Five Lectures on Determinants

§1 Foundation of Determinants

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad i,j \in [n] := \{1,2,\cdots,n\}$$

<u>Def 1.1</u> (Determinant of A)

$$\underset{\left(\underset{|A|}{\parallel}\right)}{\det A} := \sum_{\underset{\sim\ \text{permutation gp.}}{\sigma \in S_n}} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

<u>Thm 1.2</u> (Cramer's formula)

Consider $A\vec{x} = \vec{b}$ $\cdots (*)$ w/ $|A| \neq 0$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Solution to $(*)$ is given by

$$x_i = \frac{1}{|A|} \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}$$

$\hat{i}$ ← i-th column is replaced with $\vec{b}$

<u>Thm 1.3</u> (Laplace) $|A| \neq 0$, $B := A^{-1}$

$$b_{ij} = (A^{-1})_{ij} = (-1)^{i+j} \frac{|A^{ji}|}{|A|}$$

← submatrix of A deleting j-th row & i-th column

← Quiz 1. Prove it for $n=3$

<u>Cor 1.4</u> $\tilde{a}_{ij} := (-1)^{i+j} \underbrace{|A^{ij}|}_{}$, (i,j) minor, $\tilde{A} := (\tilde{a}_{ji}) \Rightarrow A\tilde{A} = \tilde{A}A = |A|$

Relation to exterior algebra

$\Lambda^n(V)$: exterior algebra of V

$v_i, v_j \in \Lambda^n(V)$ $\qquad v_i \wedge v_j = -v_j \wedge v_i \quad (v_i \wedge v_i = 0)$

Prop 1.5 $\quad w_i = \sum_{j=1}^{n} a_{ij} v_j$

$$w_1 \wedge \cdots \wedge w_n = |A|\, v_1 \wedge \cdots \wedge v_n$$

(Ex) $n=2$) $\quad w_1 \wedge w_2 = (a_{11} v_1 + a_{12} v_2) \wedge (a_{21} v_1 + a_{22} v_2)$

$$= a_{11} a_{22}\, v_1 \wedge v_2 + a_{12} a_{21} \underbrace{v_2 \wedge v_1}_{= -v_1 \wedge v_2}$$

$$= \underbrace{(a_{11} a_{22} - a_{12} a_{21})}_{= \sum_{\sigma \in S_2} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)}} v_1 \wedge v_2 = |A|\, v_1 \wedge v_2$$

Thm 1.6 (Laplace Expansion Theorem)

$I = \{ i_1, \cdots, i_r \in [n] \mid i_1 < \cdots < i_r \}$

$J = \{ j_1, \cdots, j_r \in [n] \mid j_1 < \cdots < j_r \}$

$[n] \setminus I = \{ i_{r+1}, \cdots, i_n \mid i_{r+1} < \cdots < i_n \}$

$[n] \setminus J = \{ j_{r+1}, \cdots, j_n \mid j_{r+1} < \cdots < j_n \}$

$\underset{r \times r}{A_{I,J}} := (a_{i_p, j_q})_{1 \le p, q \le r}$, $\quad \underset{(n-r) \times (n-r)}{A_{[n]\setminus I, [n]\setminus J}} = (a_{i_p, j_q})_{r+1 \le p, q \le n}$

$$|A| = \sum_{\substack{I \subset [n] \\ \# I = r}} (-1)^{\Sigma(I) + \Sigma(J)} |A_{I,J}|\, |A_{[n]\setminus I, [n]\setminus J}| \quad : \text{for a given } J$$

sum is taken over all I

$(\Sigma(I) := i_1 + \cdots + i_r)$

$$= \sum_{\substack{J \subset [n] \\ \# J = r}} (-1)^{\Sigma(I) + \Sigma(J)} |A_{I,J}|\, |A_{[n]\setminus I, [n]\setminus J}| \quad : \text{for a given } I$$

(Proof) $(n=4,\ r=2)$

$$w_1 \wedge w_2 \wedge w_3 \wedge w_4 = \overset{4\times 4}{|A|}\, v_1 \wedge v_2 \wedge v_3 \wedge v_4$$

$\parallel$ on the other hend

$(w_1 \wedge w_2) \wedge (w_3 \wedge w_4)$ (cf. Report Prob 1 (2))

$$w_1 \wedge w_2 = (a_{11} v_1 + a_{12} v_2 + a_{13} v_3 + a_{14} v_4) \wedge (a_{21} v_1 + a_{22} v_2 + a_{23} v_3 + a_{24} v_4)$$

$$= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} v_1 \wedge v_2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} v_1 \wedge v_3 + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} v_1 \wedge v_4$$

$$+ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} v_2 \wedge v_3 + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} v_2 \wedge v_4 + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} v_3 \wedge v_4$$

$w_3 \wedge w_4 =$ Quiz 2 (Calculate this)

$\therefore\ (w_1 \wedge w_2) \wedge (w_3 \wedge w_4)$

$$= \left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \right\} v_1 \wedge v_2 \wedge v_3 \wedge v_4$$

$\parallel$ expanded by "2×2 det"

$\underset{4\times 4}{|A|}$ ····· (**)

Similarly (n-form) = (r-form) $\wedge$ ((n−r)-form)

$\Downarrow$

Thm 1.6 (report)

<u>Rmk 1.7</u> $r = 1$ or $n-1 \Rightarrow$ ordinary expansion in a row / column

Thm 1.8 (Plücker relation)

$\alpha_1, \cdots, \alpha_{r-1} \in [n]$ $\quad \alpha_1 < \cdots < \alpha_{r-1}$

$\beta_1, \cdots, \beta_{r+1} \in [n]$ $\quad \beta_1 < \cdots < \beta_{r+1}$ $\quad \alpha_i \neq \beta_j \ ({}^\forall i, j)$

$$\sum_{k=1}^{r+1} (-1)^{k-1} \underbrace{\begin{vmatrix} a_{1\alpha_1} & \cdots & a_{1\alpha_{r-1}} & a_{1\beta_k} \\ \vdots & & \vdots & \vdots \\ a_{r\alpha_1} & \cdots & a_{r\alpha_{r-1}} & a_{r\beta_k} \end{vmatrix}}_{r \times r} \underbrace{\begin{vmatrix} a_{1\beta_1} & \cdots & \overset{\text{omit}}{a_{1\beta_k}} & \cdots & a_{1\beta_{r+1}} \\ \vdots & & \vdots & & \vdots \\ a_{r\beta_1} & \cdots & a_{r\beta_k} & \cdots & a_{r\beta_{r+1}} \end{vmatrix}}_{r \times r} = 0$$

Ex 1.9 $n = 4, r = 2$ $\quad \alpha = 1, \beta_1 = 2, \beta_2 = 3, \beta_3 = 4$

$$\underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}}_{k=1} - \underbrace{\begin{vmatrix} a_{11} & a_{13} \\ a_{23} & a_{23} \end{vmatrix} \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}}_{k=2} + \underbrace{\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}_{k=3} = 0 \quad \because \ (***)$$

Rmk This is a defining eq. of $Gr(2,4)$ in $\mathbb{CP}_5$

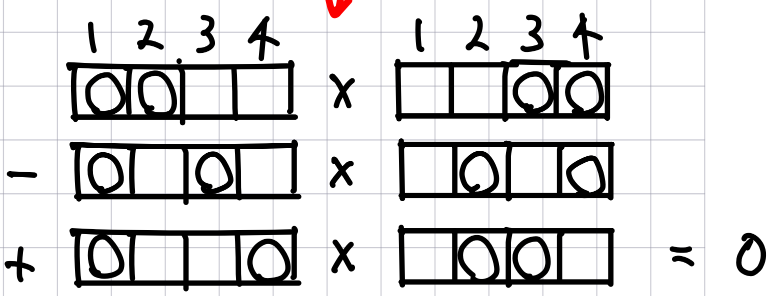
$\because$ Laplace expansion (**) for $n=4, r=2$ with

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \end{vmatrix} = 0 \qquad \blacksquare$$

This will be important (P.11)

Rmk 1.10 Maya diagram representation

(***) $\Leftrightarrow$ $(1\ 2)(3\ 4) - (1\ 3)(2, 4) + (1, 4)(2, 3) = 0$ $\Leftrightarrow$



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§2 KdV Equation and Wronskian

2.0 Introduction (→ Slide)

2.1 Definition & Properties

Def 2.1 (KdV eq.) $u = u(t, x)$: defined on (1+1)-dim space-time (t: time, x: space)

$$u_t + u_{xxx} + 6uu_x = 0 \quad \cdots ①$$

u_t, u_{xxx}: partial derivatives; $6uu_x$: non-linear term (hard to solve)

Rmk 2.2 Three coefficients are arbitrary

i.e. $① \Longleftrightarrow \tilde{u}_{\tilde{t}} + A\tilde{u}_{xxx} + B\tilde{u}\tilde{u}_x = 0$

$$\tilde{u} = \frac{6A}{B}u, \quad \tilde{t} = \frac{1}{A}t$$

Def 2.3 For $D(t,x)$, $F(t,x)$, $F(t, x\to\pm\infty) = 0$,

$D_t + F_x = 0$ is called conservation law.

D: conserved density, F: flux

Prop 2.4 $I := \int_{-\infty}^{\infty} D\,dx$ is conserved $\left(\text{i.e. } \frac{dI}{dt} = 0\right)$

∵ $$\frac{dI}{dt} = \int_{-\infty}^{\infty} D_t\,dx = -\int_{-\infty}^{\infty} F_x\,dx = -F\Big|_{x=-\infty}^{x=+\infty} = 0 \quad \blacksquare$$

Thm 2.5 KdV eq has ∞ conserved densities

e.g. $D_1 = u$, $D_2 = u^2$, $D_3 = u^3 - \frac{1}{2}u_x^2$, $\cdots$ ∵ See, e.g. [D]

2.2 Hirota trf. & Wronskian solution (τ-fcn)

Def 2.6 (Hirota trf.) $u = 2\,\partial_x^2 \log \tau(t,x) \quad \cdots ②$

Prop 2.7 KdV eq. is transformed by ② to

$$\tau\tau_{xt} - \tau_x\tau_t + 3\tau_{xx}^2 - 4\tau_x\tau_{xxx} + \tau\tau_{xxxx} = 0 \quad \cdots ③$$

(Hirota's) bilinear eq.

☺ Quiz 1: Prove this. $\left(① \xrightarrow{②} (\cdots)_x = 0 \xrightarrow{x\text{-integ. } \& \ \overset{\text{integ. const.}}{C} = 0} ③ \right)$

Def 2.8 (Wronskian)

$$\mathrm{Wr}(f_1, \cdots, f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$$

f_i : C^∞-fcns of x

$f_i^{(k)}$: k-th x-derivative of f_i

Thm 2.9 General solutions of ③ are the following Wronskians:

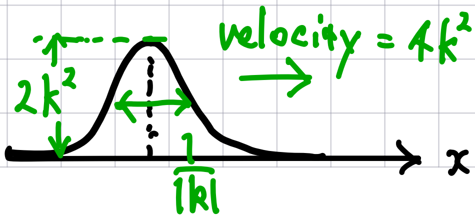
$$\tau = \mathrm{Wr}(f_1, \cdots, f_N) \qquad f_{i,t} = -4 f_{i,xxx}$$

N-soliton solutions are given by the choice:

$$f_i = e^{\theta_i} + a_i e^{-\theta_i}, \quad \theta_i := k_i x - 4k_i^3 t \qquad (a_i > 0)$$

Ex 1-soliton solution: $u = 2\partial_x^2 \log\left(e^{kx - 4k^3 t} + a e^{-kx + 4k^3 t}\right)$

$$= 2k^2 \cosh^{-2}\left(kx - 4k^3 t - \tfrac{1}{2}\log a\right)$$



Asymptotic behavior of the N-soliton solution

For simplicity, set $a_i \equiv 1$ & $0<k_1<k_2<\cdots<k_n$

Consider the comoving frame where $X_I = x-4k_I^2 t$ is finite.

$$\frac{\theta_i}{k_i} = x-4k_i^2t = \underbrace{x-4k_I^2t}_{\text{finite}} - 4(k_i^2-k_I^2)t$$

$$f_i = e^{\theta_i}+e^{-\theta_i} \xrightarrow{t\to+\infty} \begin{cases} e^{\theta_i} & i<I \\ e^{\theta_I}+e^{-\theta_I} & i=I \\ e^{-\theta_i} & i>I \end{cases}$$

($t\to-\infty$ case is similar)

$$\tau \xrightarrow{t\to\infty} \begin{vmatrix} e^{\theta_1} & \cdots & e^{\theta_I}+e^{-\theta_I} & \cdots & e^{-\theta_N} \\ k_1e^{\theta_1} & \cdots & k_Ie^{\theta_I}+(-k_I)e^{-\theta_I} & \cdots & (-k_N)e^{-\theta_N} \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1}e^{\theta_1} & \cdots & k_I^{N-1}e^{\theta_I}+(-k_I)^{N-1}e^{-\theta_I} & \cdots & (-k_N^{N-1})e^{-\theta_N} \end{vmatrix}$$

$\Delta(a_1,\cdots,a_n)$ $=$ van der Monde det $=$ $\prod_{i<j}(a_i-a_j)$

$$= \begin{vmatrix} 1 & \cdots & e^{\theta_I}+e^{-\theta_I} & \cdots & 1 \\ k_1 & & k_Ie^{\theta_I}+(-k_I)e^{-\theta_I} & & (-k_N) \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} & \cdots & k_I^{N-1}e^{\theta_I}+(-k_I)^{N-1}e^{-\theta_I} & & (-k_N)^{N-1} \end{vmatrix} \underbrace{e^{\text{linear}}}_{\text{not contribute } u}$$

$$= \underbrace{\Delta(k_1,\cdots,k_I,(-k_{I+1}),\cdots(-k_N))}_{\Delta_{I,1}}e^{\theta_I} + \underbrace{\Delta(k_1,\cdots,k_{I-1},(-k_I),\cdots,(-k_N))}_{\Delta_{I,2}}e^{-\theta_I}$$

$$u = \partial_x^2\log\tau = 2k_I^2\cosh^{-2}\left(k_Ix-4k_I^3t+\frac{1}{2}\log\frac{\Delta_{I,1}}{\Delta_{I,2}}\right)$$

(I-th) 1-soliton ! with the position shift (phase shift)

2.3 Lax form & KdV hierarchy

[MJD][D]

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Prop. 2.10 (Lax form of KdV)

Consider the linear system:

$$\begin{cases} P\psi = 0 \\ \partial_t \psi = B\psi \end{cases}$$

Lax pair $\left(\begin{array}{l} P := \partial_x^2 + u \\ B := -4\partial_x^3 - 6u\partial_x - 3u_x \end{array} \right.$

The compatible condition $[\partial_t - B, P] = 0$ ... ④

give rise to the KdV eq.

($\Leftrightarrow$ $u_t = [B, P]$)

evolution eq. or flow eq.

☺ Quiz 1 Show this

(a little bit difficult)

$[A, B] := AB - BA$

cf. Quantum Mechanics

$$\hat{x} = x, \quad \hat{p} = \frac{\hbar}{i}\frac{d}{dx} \Rightarrow [\hat{x}, \hat{p}] = i\hbar$$

(for $\forall \varphi$)

$$\Leftrightarrow \text{essentially } \left[\frac{d}{dx}, x\right] = 1 \Leftrightarrow \frac{d}{dx}(x\varphi) - x\left(\frac{d}{dx}\varphi\right) = 1 \cdot \varphi$$

cancel

Rmk 2.11

2 flow eqs. $u_t = K[u]$ and $u_s = \hat{K}[u]$ are commute

$u = u(t)$ $u = u(s)$

i.e. $\partial_s K = \partial_t \hat{K}$ ($\Leftrightarrow u_{st} = u_{ts}$)

$\Rightarrow$ u can be considered as $u = u(t, s)$

Thm 2.12 There exist ∞ commuting flow eqs. for KdV

$$\frac{\partial u}{\partial t_{2n+1}} = [B_{2n+1}, P]$$

now $u = u(x, t_3, t_5, t_7, \cdots)$

such that $\partial_{t_{2m+1}} \partial_{t_{2n+1}} u - \partial_{t_{2n+1}} \partial_{t_{2m+1}} u = 0 \quad (\forall m, n)$

(commuting flows)

This is called the KdV hierarchy

$$⑤ \begin{cases} u_{t_3} = [B_3, P] & \Rightarrow \text{(3rd) KdV eq} \quad (t_3 \equiv t,\ B_3 \equiv B \text{ in } ④) \\ u_{t_5} = [B_5, P] & \Rightarrow \text{(5th)} \quad \text{"} \quad \text{(exist)} \\ u_{t_7} = [B_7, P] & \Rightarrow \text{(7th)} \quad \text{"} \quad (\text{"}) \\ \quad \vdots & \qquad\qquad\qquad\qquad (\text{"}) \end{cases}$$

Prop 2.13

(1) ⑤ have common conserved density: $\dfrac{\partial}{\partial t_{2n+1}} D_k = 0$

conserved density in Thm 2.5

(2) ⑤ are transformed to bilinear eqs. by the Hirota trf.

3rd KdV $\longrightarrow$ bilinear eq. ③

5th KdV $\longrightarrow$ another bilinear eq.

$\vdots$ $\qquad u = 2\partial_x^2 \log \tau \qquad$ $\vdots$

(3) $\tau(x, t_3, t_5, t_7, \cdots)$ has common Wronskian solutions

$$\tau = Wr(f_1, \cdots, f_N) \qquad f_i = f_i(x, t_3, t_5, t_7, \cdots)$$

rescaled

$$\left(\text{for soliton } f_i = e^{\theta_i} + a_i e^{-\theta_i},\ \theta_i = k_i x + \sum_{n=1}^{\infty} k_i^{2n+1} t_{2n+1}\right) \quad \partial_{t_{2n+1}} f_i = \partial_x^{2n+1} f_i$$

§3 KP equation & Plücker relation

3.1 Definition & Properties

Def 3.1 (Kadomtsev-Petviashvili (KP) eq.) $u = u(x,y,t)$

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

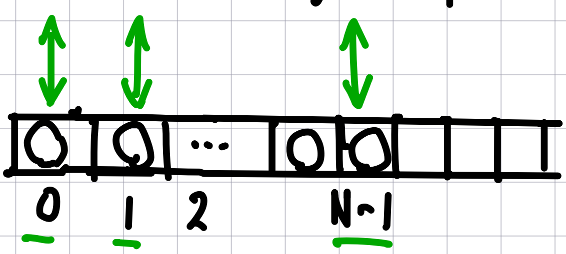
$$\Big\downarrow \ u = 2\partial_x^2 \log \tau$$

$$(\tau_{xxxx} - 4\tau_{tx} + 3\tau_{yy})\tau - 4(\tau_{xxx} - \tau_t)\tau_x + 3(\tau_{xx} - \tau_y)(\tau_{xx} + \tau_y) = 0 \quad \cdots (*)$$

Thm 3.2 General solutions to $(*)$ are given by

$$\tau = Wr(f_1, \cdots, f_N) \qquad \partial_y f_i = \partial_x^2 f_i, \ \partial_t f_i = \partial_x^3 f_i \quad \cdots ①$$

$$= \begin{vmatrix} f_1 & f_1' & \cdots & f_1^{(N-1)} \\ f_2 & f_2' & & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N' & \cdots & f_N^{(\underline{N-1})} \end{vmatrix} \leftarrow \text{transposed}$$



Maya diagram

Lemma 3.3 Derivative of determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}' = \begin{vmatrix} a' & b \\ c' & d \end{vmatrix} + \begin{vmatrix} a & b' \\ c & d' \end{vmatrix}$$

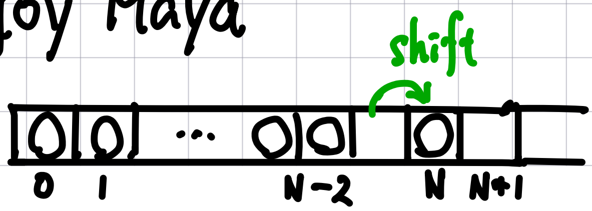
$\because \left(\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(n),n} \right)'$

Leibniz rule $\square$

3.2 Sato's observation

Let's enjoy Maya

$\underline{\tau_x} =$



$$\tau_{xx} = \begin{array}{cccccccc} 0 & & N-2 & & & N+1 & & \\ \boxed{\bigcirc} & \cdots & \boxed{\bigcirc} & \boxed{} & \boxed{} & \boxed{\bigcirc} & \boxed{} & \cdots \end{array} + \begin{array}{ccccccccc} & & N-3 & & N-1 & N & & & \\ \boxed{\bigcirc} & \cdots & \boxed{\bigcirc} & \boxed{} & \boxed{\bigcirc} & \boxed{\bigcirc} & \boxed{} & \boxed{} & \cdots \end{array}$$

$$\tau_y \overset{①}{=} \begin{array}{cccccccc} 0 & & N-2 & & & N+1 & & \\ \boxed{\bigcirc} & \cdots & \boxed{\bigcirc} & \boxed{} & \boxed{} & \boxed{\bigcirc} & \boxed{} & \cdots \end{array} - \begin{array}{ccccccccc} & & N-3 & & N-1 & N & & & \\ \boxed{\bigcirc} & \cdots & \boxed{\bigcirc} & \boxed{} & \boxed{\bigcirc} & \boxed{\bigcirc} & \boxed{} & \boxed{} & \cdots \end{array}$$

exchanged $f_i^{(N)} \leftrightarrow f_i^{(N-1)}$

$\tau_{xxx} =$

$\tau_t =$

Quiz 2 Calculate in terms of Maya diagram

$\vdots$

In summary

$$\begin{array}{rl} & \begin{array}{cccc} N-2 & N-1 & N & N+1 \end{array} \\ \tau = & \boxed{\bigcirc}\boxed{\bigcirc}\boxed{}\boxed{} \\ \tau_x = & \boxed{\bigcirc}\boxed{}\boxed{\bigcirc}\boxed{} \\ \tau_{xx} - \tau_y = 2 & \boxed{}\boxed{\bigcirc}\boxed{\bigcirc}\boxed{} \\ \tau_{xx} + \tau_y = 2 & \boxed{\bigcirc}\boxed{}\boxed{}\boxed{\bigcirc} \\ \tau_{xxx} - \tau_t = 3 & \boxed{}\boxed{\bigcirc}\boxed{}\boxed{\bigcirc} \\ \tau_{xxxx} - 4\tau_{tx} + 3\tau_{yy} = 12 & \boxed{}\boxed{}\boxed{\bigcirc}\boxed{\bigcirc} \end{array}$$

The bilinear equation (*) becomes

$$\begin{array}{rcccl} & \begin{array}{cccc} N-2 & N-1 & N & N+1 \end{array} & & \begin{array}{cccc} N-2 & N-1 & N & N+1 \end{array} & \\ & \boxed{\bigcirc}\boxed{\bigcirc}\boxed{}\boxed{} & \times & \boxed{}\boxed{}\boxed{\bigcirc}\boxed{\bigcirc} & \\ - & \boxed{\bigcirc}\boxed{}\boxed{\bigcirc}\boxed{} & \times & \boxed{}\boxed{\bigcirc}\boxed{}\boxed{\bigcirc} & \\ + & \boxed{\bigcirc}\boxed{}\boxed{}\boxed{\bigcirc} & \times & \boxed{}\boxed{\bigcirc}\boxed{\bigcirc}\boxed{} & = 0 \end{array}$$

The Plücker relation on April 13!

Rmk 3.4 The Plücker relation is obtained as follows

$$\begin{vmatrix} f & a_1 & a_2 & 0 & a_3 & a_4 \\ g & b_1 & b_2 & 0 & b_3 & b_4 \\ h & c_1 & c_2 & 0 & c_3 & c_4 \\ 0 & 0 & a_2 & f & a_3 & a_4 \\ 0 & 0 & b_2 & g & b_3 & b_4 \\ 0 & 0 & c_2 & h & c_3 & c_4 \end{vmatrix} \overset{\text{Laplace}}{=} (f\,a_1\,a_2)(f\,a_3\,a_4) - (f\,a_1\,a_3)(f\,a_2\,a_4) + (f\,a_1\,a_4)(f\,a_2\,a_3) = 0$$

add to Ex 1.9

$\downarrow$ extend to [H]

$f_1, \cdots, f_N \Leftrightarrow$ previous discussion

Thm 3.5 (e.g. [MJD], Sato's lecture note)

(1) There exists KP hierarchy

$$u_{t_3} = K_3[u]$$
$$u_{t_4} = K_4[u]$$
$$\vdots$$

infinite commuting flows

$\Rightarrow u = u(x, t_2, t_3, t_4, \cdots)$

(2) $\downarrow u = 2\partial_x^2 \log \tau$

infinite bilinear eqs.

||| Observation

infinite Plücker relations $\to$ describe an infinite-dim Grassmann manifold.

Rmk 3.6

- (Noether) "Symmetry $\Rightarrow$ conservation law" (origin of integrability)
- (Liouville) "conserved quantities (as many as DOF) $\Rightarrow$ integrable"