

## 2.3 Lax form & KdV hierarchy

[MJD][D]

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Prop. 2.10 (Lax form of KdV)

Consider the linear system:

$$\begin{cases} P\psi = 0 \\ \partial_t \psi = B\psi \end{cases} \quad \begin{array}{l} \text{Lax pair} \\ \text{pair} \end{array} \quad \begin{aligned} P &:= \partial_x^2 + u \\ B &:= -4\partial_x^3 - 6u\partial_x - 3u_x \end{aligned}$$

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The compatible condition  $[\partial_t - B, P] = 0 \dots \textcircled{4}$

give rise to the KdV eq.

$$(\uparrow \quad u_t = [B, P])$$

evolution eq. or flow eq.

∴ Quiz 1 Show this ↑

(a little bit difficult)

$$[A, B] := AB - BA$$

cf. Quantum Mechanics

$$\hat{x} \approx x, \hat{p} = \frac{\hbar}{i} \frac{d}{dx} \Rightarrow [\hat{x}, \hat{p}] = i\hbar$$

(for  $\psi$ )

$$\Leftrightarrow \text{essentially } \left[ \frac{d}{dx}, x \right] = 1 \Leftrightarrow \frac{d}{dx} \left( x\varphi \right) - x \left( \frac{d}{dx} \varphi \right) = 1 \cdot \varphi$$

$\cancel{x}$   $\cancel{\varphi}$  cancel

Rmk 2.11

2 flow eqs.  $u_t = K[u]$  and  $u_s = \hat{K}[u]$  are commute

i.e.  $u_{st} = u_{ts}$

⇒  $u$  can be considered as  $u = u(t, s)$

$$u = u(t)$$

$$u = u(s)$$

Thm 2.12 There exist  $\infty$  commuting flow eqs. for KdV ⑨

$$\frac{\partial u}{\partial t_{2n+1}} = [B_{2n+1}, P]$$

now  $u = u(x, t_3, t_5, t_7, \dots)$



such that  $\partial_{t_{2m+1}} \partial_{t_{2n+1}} u - \partial_{t_{2n+1}} \partial_{t_{2m+1}} u = 0 \quad (\forall m, n)$

(commuting flows)

This is called the KdV hierarchy

$$\left. \begin{array}{l} u_{t_3} = [B_3, P] \Rightarrow (3\text{rd}) \text{ KdV eq } (t_3 \equiv t, B_3 \equiv B) \\ u_{t_5} = [B_5, P] \Rightarrow (5\text{th}) \quad " \quad (\text{exist}) \\ u_{t_7} = [B_7, P] \Rightarrow (7\text{th}) \quad " \quad (" ) \\ \vdots \quad \quad \quad \quad (" ) \end{array} \right\} \text{ in ④}$$

Prop 2.13

conserved density in Thm 2.5

(1) ⑤ have common conserved density:  $\frac{\partial}{\partial t_{2n+1}} D_k = 0$

(2) ⑤ are transformed to bilinear eqs. by the Hirota trf.

3rd KdV  $\longrightarrow$  bilinear eq. ③

5th KdV  $\longrightarrow$  another bilinear eq.

$$\vdots \quad u = 2 \partial_x^2 \log T \quad \vdots$$

(3)  $T(x, t_3, t_5, t_7, \dots)$  has common Wronskian solutions

$$T = \text{Wr}(f_1, \dots, f_N) \quad f_i = f_i(x, t_3, t_5, t_7, \dots) \quad \text{rescaled}$$

$$(\text{for soliton} \quad f_i = 1 + e^{kx + k^3 t_3 + \dots + k^{2n+1} t_{2n+1} + \dots}) \quad \partial_{t_{2n+1}} f_i = \partial_x^{2n+1} f_i$$

# § 3 KP equation & Plücker relation

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## 3.1 Definition & Properties

Def 3.1 (Kadomtsev-Petviashvili (KP) eq.)  $U = U(x, y, t)$

$$(-4U_t + U_{xxx} + 6UU_x)_x + 3U_{yy} = 0$$

$$\downarrow U = 2 \partial_x^2 \log T$$

$$(T_{xxxx} - 4T_{tx} + 3T_{yy})T - 4(T_{xxx} - T_t)T_x + 3(T_{xx} - T_y)(T_{xx} + T_y) = 0$$

... (\*)

Thm 3.2 General solutions to (\*) are given by

$$T = Wr(f_1, \dots, f_N) \quad \partial_y f_i = \partial_x^2 f_i, \quad \partial_t f_i = \partial_x^3 f_i \quad \dots \textcircled{1}$$

$$= \begin{vmatrix} f_1 & f_1' & \cdots & f_1^{(N-1)} \\ f_2 & f_2' & & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N' & \cdots & f_N^{(N-1)} \end{vmatrix} \quad \text{← transpose}$$


Maya diagram

## 3.2 Sato's observation

Let's enjoy Maya

$$T_x = \begin{array}{cccccc|c} [0|0] & \cdots & [0|0] & [0] & & & \\ \hline 0 & 1 & 2 & N-1 & & & \end{array} \quad \text{shift}$$

cf. Derivative of determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}' = \begin{vmatrix} a' & b \\ c' & d \end{vmatrix} + \begin{vmatrix} a & b \\ c & d' \end{vmatrix}$$

$$T_{xx} = \overbrace{[0 \dots 0 | 1 | 0]}_0^N + \overbrace{[0 \dots 0 | 0 | 0]}_{N-3}^N$$

$$T_y = \overbrace{[0 \dots 0 | 1 | 0]}_0^N - \overbrace{[0 \dots 0 | 0 | 0]}_{N-3}^N$$

$$T_{xxx} = ]$$

Quiz 2 Calculate in terms of Maya diagram

$$T_t =$$

⋮

In summary

$$T = \boxed{\begin{matrix} & N-2 & N-1 & N & N+1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}}$$

$$T_x = \boxed{\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}}$$

$$T_{xx} - T_y = 2 \boxed{\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}}$$

$$T_{xx} + T_y = 2 \boxed{\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}}$$

$$T_{xxx} - T_t = 3 \boxed{\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}}$$

$$T_{xxx} - 4T_x + 3T_y = 12 \boxed{\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}}$$

The bilinear equation (\*) becomes

$$\begin{array}{c} \boxed{\begin{matrix} & N-2 & N-1 & N & N+1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}} \times \boxed{\begin{matrix} & N-2 & N-1 & N & N+1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}} \\ - \boxed{\begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix}} \times \boxed{\begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix}} \\ + \boxed{\begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix}} \times \boxed{\begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix}} \end{array}$$

The Plücker relation  
on April 13!

Rmk 3.3 The Plücker relation is obtained as follows

$$\left| \begin{array}{cc|cc} f & a_1 & a_2 & 0 & a_3 & a_4 \\ g & b_1 & b_2 & 0 & b_3 & b_4 \\ h & c_1 & c_2 & 0 & c_3 & c_4 \\ 0 & 0 & a_2 & f & a_3 & a_4 \\ 0 & 0 & b_2 & g & b_3 & b_4 \\ 0 & 0 & c_2 & h & c_3 & c_4 \end{array} \right| = (f a_1 a_2)(f a_3 a_4) - (f a_1 a_3)(f a_2 a_4)$$

+ (f a\_1 a\_4)(f a\_2 a\_3) = 0

$\sim$

↓ extend to [H]

add to Ex 1.9

$f_1, \dots, f_N \Leftrightarrow$  previous discussion

Thm 3.5 (e.g. [MJD], Sato's lecture note)

(1) There exists KP hierarchy

$$u_{t_3} = K_3[u] \quad \text{infinite commuting flows}$$

$$u_{t_4} = K_4[u] \quad \Rightarrow \quad u = u(x, t_2, t_3, t_4, \dots)$$

⋮

$$(2) \quad \downarrow \quad u = 2 \partial_x^2 \log T$$

infinite bilinear egs.

III Observation

infinite Plücker relations  $\rightarrow$  describe an infinite-dim Grassmann manifold.

Rmk 3.6

- (Noether) "Symmetry  $\Rightarrow$  conservation law" (origin of integrability)
- (Liouville) "conserved quantities  $\Rightarrow$  integrable" (as many as DOF)