \tau_y) = 0 \quad \dots (*)$$

Thm 3.2 General solutions to (\*) are given by

$$\tau = W_r(f_1, \dots, f_N) \quad \partial_y f_i = \partial_x^2 f_i, \quad \partial_t f_i = \partial_x^3 f_i \quad \dots \textcircled{1}$$

$$= \begin{vmatrix} f_1 & f_1' & \dots & f_1^{(N-1)} \\ f_2 & f_2' & & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N' & \dots & f_N^{(N-1)} \end{vmatrix} \quad \leftarrow \text{transposed}$$



Lemma 3.3 Derivative of determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}' = \begin{vmatrix} a' & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c & d' \end{vmatrix}$$

## 3.2 Sato's observation

Let's enjoy Maya



$$\textcircled{2} \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1),1} \dots a_{\sigma(n),n} \right) \quad \text{Leibniz rule} \quad \square$$

$$\tau_{xx} = \begin{array}{c} \boxed{0 \dots 0} \quad \boxed{0} \\ \text{0} \quad \text{N-2} \quad \text{N+1} \end{array} + \begin{array}{c} \boxed{0 \dots 0} \quad \boxed{0} \quad \boxed{0} \quad \boxed{1} \\ \text{N-3} \quad \text{N-1} \quad \text{N} \end{array} \quad \boxed{11}$$

$$\tau_y = \begin{array}{c} \boxed{0 \dots 0} \quad \boxed{0} \\ \text{0} \quad \text{N-2} \quad \text{N+1} \end{array} - \begin{array}{c} \boxed{0 \dots 0} \quad \boxed{0} \quad \boxed{0} \quad \boxed{1} \\ \text{N-3} \quad \text{N-1} \quad \text{N} \end{array}$$

exchanged  $f_i^{(N)} \rightarrow f_i^{(N+1)}$

$$\tau_{xxx} =$$

$$\tau_t =$$

⋮

Quiz 2 Calculate in terms of Maya diagram

In summary

$$\tau = \begin{array}{c} \text{N-2 N-1 N N+1} \\ \boxed{0} \boxed{0} \quad \boxed{\quad} \quad \boxed{\quad} \end{array}$$

$$\tau_x = \begin{array}{c} \boxed{0} \quad \boxed{\quad} \quad \boxed{0} \quad \boxed{\quad} \end{array}$$

$$\tau_{xx} - \tau_y = 2 \begin{array}{c} \boxed{\quad} \boxed{0} \boxed{0} \quad \boxed{\quad} \end{array}$$

$$\tau_{xx} + \tau_y = 2 \begin{array}{c} \boxed{0} \quad \boxed{\quad} \quad \boxed{\quad} \quad \boxed{0} \end{array}$$

$$\tau_{xxx} - \tau_t = 3 \begin{array}{c} \boxed{\quad} \boxed{0} \quad \boxed{\quad} \quad \boxed{0} \end{array}$$

$$\tau_{xxx} - 4\tau_x + 3\tau_y = 12 \begin{array}{c} \boxed{\quad} \quad \boxed{\quad} \quad \boxed{0} \boxed{0} \end{array}$$

The bilinear equation (\*) becomes

$$\begin{array}{c} \text{N-2 N-1 N N+1} \\ \boxed{0} \boxed{0} \quad \boxed{\quad} \quad \boxed{\quad} \end{array} \times \begin{array}{c} \text{N-2 N-1 N N+1} \\ \boxed{\quad} \quad \boxed{\quad} \quad \boxed{0} \boxed{0} \end{array} \\ - \begin{array}{c} \boxed{0} \quad \boxed{\quad} \quad \boxed{0} \quad \boxed{\quad} \end{array} \times \begin{array}{c} \boxed{\quad} \quad \boxed{0} \quad \boxed{\quad} \quad \boxed{0} \end{array} \\ + \begin{array}{c} \boxed{0} \quad \boxed{\quad} \quad \boxed{\quad} \quad \boxed{0} \end{array} \times \begin{array}{c} \boxed{\quad} \quad \boxed{0} \boxed{0} \quad \boxed{\quad} \end{array} = 0$$

The Plücker relation  
on April 13 🎉

Rmk 3.4 The Plücker relation is obtained as follows [12]

$$\begin{vmatrix}
 f & a_1 & a_2 & 0 & a_3 & a_4 \\
 g & b_1 & b_2 & 0 & b_3 & b_4 \\
 h & c_1 & c_2 & 0 & c_3 & c_4 \\
 0 & 0 & a_2 & f & a_3 & a_4 \\
 0 & 0 & b_2 & g & b_3 & b_4 \\
 0 & 0 & c_2 & h & c_3 & c_4
 \end{vmatrix}
 = (f a_1 a_2)(f a_3 a_4) - (f a_1 a_3)(f a_2 a_4)$$

Laplace

$$+ (f a_1 a_4)(f a_2 a_3) = 0$$

$\downarrow$  extend to  $[H]$   
 $f_1, \dots, f_n \Leftrightarrow$  previous discussion

$\uparrow$  added to Ex 1.9

Thm 3.5 (e.g. [MJD], Sato's lecture note)

(1) There exists KP hierarchy

$$u_{t_3} = K_3[u] \quad \text{infinite commuting flows}$$

$$u_{t_4} = K_4[u] \Rightarrow u = u(x, t_2, t_3, t_4, \dots)$$

⋮

(2)  $\downarrow u = 2 \partial_x^2 \log \tau$

infinite bilinear eqs.

III Observation

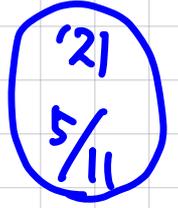
infinite Plücker relations  $\rightarrow$  describe an infinite-dim Grassmann manifold.

Rmk 3.6

- (Noether) "Symmetry  $\Rightarrow$  conservation law" (origin of integrability)
- (Liouville) "conserved quantities  $\Rightarrow$  integrable" (as many as DOF)

# §4 Quasideterminant

## 4.0 Motivation (Slide)



## 4.1 Definition & Properties

$$A = (a_{ij})_{1 \leq i, j \leq n} \quad a_{ij} \in \text{Division Ring (斜体)}$$

$\mathbb{H}^{-1}$  is assumed to exist. (e.g.  $\mathbb{H}$ ) *noncommutative*

### Def 4.1

Let  $A = (a_{ij})$  be an  $n \times n$  square matrix, and  $B = (b_{ij})$  be  $A^{-1}$ .  $b_{ji}^{-1}$  is  $(i, j)$ -quasideterminant of  $A$  and represented:

$$b_{ji}^{-1} =: |A|_{ij} \quad \text{or} \quad \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \boxed{a_{ij}} & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \quad \left( \begin{array}{l} \text{com.} \\ \rightarrow \\ \text{limit} \end{array} \begin{array}{l} (-1)^{i+j} \\ \frac{|A|}{|A_{ij}|} \end{array} \right) \quad \text{Thm. 3}$$

### Rmk 4.2 Block decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1} B S^{-1} C A^{-1} & -A^{-1} B S^{-1} \\ -S^{-1} C A^{-1} & S^{-1} \end{pmatrix} \quad \dots \textcircled{1}$$

w/  $S := D - C A^{-1} B$  (Schur complement)

$$D \equiv a_{ij} \Rightarrow S \equiv |A|_{ij} = a_{ij} - \begin{array}{c} \downarrow i \\ \boxed{A_{ij}} \\ \uparrow j \end{array} \begin{array}{c} \boxed{j\text{-th column}} \\ \downarrow j \end{array}$$

*deleting i-th row & j-th column*

### Ex 4.3

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$$n=1) \quad |A| = a$$

$$n=2) \quad \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} a_{22}^{-1} a_{21}$$

$$\begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11} a_{21}^{-1} a_{22}$$

⋮

$$\therefore \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \stackrel{\text{all squared}}{=} \begin{pmatrix} (A - BD^{-1}C)^{-1} & (C - DB^{-1}A)^{-1} \\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \dots \textcircled{2}$$

( $\begin{matrix} 11 \\ 0 \end{matrix}$  later)

$$n=3) \quad \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \stackrel{\text{blocked}}{=} a_{11} - (a_{12} \ a_{13}) \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$$

$$\stackrel{\textcircled{2}}{=} a_{11} - a_{12} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} a_{31}$$

$$- a_{13} \begin{vmatrix} a_{22} & \boxed{a_{23}} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} a_{31} \dots \textcircled{3}$$

### Quiz 1

Calculate  $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$  in similar way

### Rmk 4.4

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$$\begin{vmatrix} \text{hatched} & \text{yellow} & \text{hatched} \\ \text{blue} & a_{ij} & \text{blue} \\ \text{hatched} & \text{yellow} & \text{hatched} \end{vmatrix} = \begin{vmatrix} \text{hatched} & \text{yellow} \\ \text{blue} & a_{ij} \end{vmatrix}$$

From now on, we use this "canonical" position

### Prop 4.5

(i) For any invertible elements  $\lambda_k, r_k$  ( $1 \leq k \leq n$ ):

$$\begin{vmatrix} a_{11}r_1 & a_{12}r_2 & \dots & a_{1n}r_n \\ \vdots & \vdots & & \vdots \\ a_{n1}r_1 & a_{n2}r_2 & \dots & \boxed{a_{nn}r_n} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & \boxed{a_{nn}} \end{vmatrix} r_n$$

$$\begin{vmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \dots & \lambda_1 a_{1n} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \dots & \lambda_2 a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_n a_{n1} & \lambda_n a_{n2} & \dots & \boxed{\lambda_n a_{nn}} \end{vmatrix} = \lambda_n \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & \boxed{a_{nn}} \end{vmatrix}$$

(ii)  $\vec{a}_k := \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}$  ( $1 \leq k \leq n-1$ )

$k$ th column

(There is row version)

$\vec{a}_\ell$  is added to  $\vec{a}_k$  ( $\ell = 1, \dots, n-1$ )

$$\begin{vmatrix} \vec{a}_1 & \dots & \vec{a}_k + \vec{a}_\ell & \dots & \vec{a}_\ell & \dots & \vec{a}_n \end{vmatrix}_{nn} = \begin{vmatrix} \vec{a}_1 & \dots & \vec{a}_k & \dots & \vec{a}_\ell & \dots & \vec{a}_n \end{vmatrix}_{nn}$$

☺ By definition (see ③) ▣

# 4.2 Useful Identity

## Thm 4.6 (NC <sup>NonCommutative</sup> Jacobi identity)

$$\begin{matrix}
 N \\
 1 \\
 1
 \end{matrix}
 \left( \begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 E & h & i & 
 \end{array} \right) = \begin{array}{c}
 \left| \begin{array}{cc|c}
 A & C & \\
 E & i & 
 \end{array} \right| - \left| \begin{array}{cc|c}
 A & B & \\
 E & h & 
 \end{array} \right| \left| \begin{array}{c|c}
 A & B \\
 D & f 
 \end{array} \right|^{-1} \left| \begin{array}{cc|c}
 A & C & \\
 D & g & 
 \end{array} \right| \dots \textcircled{4}
 \end{array}$$

$(N+2) \times (N+2)$        $\rightarrow$  "i-hf<sup>-1</sup>g" (Schur comp.)       $(N+1) \times (N+1)$

☹ By definition (or report?) ▣

## Cor 4.7 (homological relation) for "moving box"

$$\left( \begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 E & h & i & 
 \end{array} \right) = \left( \begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 E & h & i & 
 \end{array} \right) \left( \begin{array}{c|c}
 A & B & C \\
 0 & 0 & 1 
 \end{array} \right) \dots \textcircled{5}$$

$$\left( \begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 E & h & i & 
 \end{array} \right) = \left( \begin{array}{ccc|c}
 A & B & 0 & \\
 D & f & 0 & \\
 E & h & 1 & 
 \end{array} \right) \left( \begin{array}{c|c}
 A & B & C \\
 D & f & g \\
 E & h & i 
 \end{array} \right) \dots \textcircled{6}$$

☹

$$\textcircled{5}) \left( \begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 E & h & i & 
 \end{array} \right) \left| \begin{array}{c|c}
 A & C \\
 D & g 
 \end{array} \right|^{-1} \stackrel{\textcircled{4}}{=} \left| \begin{array}{cc|c}
 A & C & \\
 E & i & 
 \end{array} \right| \left| \begin{array}{c|c}
 A & C \\
 D & g 
 \end{array} \right|^{-1} - \left| \begin{array}{cc|c}
 A & B & \\
 E & h & 
 \end{array} \right| \left| \begin{array}{c|c}
 A & B \\
 D & f 
 \end{array} \right|^{-1}$$

$$\stackrel{\textcircled{4}}{=} - \left( \begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 E & h & i & 
 \end{array} \right) \left| \begin{array}{c|c}
 A & B \\
 D & f 
 \end{array} \right|^{-1}$$

On the other hand

$$\left( \begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 0 & 0 & 1 & 
 \end{array} \right) \stackrel{\textcircled{4}}{=} - \left( \begin{array}{c|c}
 A & C \\
 D & g 
 \end{array} \right) \left| \begin{array}{cc|c}
 A & B & \\
 D & f & 
 \end{array} \right|^{-1} \quad \blacksquare$$

Quiz 2  
Prove ⑥

Commutative limit of Thm 4.6:

$$\left| \begin{array}{ccc} A & B & C \\ D & f & g \\ E & h & i \end{array} \right| |A| = \left| \begin{array}{cc|c} A & B & C \\ D & f & g \end{array} \right| \left| \begin{array}{c|c} A & C \\ E & i \end{array} \right| - \left| \begin{array}{cc|c} A & B & C \\ E & h & D & g \end{array} \right|$$

Jacobi id (weak version)  
of Report 2(2)

Return back to ① = ②:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} + a^{-1} b s^{-1} c a^{-1} & -a^{-1} b s^{-1} \\ -s^{-1} c a^{-1} & s^{-1} \end{pmatrix} = \begin{pmatrix} (a - b d^{-1} c)^{-1} & (c - d b^{-1} a)^{-1} \\ (b - a c^{-1} d)^{-1} & (d - c a^{-1} b)^{-1} \end{pmatrix}$$

(1-2) component:

$$(c - d b^{-1} a)^{-1} = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right|^{-1} \stackrel{\text{hom}}{=} \underbrace{\left| \begin{array}{c|c} a & b \\ \hline 0 & 1 \end{array} \right|^{-1}}_{-a^{-1} b} \underbrace{\left| \begin{array}{c|c} a & b \\ \hline c & d \end{array} \right|^{-1}}_{s^{-1}} \quad \text{OK}$$

(1-1)

$$(a - b d^{-1} c)^{-1} = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right|^{-1} \stackrel{\text{③}}{=} \left| \begin{array}{ccc} 0 & 1 & 0 \\ 1 & a & b \\ 0 & c & d \end{array} \right|$$

$$\stackrel{\text{Jac}}{=} \underbrace{\left| \begin{array}{cc} 0 & 1 \\ 1 & a \end{array} \right|^{-1}}_{-a^{-1}} + \underbrace{\left| \begin{array}{cc|c} 1 & 0 & a & b \\ a & b & c & d \end{array} \right|^{-1}}_{-a^{-1} b} \underbrace{\left| \begin{array}{c|c} a & b \\ \hline 0 & c \end{array} \right|^{-1}}_{s^{-1}} \underbrace{\left| \begin{array}{c|c} 1 & a \\ \hline 0 & c \end{array} \right|^{-1}}_{-c a^{-1}} \quad \text{OK}$$

# §5 Non-Commutative (NC) Integrable Equations

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## 5.1 Preliminary

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5/18

### Lemma 5.1

$$\begin{vmatrix} A & B \\ C & d \end{vmatrix}' \stackrel{\substack{\uparrow n \\ \downarrow i}}{=} \begin{vmatrix} A & B \\ C' & d' \end{vmatrix} + \sum_{k=1}^n \begin{vmatrix} A & e_k \\ C & 0 \end{vmatrix} \begin{vmatrix} A & B \\ (A^k)' & (B^k)' \end{vmatrix} \dots \textcircled{1}$$

derivative
k-th row of A

$$= \begin{vmatrix} A & B' \\ C & d' \end{vmatrix} + \sum_{k=1}^n \begin{vmatrix} A & (A^k)' \\ C & (C^k)' \end{vmatrix} \begin{vmatrix} A & B \\ {}^t e_k & 0 \end{vmatrix} \dots \textcircled{2}$$

k-th column of A
k-th

$(\odot) (A \cdot A^{-1})' = 0 \rightarrow (A^{-1})' = -A^{-1} A' A^{-1}$

$\underbrace{{}^t e_k}_{(0 \dots 1 \dots 0)}$

$$\begin{aligned} \odot \begin{vmatrix} A & B \\ C & d \end{vmatrix}' &= (d - CA^{-1}B)' = d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B' \\ &= d' - C'A^{-1}B \\ &\quad + \sum_{k=1}^n (CA^{-1}e_k)({}^t e_k A' A^{-1}B) - \sum_{k=1}^n CA^{-1}e_k (B^k)' \end{aligned}$$

$\sum_{k=1}^n e_k {}^t e_k$ : unit matrix

### Def 5.2 (Quasi Wronskian)

Quiz 1. Prove  $\textcircled{2}$

$$\begin{vmatrix} f_1 & \dots & f_n \\ f_1' & & f_n' \\ \vdots & & \vdots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

$\odot$   $\begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} = 0$

same
same

### Rmk 5.3

$\begin{vmatrix} A & B \\ C & d \end{vmatrix}' \rightarrow$  Only  $k=n$  survives in RHS of  $\textcircled{1}$

Q-Wronskian

## 5.2 NC KdV eq.

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$u = u(t, x) : \mathbb{H}$ -valued fcn.

Def 5.4 (NC KdV eq.)

$$u_t + u_{xxx} + 3(u_x u + u u_x) = 0 \quad \dots \textcircled{3}$$

Prop 5.5  $\textcircled{3}$  is derived from compatibility condition of the linear system :

essentially the same as Def 2.10

$$\begin{cases} L \phi = 0 \\ M \phi = 0 \end{cases} \dots \textcircled{4}$$

$\nearrow \mathbb{H}$ -valued

$$\begin{cases} L := \partial_x^2 + u - \lambda \\ M := \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x \end{cases}$$

$\lambda$  spectral parameter

Def 5.6 (Darboux trf.)

$$\theta : \text{special sol. of } \textcircled{4} \text{ i.e. } \begin{cases} L\theta = 0 \\ M\theta = 0 \end{cases} \dots \textcircled{5}$$

$$G_\theta := \theta \partial_x \theta^{-1} = \partial_x - \theta_x \theta^{-1} \dots \textcircled{6} \quad \leftarrow \partial_x \theta^{-1} = -\theta^{-1} \theta_x \theta^{-1}$$

$$(\Leftrightarrow G_\theta f = \theta \partial_x (\theta^{-1} f) = \partial_x f - \theta_x \theta^{-1} f)$$

The following trf. is called the Darboux trf.

$$(D) \begin{cases} L \mapsto \tilde{L} := G_\theta L G_\theta^{-1} (= \partial_x^2 + \tilde{u} - \lambda : \text{form invariant}) \\ M \mapsto \tilde{M} := G_\theta M G_\theta^{-1} \\ \phi \mapsto \tilde{\phi} := G_\theta \phi \stackrel{\textcircled{6}}{=} \phi_x - \theta_x \theta^{-1} \phi \end{cases}$$

Prop 5.7 (D) induces  $\tilde{u} = u + 2(\theta_x \theta^{-1})_x \dots \textcircled{7}$

$$\textcircled{8} (D) \Leftrightarrow \tilde{L} G_\theta = G_\theta L, \tilde{M} G_\theta = G_\theta M \stackrel{\textcircled{6}}{\Rightarrow} \textcircled{7} \quad \square$$



$n+1 \rightarrow n+2$ )

only  $k=n$  survive (Rmk 5.3) 21

$$\phi_{[n+1]}^{(1)} = \begin{pmatrix} \textcircled{1} & \phi \\ \vdots & \vdots \\ \textcircled{1}^{(n-1)} & \phi^{(n-1)} \\ \textcircled{1}^{(n+1)} & \boxed{\phi^{(n+1)}} \end{pmatrix} + \begin{pmatrix} \textcircled{1} & 0 \\ \vdots & \vdots \\ \textcircled{1}^{(n-1)} & 0 \\ \textcircled{1}^{(n)} & \boxed{0} \end{pmatrix} \phi_{[n+1]}$$

$$\theta_{[n+1]}^{(1)} = \begin{pmatrix} \textcircled{1} & \theta_{n+1} \\ \vdots & \vdots \\ \textcircled{1}^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \textcircled{1}^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{pmatrix} + \begin{pmatrix} \textcircled{1} & 0 \\ \vdots & \vdots \\ \textcircled{1}^{(n-1)} & 0 \\ \textcircled{1}^{(n)} & \boxed{1} \end{pmatrix} \theta_{[n+1]}$$

$$\therefore \phi_{[n+1]}^{(1)} - \theta_{[n+1]}^{(1)} \theta_{[n+1]}^{-1} \phi_{[n+1]} = \begin{pmatrix} \textcircled{1} & \phi \\ \vdots & \vdots \\ \textcircled{1}^{(n-1)} & \phi^{(n-1)} \\ \textcircled{1}^{(n+1)} & \boxed{\phi^{(n+1)}} \end{pmatrix} - \begin{pmatrix} \textcircled{1} & \theta_{n+1} \\ \vdots & \vdots \\ \textcircled{1}^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \textcircled{1}^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{pmatrix} \theta_{[n+1]}^{-1} \phi_{[n+1]}$$

On the other hand

$$\phi_{[n+2]} = \begin{pmatrix} \textcircled{1} & \theta_{n+1} & \phi \\ \vdots & \vdots & \vdots \\ \textcircled{1}^{(n-1)} & \theta_{n+1}^{(n-1)} & \phi^{(n-1)} \\ \textcircled{1}^{(n)} & \theta_{n+1}^{(n)} & \phi^{(n)} \\ \textcircled{1}^{(n+1)} & \theta_{n+1}^{(n+1)} & \boxed{\phi^{(n+1)}} \end{pmatrix} = \begin{matrix} \text{Jacobi} \\ \square \end{matrix}$$

Thm 5.9  $U_{[n+1]} = 2 \left( \sum_{k=1}^n \theta_{[k]}^{(1)} \theta_{[k]}^{-1} \right)_x = -2$

[Etingof-Gelfand-Retakh, 1997]

$$\begin{pmatrix} \textcircled{1} & 0 \\ \vdots & \vdots \\ \textcircled{1}^{(n-2)} & 0 \\ \textcircled{1}^{(n-1)} & 0 \\ \textcircled{1}^{(n)} & \boxed{0} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}_x$$

☹  
Quiz 2  
Prove this (induction)



Rmk 5.10

Quasi Wronskian

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$$U_{(n+1)} \stackrel{\text{Jacobi}}{=} + 2 \partial_x \left[ \begin{array}{c|c} \begin{array}{c} \text{gapped} \\ \theta_1^{(n)} \dots \theta_{n-1}^{(n)} \theta_n^{(n)} \end{array} & \begin{array}{c} \theta_n \\ \vdots \\ \theta_n^{(n-2)} \\ \theta_n^{(n-1)} \end{array} \\ \hline \begin{array}{c} \theta_1^{(n-1)} \dots \theta_{n-1}^{(n-1)} \theta_n^{(n-1)} \end{array} & \begin{array}{c} \theta_n \\ \vdots \\ \theta_n^{(n-2)} \\ \theta_n^{(n-1)} \end{array} \end{array} \right]^{-1}$$

↓ commutative limit

$$= 2 \partial_x \left\{ \text{Wr}(\theta_1, \dots, \theta_n) \text{Wr}(\theta_1, \dots, \theta_n)^{-1} \right\}$$

$$= 2 \partial_x^2 \log \text{Wr}(\theta_1, \dots, \theta_n) \quad \text{Hirota trf. \& Wronskian sol.}$$

Quiz 3 Write your opinions, impression etc. for Part I lecture  
(Constructive and Critical comments are welcome.)

Thank You Very Much!