§ 2 KdV equation and Wronskian
2.0 Introduction

(2021/4/20)

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- **Refs:**

[D] L. Dickey, ``Soliton Equations and Hamiltonian Systems,'' (World Sci.)

[MJD] Miwa,Jimbo,Date, Solitons (Cambridge UP)

[H] R.Hirota, The Direct Method in Soliton Theory (Cambridge UP)

Brief history of soliton theory

- **1834：Scott Russell**

  Observation of solitary wave at a canal and experiment by using water tank

- **1895：Korteweg de Vries**

  Derivation of soliton eq. from equation of shallow water wave (浅水波)

- **1955：Fermi, Pasta, Ulam**

  Numerical observation of nonlinear lattice

- **1965：Zabusky, Kruskal**

  Numerical simulation of KdV eq.

  (Soliton=Solitary wave + -ton)

- 1967：Gardner, Green, Kruskal, Miura

  Solve initial value problem for KdV eq.
- 1971年：Hirota（広田良吾）

  Direct method to solve KdV eq.
- 1980～：Sato（佐藤幹夫、et al）

  Sato' s theory of soliton (clarifies symmetry of KP eq.)

Subsequent development

- application to conformal field theory, 2-dim quantum gravity, string theory,…
- Discritization,
- Noncommutative extension
- Higher-dim extension, ……………

Birthplace of soliton research （The Union canal, Edinburgh）

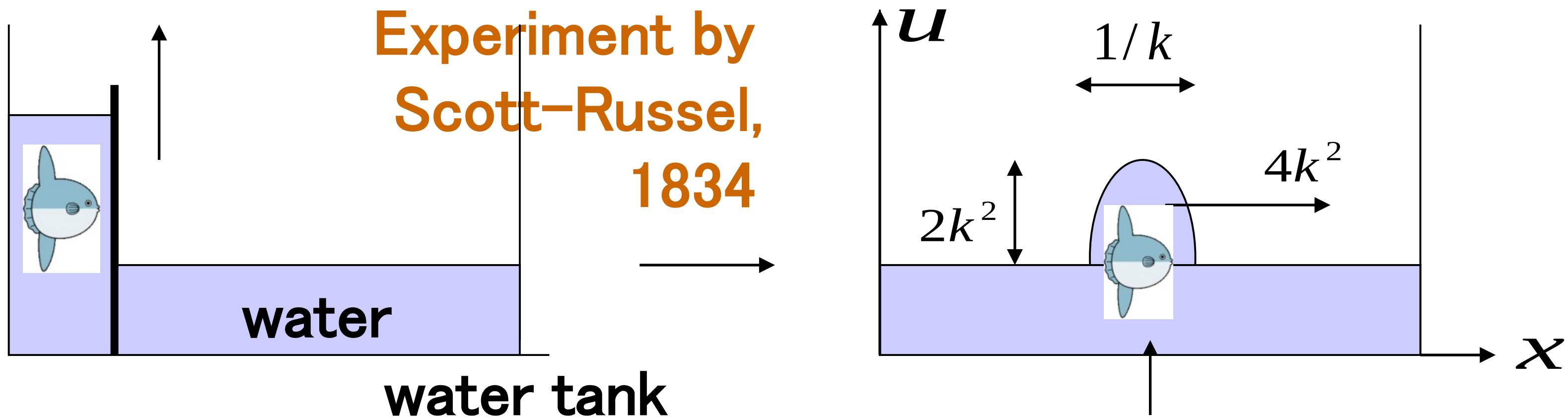
The Union canal
In Edinburgh, Scotland
（photo by H中, 2005）

Attempt to re-creat the soliton in 1995 (Photo from Nature v. 376, 3 Aug 1995, p.373)

Soliton Solutions

- **KdV equation : describe shallow water wave**

Experiment by Scott–Russel, 1834



solitary wave = soliton

This configuration satisfies

$$u = 2k^2 \cosh^{-2}(kx - 4k^3 t)$$

$$\dot{u} + u''' + 6u'u = 0$$: **KdV eq.** [Korteweg–de Vries, 1895]

This is a typical integrable equation.

(hard to solve in general)

Let's solve it now !

- **Hirota's method** [PRL27(1971)1192]

$$\dot{u} + u''' + 6u'u = 0$$: naively hard to solve

$$\downarrow \quad u = 2\partial_x^2 \log \tau$$

$$\tau\dot{\tau}' - \tau'\dot{\tau} + 3\tau''\tau'' - 4\tau'\tau''' + \tau\tau'''' = 0$$

Hirota's bilinear relation : more complicated ?

A solution: $\tau = 1 + e^{2(kx - \omega t)}, \quad \omega = 4k^3$

$$\rightarrow \quad u = 2k^2 \cosh^{-2}(kx - 4k^3 t)$$: The solitary wave !
(1-soliton solution)

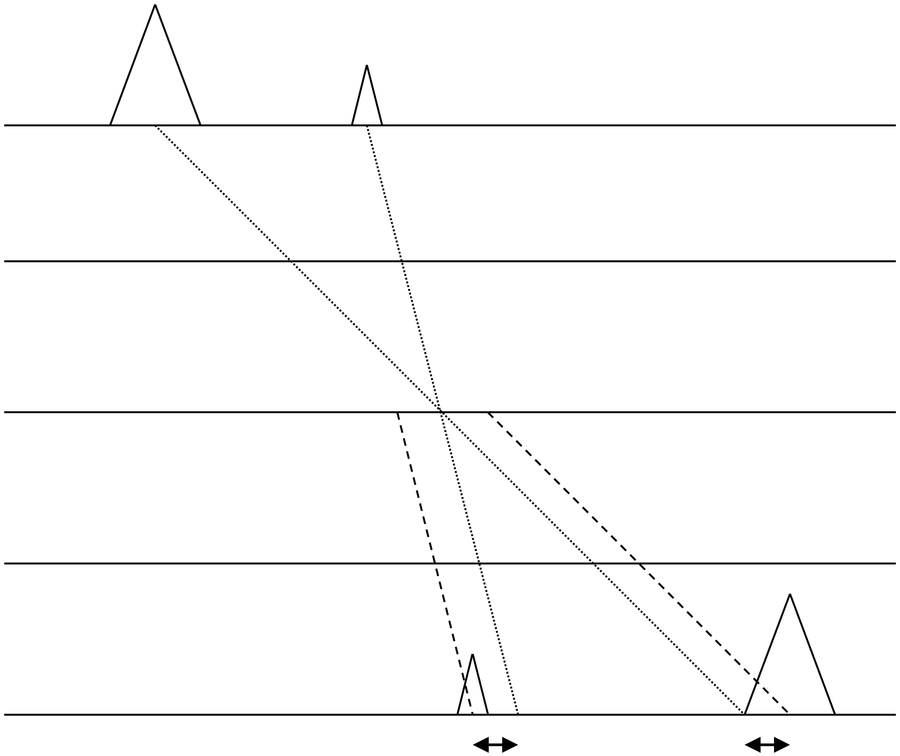
2-soliton solution

$$\tau = 1 + A_1 e^{2\theta_1} + A_2 e^{2\theta_2} + B A_1 A_2 e^{2(\theta_1+\theta_2)}$$

= A determinant of Wronski matrix (general property of soliton sols.)
``tau-functions”

$$\theta_i = k_i x - 4k_i^3 t, \quad B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

Scattering process



The shape and velocity is preserved ! (stable)

The positions are shifted ! (Phase shift)

Break：Ryogo HIROTA san

R.Hirota
(1932～2015)
(享年age 82)
[Wikipedia]

- Prof. in Waseda (at RCA kisoken also)
- Hirota's bilinearization
- Discretization of soliton equations (in order to ``put'' the PC)
- 「Modern``Japanese mathematician''(和算家」(originality)

KdV hierarchy and conservation law

- Infinite conserved densities σ

$$\sigma_1 = u, \sigma_2 = u^2, \sigma_3 = u^3 - \frac{1}{2}u_x^2, \cdots$$

Conserved densities
(spatial integration: time indep)

- infinite evolution eqs. which possess the σ's

KdV hierarchy
(infinite commuting flow)

$$u_{t_3} = -u_{xxx} - 6u_x u$$

$$u_{t_5} = -u_{xxxxx} + 10(u_{xxx}u + 2u_{xx}u_x) - 30u_x u^2$$

$$u_{t_7} = -u_{xxxxxxx} + \cdots$$

$$\vdots$$

$$u = u(x, t_3, t_5, t_7, \cdots)$$

- Hirota trf. for the KdV hierarchy $u = 2\partial_x^2 \log\tau$

→infinite bilinear relations $\tau = \tau(x, t_3, t_5, t_7, \cdots)$

Summary

KdV hier.(∞evo.eqs.)→∞bilinear relations

$$u_{t_3} = \cdots$$
$$u_{t_4} = \cdots$$
$$u_{t_5} = \cdots$$
$$\vdots$$

Hirota trf. →

$$u = 2\partial_x^2 \log\tau$$

$$\sum \tau_* \cdot \tau_* = 0$$
$$\sum \tau_* \cdot \tau_* = 0$$
$$\sum \tau_* \cdot \tau_* = 0$$
$$\vdots$$

$$u = u(t_1, t_3, t_5, \cdots)$$

$$\tau = \tau(t_1, t_3, t_5, \cdots)$$

〈∞conserved quant.〉

〈integrability〉