

§4 Quasideterminant

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4.0 Motivation (Slide)

$$\begin{pmatrix} 2 \\ 5/11 \end{pmatrix}$$

4.1 Definition & Properties

$$A = (a_{ij})_{1 \leq i, j \leq n} \quad a_{ij} \in \text{Division Ring (斜体)}$$

\mathbb{H}^{-1} is assumed to exist. (e.g. \mathbb{H}) *noncommutative*

Def 4.1

Let $A = (a_{ij})$ be an $n \times n$ square matrix, and $B = (b_{ij})$ be A^{-1} .

b_{ji}^{-1} is (i, j) -quasideterminant of A and represented:

$$b_{ji}^{-1} =: |A|_{ij} \quad \text{or} \quad \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \boxed{a_{ij}} & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \quad \left(\begin{array}{l} \text{com.} \\ \rightarrow \\ \text{limit} \end{array} \begin{array}{l} (-1)^{i+j} \\ \text{Thm 1.3} \end{array} \frac{|A|}{|A_{ij}|} \right)$$

suffix or box \rightarrow

Rmk 4.2 Block decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1} B S^{-1} C A^{-1} & -A^{-1} B S^{-1} \\ -S^{-1} C A^{-1} & S^{-1} \end{pmatrix} \quad \dots \textcircled{1}$$

square \uparrow *square*

w/ $S := D - C A^{-1} B$ (Schur complement)

$$D \equiv a_{ij} \Rightarrow S \equiv |A|_{ij} = a_{ij} - \begin{array}{c} \downarrow i \\ \boxed{A_{ij}} \\ \uparrow j \end{array} \begin{array}{c} \boxed{A_{ij}}^{-1} \\ \uparrow j \end{array}$$

deleting i-th row & j-th column

Ex 4.3

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$$n=1) \quad |A| = a$$

$$n=2) \quad \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} a_{22}^{-1} a_{21}$$

$$\begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11} a_{21}^{-1} a_{22}$$

⋮

$$\therefore \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \stackrel{\text{all squared}}{=} \begin{pmatrix} (A - BD^{-1}C)^{-1} & (C - DB^{-1}A)^{-1} \\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \dots \textcircled{2}$$

($\begin{matrix} \text{||} \\ \text{0} \end{matrix}$ later)

$$n=3) \quad \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \stackrel{\text{blocked}}{=} a_{11} - (a_{12} \ a_{13}) \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$$

$$\stackrel{\textcircled{2}}{=} a_{11} - a_{12} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} a_{31}$$

$$- a_{13} \begin{vmatrix} a_{22} & \boxed{a_{23}} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} a_{31} \dots \textcircled{3}$$

Quiz 1

Calculate $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ in similar way

Rmk 4.4

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$$\begin{vmatrix} \text{hatched} & \text{yellow} & \text{hatched} \\ \text{blue} & a_{ij} & \text{blue} \\ \text{hatched} & \text{yellow} & \text{hatched} \end{vmatrix} = \begin{vmatrix} \text{hatched} & \text{yellow} \\ \text{blue} & a_{ij} \end{vmatrix}$$

From now on, we use this "canonical" position

Prop 4.5

(i) For any invertible elements λ_k, r_k ($1 \leq k \leq n$):

$$\begin{vmatrix} a_{11}r_1 & a_{12}r_2 & \dots & a_{1n}r_n \\ \vdots & \vdots & & \vdots \\ a_{n1}r_1 & a_{n2}r_2 & \dots & \boxed{a_{nn}r_n} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & \boxed{a_{nn}} \end{vmatrix} r_n$$

$$\begin{vmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \dots & \lambda_1 a_{1n} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \dots & \lambda_2 a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_n a_{n1} & \lambda_n a_{n2} & \dots & \boxed{\lambda_n a_{nn}} \end{vmatrix} = \lambda_n \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & \boxed{a_{nn}} \end{vmatrix}$$

(ii) $\vec{a}_k := \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}$ ($1 \leq k \leq n-1$)

k th column

(There is row version)

\vec{a}_ℓ is added to \vec{a}_k ($\ell=1, \dots, n-1$)

$$\begin{vmatrix} \vec{a}_1 & \dots & \vec{a}_k + \vec{a}_\ell & \dots & \vec{a}_\ell & \dots & \vec{a}_n \end{vmatrix}_{nn} = \begin{vmatrix} \vec{a}_1 & \dots & \vec{a}_k & \dots & \vec{a}_\ell & \dots & \vec{a}_n \end{vmatrix}_{nn}$$

☺ By definition (see ③) ▣

4.2 Useful Identity

Thm 4.6 (NC ^{NonCommutative} Jacobi identity)

$$\begin{matrix}
 N \\
 1 \\
 1
 \end{matrix}
 \left(\begin{array}{ccc|c}
 A & B & C & \\
 D & f & g & \\
 E & h & i &
 \end{array} \right) = \begin{array}{c|c}
 A & C \\
 E & i
 \end{array} - \begin{array}{c|c}
 A & B \\
 E & h
 \end{array} \begin{array}{c|c}
 A & B \\
 D & f
 \end{array}^{-1} \begin{array}{c|c}
 A & C \\
 D & g
 \end{array} \dots \textcircled{4}$$

$(N+2) \times (N+2)$ \rightarrow "i-hf⁻¹g" (Schur comp.) $(N+1) \times (N+1)$

☹ By definition (or report?) ▣

Cor 4.7 (homological relation) for "moving box"

$$\begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 E & h & i
 \end{array} = \begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 E & h & i
 \end{array} \begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 0 & 0 & 1
 \end{array} \dots \textcircled{5}$$

$$\begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 E & h & i
 \end{array} = \begin{array}{c|c|c}
 A & B & 0 \\
 D & f & 0 \\
 E & h & 1
 \end{array} \begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 E & h & i
 \end{array} \dots \textcircled{6}$$

☹

$$\textcircled{5}) \begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 E & h & i
 \end{array} \begin{array}{c|c}
 A & C \\
 D & g
 \end{array}^{-1} \begin{array}{c|c}
 A & C \\
 E & i
 \end{array} \begin{array}{c|c}
 A & C \\
 D & g
 \end{array}^{-1} - \begin{array}{c|c}
 A & B \\
 E & h
 \end{array} \begin{array}{c|c}
 A & B \\
 D & f
 \end{array}^{-1}$$

$$\begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 E & h & i
 \end{array} \begin{array}{c|c}
 A & B \\
 D & f
 \end{array}^{-1}$$

On the other hand

$$\begin{array}{c|c|c}
 A & B & C \\
 D & f & g \\
 0 & 0 & 1
 \end{array} \begin{array}{c|c}
 A & C \\
 D & g
 \end{array}^{-1} \begin{array}{c|c}
 A & B \\
 D & f
 \end{array} \dots \textcircled{6}$$

Quiz 2
Prove ⑥

Return back to ① = ②:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} + a^{-1} b s^{-1} c a^{-1} & -a^{-1} b s^{-1} \\ -s^{-1} c a^{-1} & s^{-1} \end{pmatrix} = \begin{pmatrix} (a - b d^{-1} c)^{-1} & (c - d b^{-1} a)^{-1} \\ (b - a c^{-1} d)^{-1} & (d - c a^{-1} b)^{-1} \end{pmatrix}$$

(1-2) component:

$$(c - d b^{-1} a)^{-1} = \begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix}^{-1} \stackrel{\text{hom}}{=} \underbrace{\begin{vmatrix} a & b \\ \boxed{0} & 1 \end{vmatrix}^{-1}}_{-a^{-1} b} \underbrace{\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix}^{-1}}_{s^{-1}} \quad \text{OK}$$

(1-1)

lemma $\begin{vmatrix} A & B & 0 \\ C & d & 1 \\ 0 & 1 & \boxed{0} \end{vmatrix} = - \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}^{-1}$

$$(a - b d^{-1} c)^{-1} = \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix}^{-1} \downarrow = - \begin{vmatrix} \boxed{0} & 1 & \boxed{0} \\ 1 & a & b \\ \boxed{0} & c & \boxed{d} \end{vmatrix}$$

$$\stackrel{\text{Jac}}{=} - \underbrace{\begin{vmatrix} \boxed{0} & 1 \\ 1 & a \end{vmatrix}}_{-a^{-1}} + \underbrace{\begin{vmatrix} 1 & \boxed{0} \\ a & b \end{vmatrix}}_{-a^{-1} b} \underbrace{\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix}^{-1}}_{s^{-1}} \underbrace{\begin{vmatrix} 1 & a \\ \boxed{0} & c \end{vmatrix}}_{-c a^{-1}} \quad \text{OK}$$