

Five Lectures on Determinants

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§1 Foundation of Determinants

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad i, j \in [n] := \{1, 2, \dots, n\}$$

Def 1.1 (Determinant of A)

$$\det A := \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1, \sigma(1)} a_{2, \sigma(2)} \dots a_{n, \sigma(n)}$$

$\left(\begin{smallmatrix} \text{''' } \\ |A| \end{smallmatrix} \right)$ — permutation gp.

Thm 1.2 (Cramer's formula)

Consider $A \vec{x} = \vec{b}$... (*) w/ $|A| \neq 0$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Solution to (*) is given by

$$x_i = \frac{1}{|A|} \begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{ni} & \dots & b_n & \dots & a_{nn} \end{vmatrix}$$

$\leftarrow i\text{-th column is replaced with } \vec{b}$

Thm 1.3 (Laplace) $|A| \neq 0$, $B := A^{-1}$

$$b_{ij} = (A^{-1})_{ij} = (-1)^{i+j} \frac{|A^{ji}|}{|A|}$$

\leftarrow submatrix of A deleting j-th row & i-th column

\leftarrow Quiz 1. Prove it for $n=3$

Cor 1.4 $\tilde{a}_{ij} := (-1)^{i+j} |A^{ji}|$, $\tilde{A} := (\tilde{a}_{ji}) \Rightarrow A \tilde{A} = \tilde{A} A = |A|$

Relation to exterior algebra

$\Lambda^n(V)$: exterior algebra of V

$$v_i, v_j \quad v_i \wedge v_j = -v_j \wedge v_i \quad (v_i \wedge v_i = 0)$$

Prop 1.5 $w_i = \sum_{j=1}^n a_{ij} v_j$

$$w_1 \wedge \dots \wedge w_n = |A| v_1 \wedge \dots \wedge v_n$$

(Ex) $n=2$) $w_1 \wedge w_2 = (a_{11} v_1 + a_{12} v_2) \wedge (a_{21} v_1 + a_{22} v_2)$

$$= a_{11} a_{22} v_1 \wedge v_2 + a_{12} a_{21} \overbrace{v_2 \wedge v_1}^{-v_1 \wedge v_2}$$

$$= \underbrace{(a_{11} a_{22} - a_{12} a_{21})}_{\sum_{\sigma \in S_2} \text{sgn}(\sigma) a_{1, \sigma(1)} a_{2, \sigma(2)}} v_1 \wedge v_2 = |A| v_1 \wedge v_2$$

Thm 1.6 (Laplace Expansion Theorem)

$$I = \{i_1, \dots, i_r \in [n] \mid i_1 < \dots < i_r\}$$

$$J = \{j_1, \dots, j_r \in [n] \mid j_1 < \dots < j_r\}$$

$$[n] \setminus I = \{i_{r+1}, \dots, i_n \mid i_{r+1} < \dots < i_n\}$$

$$[n] \setminus J = \{j_{r+1}, \dots, j_n \mid j_{r+1} < \dots < j_n\}$$

$$A_{I,J} := (a_{ip, j_q})_{1 \leq p, q \leq r}, \quad A_{([n] \setminus I, [n] \setminus J)} = (a_{ip, j_q})_{r+1 \leq p, q \leq n}$$

$r \times r$ $(n-r) \times (n-r)$

$$|A| = \sum_{\substack{I \subset [n] \\ \#I=r}} (-1)^{\Sigma(I) + \Sigma(J)} |A_{I,J}| |A_{([n] \setminus I, [n] \setminus J)}| : \text{for a given } J$$

sum is taken over all I $(\Sigma(I) := i_1 + \dots + i_r)$

$$= \sum_{\substack{J \subset [n] \\ \#J=r}} (-1)^{\Sigma(I) + \Sigma(J)} |A_{I,J}| |A_{([n] \setminus I, [n] \setminus J)}| : \text{for a given } I$$

(Proof) ($n=4, r=2$)

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$$W_1 \wedge W_2 \wedge W_3 \wedge W_4 = |A| V_1 \wedge V_2 \wedge V_3 \wedge V_4$$

|| on the other hand

$$(W_1 \wedge W_2) \wedge (W_3 \wedge W_4) \quad (\text{cf. Report Prob 1 (2)})$$

$$\begin{aligned} W_1 \wedge W_2 &= (a_{11}V_1 + a_{12}V_2 + a_{13}V_3 + a_{14}V_4) \wedge (a_{21}V_1 + a_{22}V_2 + a_{23}V_3 + a_{24}V_4) \\ &= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} V_1 \wedge V_2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} V_1 \wedge V_3 + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} V_1 \wedge V_4 \\ &\quad + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} V_2 \wedge V_3 + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} V_2 \wedge V_4 + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} V_3 \wedge V_4 \end{aligned}$$

$$W_3 \wedge W_4 = \text{Quiz 2 (Calculate this)}$$

$$\therefore (W_1 \wedge W_2) \wedge (W_3 \wedge W_4)$$

$$\begin{aligned} &= \left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \right. \\ &\quad \left. + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \right\} \\ &\quad \begin{matrix} // \text{ expanded by "2x2 det"} \\ \dots (**) \end{matrix} \quad V_1 \wedge V_2 \wedge V_3 \wedge V_4 \end{aligned}$$

Similarly $(n\text{-form}) = (r\text{-form}) \wedge ((n-r)\text{-form})$

↓
Thm 1.6 (report)

Rmk 1.7 $r=1$ or $n-1 \Rightarrow$ ordinary expansion in a $\begin{matrix} \text{row} \\ \text{column} \end{matrix}$

Thm 1.8 (Plücker relation)

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$$\alpha_1, \dots, \alpha_{r-1} \in [n] \quad \alpha_1 < \dots < \alpha_{r-1}$$

$$\beta_1, \dots, \beta_{r+1} \in [n] \quad \beta_1 < \dots < \beta_{r+1}$$

unnecessary
 ~~$\alpha_i = \beta_j \ (i, j)$~~

$$\sum_{k=1}^{r+1} (-1)^{k-1} \begin{vmatrix} a_{1\alpha_1} & \dots & a_{1\alpha_{r-1}} & a_{1\beta_k} \\ \vdots & & \vdots & \vdots \\ a_{r\alpha_1} & \dots & a_{r\alpha_{r-1}} & a_{r\beta_k} \end{vmatrix} \begin{vmatrix} a_{1\beta_1} & \dots & \overset{\text{omit}}{a_{1\beta_k}} & \dots & a_{1\beta_{r+1}} \\ \vdots & & \vdots & & \vdots \\ a_{r\beta_1} & \dots & a_{r\beta_k} & \dots & a_{r\beta_{r+1}} \end{vmatrix} = 0$$

$r \times r$ $r \times r$

Ex 1.9 $n=4, r=2$

$$\alpha_1 = 1, \beta_1 = 2, \beta_2 = 3, \beta_3 = 4$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{23} & a_{23} \end{vmatrix} \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0$$

$k=1$ $k=2$ $k=3$ \vdots

Rmk This is a defining eq. of $Gr(2,4)$ in CP^5 (***)

☺ Laplace expansion (***) for $n=4, r=2$ with

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \end{vmatrix} = 0$$

This will be important (P.11)

Rmk 1.10 Maya diagram representation

$$\begin{aligned}
 & (1\ 2)(3\ 4) \\
 (***) \Leftrightarrow & -(1\ 3)(2\ 4) \Leftrightarrow \\
 & + (1\ 4)(2\ 3) = 0
 \end{aligned}
 \Leftrightarrow
 \begin{aligned}
 & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \\
 & - \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \\
 & + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \square & \square & \square & \square \\ \hline \end{array} = 0
 \end{aligned}$$

§2 KdV Equation and Wronskian

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2.0 Introduction (\rightarrow Slide)

2.1 Definition & Properties

Def 2.1 (KdV eq.) $u = u(t, x)$: defined on (1+1)-dim space-time

$$u_t + u_{xxx} + 6u u_x = 0 \quad \dots \textcircled{1}$$

↑ partial derivatives ↑ time space ~~~~~ non-linear term (hard to solve)

Rmk 2.2 Three coefficients are arbitrary

i.e. $\textcircled{1} \iff \tilde{u}_{\tilde{t}} + A \tilde{u}_{xxx} + B \tilde{u} \tilde{u}_x = 0$

$$\tilde{u} = \frac{6A}{B} u$$
$$\tilde{t} = \frac{1}{A} t$$

Def 2.3 For $D(t, x)$, $F(t, x)$, $F(t, x \rightarrow \pm\infty) = 0$,

$D_t + F_x = 0$ is called conservation law.

D : conserved density, F : flux

Prop 2.4 $I := \int_{-\infty}^{\infty} D dx$ is conserved (i.e. $\frac{dI}{dt} = 0$)

$$\textcircled{\ominus} \frac{dI}{dt} = \int_{-\infty}^{\infty} D_t dx = - \int_{-\infty}^{\infty} F_x dx = -F \Big|_{x=-\infty}^{x=+\infty} = 0 \quad \square$$

Thm 2.5 KdV eq has ∞ conserved densities

c.g. $D_1 = u$, $D_2 = u^2$, $D_3 = u^3 - \frac{1}{2} u_x^2$, ... $\textcircled{\ominus}$ See, e.g. [D]

2.2 Hirota trf. & Wronskian solution (τ -fcn) ⑥

Def 2.6 (Hirota trf.) $u = 2 \partial_x^2 \log \tau(t, x) \dots$ ②

Prop 2.7 KdV eq. is transformed by ② to

$$\tau \tau_{xt} - \tau_x \tau_t + 3 \tau_{xx}^2 - 4 \tau_x \tau_{xxx} + \tau \tau_{xxxx} = 0 \dots$$
 ③

(Hirota's) bilinear eq.

☺ **Quiz 1: Prove this.** (① $\xrightarrow{②}$ $(\dots)_x = 0 \xrightarrow{x\text{-integ. \& } C=0}$ ③)

Def 2.8 (Wronskian)

$$\text{Wr}(f_1, \dots, f_n) := \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix} \quad \begin{array}{l} f_i : C^\infty \text{ fcn of } x \\ f_i^{(k)} : k\text{-th } x\text{-derivative} \\ \text{of } f_i \end{array}$$

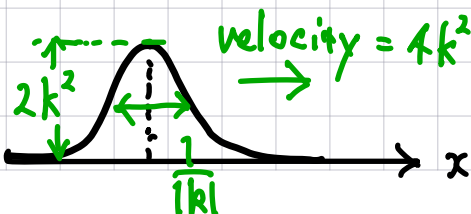
Thm 2.9 General solutions of ③ are the following Wronskians:

$$\tau = \text{Wr}(f_1, \dots, f_N) \quad f_{i,t} = -4 f_{i,xxx}$$

N-soliton solutions are given by the choice:

$$f_i = e^{\theta_i} + a_i e^{-\theta_i}, \quad \theta_i := k_i x - 4k_i^3 t \quad (a_i > 0)$$

Ex 1-soliton solution: $u = 2 \partial_x^2 \log (e^{kx - 4k^3 t} + a e^{-kx + 4k^3 t})$



$$= 2k^2 \text{cosh}^{-2} (kx - 4k^3 t - \frac{1}{2} \log a)$$

Asymptotic behavior of the N-soliton solution

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For simplicity, set $a_i \equiv 1$ & $0 < k_1 < k_2 < \dots < k_N$

Consider the comoving frame where $X_I = x - 4k_I^2 t$ is finite.

$$\frac{\theta_i}{k_i} = x - 4k_i^2 t = \underbrace{x - 4k_I^2 t}_{\text{finite}} - 4(k_i^2 - k_I^2)t$$

$$f_i = e^{\theta_i} + e^{-\theta_i} \xrightarrow{t \rightarrow +\infty} \begin{cases} e^{\theta_i} & i < I \\ e^{\theta_I} + e^{-\theta_I} & i = I \\ e^{-\theta_i} & i > I \end{cases}$$

($t \rightarrow -\infty$ case)
is similar

$$\tau \xrightarrow{t \rightarrow \infty} \begin{vmatrix} e^{\theta_1} & \dots & e^{\theta_I} + e^{-\theta_I} & \dots & e^{-\theta_N} \\ k_1 e^{\theta_1} & \dots & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & \dots & (-k_N) e^{-\theta_N} \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} e^{\theta_1} & \dots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & \dots & (-k_N)^{N-1} e^{-\theta_N} \\ 1 & \dots & e^{\theta_I} + e^{-\theta_I} & \dots & 1 \\ k_1 & & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & & (-k_N) \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} & \dots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & & (-k_N)^{N-1} \end{vmatrix}$$

$\Delta(a_1, \dots, a_N)$
" "
van der Monde det
" "
 $\prod_{i < j} (a_i - a_j)$

e^{linear}
not contribute u

$$= \underbrace{\Delta(k_1, \dots, k_I, (-k_{I+1}), \dots, (-k_N))}_{\Delta_{I,1}} e^{\theta_I} + \underbrace{\Delta(k_1, \dots, k_{I-1}, (-k_I), \dots, (-k_N))}_{\Delta_{I,2}} e^{-\theta_I}$$

$$u = \partial_x^2 \log \tau = 2k_I^2 \cosh^{-2} \left(k_I x - 4k_I^3 t + \frac{1}{2} \log \frac{\Delta_{I,1}}{\Delta_{I,2}} \right)$$

(I-th) 1-soliton with the position shift (phase shift)

2.3 Lax form & KaV hierarchy

[MJD][D]

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Prop. 2.10 (Lax form of KaV)

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Consider the linear system:

$$\begin{cases} P\psi = 0 \\ \partial_t \psi = B\psi \end{cases} \quad \begin{array}{l} \text{Lax} \\ \text{pair} \end{array} \quad \begin{cases} P := \partial_x^2 + u \\ B := -4\partial_x^3 - 6u\partial_x - 3u_x \end{cases}$$

The compatible condition $[\partial_t - B, P] = 0 \dots \textcircled{4}$

give rise to the KaV eq.

($\nearrow u_t = [B, P]$)
evolution eq. or flow eq.

☹ Quiz 1 Show this \uparrow
(a little bit difficult)

$$[A, B] := AB - BA$$

of. Quantum Mechanics

$$\hat{x} = x, \hat{p} = \frac{\hbar}{i} \frac{d}{dx} \Rightarrow [\hat{x}, \hat{p}] = i\hbar$$

(for $\forall \varphi$)

$$\Leftrightarrow \text{essentially } \left[\frac{d}{dx}, x \right] = 1 \Leftrightarrow \frac{d}{dx} (x\varphi) - x \left(\frac{d}{dx} \varphi \right) = 1 \cdot \varphi$$

\swarrow cancel \nwarrow

Rmk 2.11

2 flow eqs. $u_t = K[u]$ and $u_s = \hat{K}[u]$ are commute

$$\text{i.e. } \partial_s K = \partial_t \hat{K} \quad (\Leftrightarrow u_{st} = u_{ts})$$

$\Rightarrow u$ can be considered as $u = u(t, s)$

Thm 2.12 There exist ∞ commuting flow eqs. for KdV 9

$$\frac{\partial u}{\partial t_{2n+1}} = [B_{2n+1}, P] \quad \text{now } u = u(x, t_3, t_5, t_7, \dots)$$

such that $\partial_{t_{2m+1}} \partial_{t_{2n+1}} u - \partial_{t_{2n+1}} \partial_{t_{2m+1}} u = 0 \quad (\forall m, n)$
 (commuting flows)

This is called the KdV hierarchy

$$\textcircled{5} \begin{cases} u_{t_3} = [B_3, P] \Rightarrow \text{(3rd) KdV eq} & (t_3 \equiv t, B_3 \equiv B \text{ in } \textcircled{4}) \\ u_{t_5} = [B_5, P] \Rightarrow \text{(5th) " (exist)} \\ u_{t_7} = [B_7, P] \Rightarrow \text{(7th) " (")} \\ \vdots & \text{(")} \end{cases}$$

Prop 2.13

(1) $\textcircled{5}$ have common conserved density: $\frac{\partial}{\partial t_{2n+1}} \mathcal{D}_k = 0$ (conserved density in Thm 2.5)

(2) $\textcircled{5}$ are transformed to bilinear eqs. by the Hirota trf.

3rd KdV \longrightarrow bilinear eq. $\textcircled{3}$

5th KdV \longrightarrow another bilinear eq.

$$\vdots \quad u = 2 \partial_x^2 \log \tau \quad \vdots$$

(3) $\tau(x, t_3, t_5, t_7, \dots)$ has common Wronskian solutions

$$\tau = \text{Wr}(f_1, \dots, f_N) \quad f_i = f_i(x, t_3, t_5, t_7, \dots) \quad \text{rescaled}$$

$$\text{(for soliton } f_i = e^{\theta_i} + a_i e^{-\theta_i}, \theta_i = k_i x + \sum_{n=1}^{\infty} k_i^{2n+1} t_{2n+1}) \quad \partial_{t_{2n+1}} f_i = \partial_x^{2n+1} f_i$$

§3 KP equation & Plücker relation



3.1 Definition & Properties

Def 3.1 (Kadomtsev-Petviashvili (KP) eq.) $u = u(x, y, t)$

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

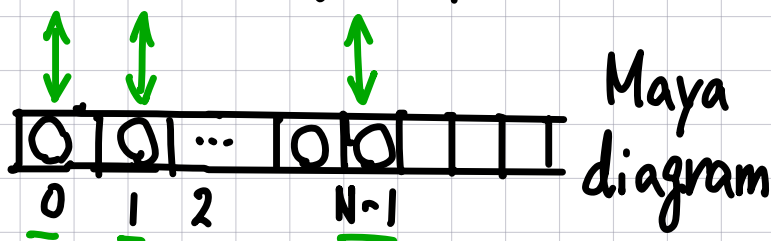
$$\downarrow u = 2 \partial_x^2 \log \tau$$

$$(\tau_{xxxx} - 4\tau_{tx} + 3\tau_{yy})\tau - 4(\tau_{xxx} - \tau_t)\tau_x + 3(\tau_{xx} - \tau_y)(\tau_{xx} + \tau_y) = 0 \quad \dots (*)$$

Thm 3.2 General solutions to (*) are given by

$$\tau = W_r(f_1, \dots, f_N) \quad \partial_y f_i = \partial_x^2 f_i, \quad \partial_t f_i = \partial_x^3 f_i \quad \dots \textcircled{1}$$

$$= \begin{vmatrix} f_1 & f_1' & \dots & f_1^{(N-1)} \\ f_2 & f_2' & & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N' & \dots & f_N^{(N-1)} \end{vmatrix} \quad \leftarrow \text{transposed}$$

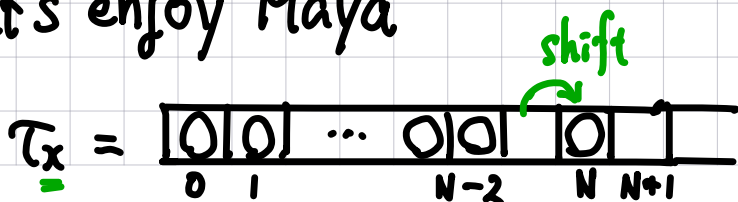


Lemma 3.3 Derivative of determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}' = \begin{vmatrix} a' & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c & d' \end{vmatrix}$$

3.2 Sato's observation

Let's enjoy Maya



$$\textcircled{2} \left(\sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1),1} \dots a_{\sigma(n),n} \right) \quad \text{Leibniz rule} \quad \square$$

$$\tau_{xx} = \begin{array}{|c|c|c|c|} \hline 0 & \dots & 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 0 & \dots & 0 & 0 \\ \hline \end{array} \quad \text{II}$$

$\begin{array}{cccc} & & N-2 & N+1 \\ & & \circ & \circ \end{array}$
 $\begin{array}{cccc} & & N-3 & N-1 \\ & & \circ & \circ \end{array}$

$$\tau_y = \begin{array}{|c|c|c|c|} \hline 0 & \dots & 0 & 0 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 0 & \dots & 0 & 0 \\ \hline \end{array}$$

$\begin{array}{cccc} \circ & & & \\ & & N-2 & N+1 \end{array}$
 $\begin{array}{cccc} & & N-3 & N-1 \\ & & \circ & \circ \end{array}$

exchanged $f_i^{(N)} \rightarrow f_i^{(N+1)}$

$$\tau_{xxx} =$$

$$\tau_t =$$

Quiz 2 Calculate in terms of Maya diagram

⋮

In summary

$$\tau = \begin{array}{|c|c|c|c|} \hline & N-2 & N-1 & N & N+1 \\ \hline 0 & 0 & & & \end{array}$$

$$\tau_x = \begin{array}{|c|c|c|c|} \hline 0 & & 0 & \\ \hline \end{array}$$

$$\tau_{xx} - \tau_y = 2 \begin{array}{|c|c|c|c|} \hline & 0 & 0 & \\ \hline \end{array}$$

$$\tau_{xx} + \tau_y = 2 \begin{array}{|c|c|c|c|} \hline 0 & & & 0 \\ \hline \end{array}$$

$$\tau_{xxx} - \tau_t = 3 \begin{array}{|c|c|c|c|} \hline & 0 & & 0 \\ \hline \end{array}$$

$$\tau_{xxx} - 4\tau_x + 3\tau_y = 12 \begin{array}{|c|c|c|c|} \hline & & 0 & 0 \\ \hline \end{array}$$

The bilinear equation (*) becomes

$$\begin{array}{|c|c|c|c|} \hline & N-2 & N-1 & N & N+1 \\ \hline 0 & 0 & & & \end{array} \times \begin{array}{|c|c|c|c|} \hline & N-2 & N-1 & N & N+1 \\ \hline & & & 0 & 0 \\ \hline \end{array} \\ - \begin{array}{|c|c|c|c|} \hline 0 & & 0 & & \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline & & 0 & & 0 \\ \hline \end{array} \\ + \begin{array}{|c|c|c|c|} \hline 0 & & & 0 & \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline & & 0 & 0 & \\ \hline \end{array} = 0$$

The Plücker relation

on April 13 🎉

Rmk 3.4 The Plücker relation is obtained as follows [12]

$$\begin{vmatrix}
 f & a_1 & a_2 & 0 & a_3 & a_4 \\
 g & b_1 & b_2 & 0 & b_3 & b_4 \\
 h & c_1 & c_2 & 0 & c_3 & c_4 \\
 0 & 0 & a_2 & f & a_3 & a_4 \\
 0 & 0 & b_2 & g & b_3 & b_4 \\
 0 & 0 & c_2 & h & c_3 & c_4
 \end{vmatrix}
 = (f a_1 a_2)(f a_3 a_4) - (f a_1 a_3)(f a_2 a_4)$$

Laplace

$$+ (f a_1 a_4)(f a_2 a_3) = 0$$

\downarrow extend to $[H]$
 $f_1, \dots, f_n \Leftrightarrow$ previous discussion

\uparrow add to Ex 1.9

Thm 3.5 (e.g. [MJD], Sato's lecture note)

(1) There exists KP hierarchy

$$u_{t_3} = K_3[u] \quad \text{infinite commuting flows}$$

$$u_{t_4} = K_4[u]$$

⋮

$$\Rightarrow u = u(x, t_2, t_3, t_4, \dots)$$

(2) $\downarrow u = 2 \partial_x^2 \log \tau$

infinite bilinear eqs.

III Observation

infinite Plücker relations \rightarrow describe an infinite-dim Grassmann manifold.

Rmk 3.6

- (Noether) "Symmetry \Rightarrow conservation law" (origin of integrability)
- (Liouville) "conserved quantities \Rightarrow integrable" (as many as DOF)