

§ 2 KdV Equation and Wronskian

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2.0 Introduction (\rightarrow Slide)

2.1 Definition & Properties

Def 2.1 (KdV eq.) $u = u(t, x)$: defined on $(1+1)$ -dim space-time

$$u_t + u_{xxx} + 6uu_x = 0 \quad \dots \textcircled{1}$$

\nwarrow partial derivatives \nwarrow non-linear term (hard to solve)

Rmk 2.2 Three coefficients are arbitrary

i.e. $\textcircled{1} \iff \tilde{u}_{\tilde{t}} + A u_{xxx} + Bu u_x = 0$

$$\tilde{u} = \frac{6A}{B} u$$
$$\tilde{t} = \frac{1}{A} t$$

Def 2.3 For $D(t, x)$, $F(t, x)$, $F(t, x \rightarrow \pm\infty) = 0$,

$D_t + F_x = 0$ is called conservation law.

D : conserved density, F : flux

Prop 2.4 $I := \int_{-\infty}^{\infty} D dx$ is conserved (i.e. $\frac{dI}{dt} = 0$)

$$\textcircled{2} \quad \frac{dI}{dt} = \int_{-\infty}^{\infty} D_t dx \approx - \int_{-\infty}^{\infty} F_x dx = -F \left|_{x=-\infty}^{x=+\infty} \right. = 0 \quad \blacksquare$$

Thm 2.5 KdV eq has ∞ conserved densities

c.g. $D_1 = u$, $D_2 = u^2$, $D_3 = u^3 - \frac{1}{2}u_x^2$, ...

$\textcircled{3}$ See, ej. [D])

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2.2 Hirota trf. & Wronskian solution (τ -fn) [6]

Def 2.6 (Hirota trf.) $U = 2 \partial_x^2 \log \tau(t, x) \dots \textcircled{2}$

Prop 2.7 KdV eq. is transformed by $\textcircled{2}$ to

$$\tau \tau_{xt} - \tau_x \tau_t + 3 \tau_{xx}^2 - 4 \tau_x \tau_{xxx} + \tau \tau_{xxxx} = 0 \dots \textcircled{3}$$

(Hirota's) bilinear eq. integ. const.

☺ Quiz 1: Prove this. $(\textcircled{1} \xrightarrow{\textcircled{2}} (\dots)_x = 0 \xrightarrow{x\text{-integ. \& } C=0} \textcircled{3})$

Def 2.8 (Wronskian)

$$Wr(f_1, \dots, f_n) := \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f'_1 & f'_2 & \dots & f'_n \\ \vdots & \vdots & & \vdots \\ f_i^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \quad \begin{array}{l} f_i : C^\infty \text{ funcs of } x \\ f_i^{(k)} : k\text{-th } x\text{-derivative} \\ \text{of } f_i \end{array}$$

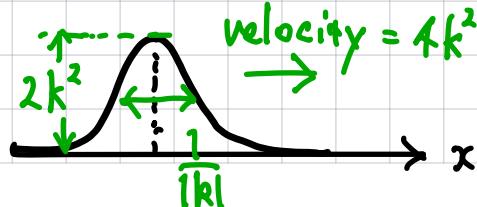
Thm 2.9 General solutions of $\textcircled{3}$ are the following Wronskians:

$$\tau = Wr(f_1, \dots, f_N) \quad f_i, t = 4 f_i,_{xxx}$$

N-soliton solutions are given by the choice:

$$f_i = e^{\theta_i} + a_i e^{-\theta_i}, \quad \theta_i := k_i x - 4 k_i t \quad (a_i > 0)$$

Ex 1-soliton solution: $U = 2 \partial_x^2 \log (e^{kx - 4kt} + a e^{-kx + 4kt})$



$$= 2k^2 \cosh^{-2} \left(kx - 4kt - \frac{1}{2} \log a \right)$$

Asymptotic behavior of the N-soliton solution

For simplicity, set $a_i \equiv 1$ & $0 < k_1 < k_2 < \dots < k_N$

Consider the comoving frame where $X_I = x - 4k_I^2 t$ is finite.

$$\frac{\theta_i}{k_i} = x - 4k_i^2 t = \underbrace{x - 4k_I^2 t}_{\text{finite}} - 4(k_i^2 - k_I^2)t$$

$$f_i = e^{\theta_i} + e^{-\theta_i} \xrightarrow{t \rightarrow +\infty} \begin{cases} e^{\theta_i} & i < I \\ e^{\theta_I} + e^{-\theta_I} & i = I \\ e^{-\theta_i} & i > I \end{cases}$$

($t \rightarrow -\infty$ case
is similar)

$$\begin{aligned} T \xrightarrow{t \rightarrow \infty} & \left| \begin{array}{cccccc} e^{\theta_1} & \dots & e^{\theta_I} + e^{-\theta_I} & \dots & e^{-\theta_N} \\ k_1 e^{\theta_1} & \dots & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & \dots & (-k_N) e^{-\theta_N} \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} e^{\theta_1} & \dots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & \dots & (-k_N)^{N-1} e^{-\theta_N} \end{array} \right| \\ = & \left| \begin{array}{cccccc} 1 & \dots & e^{\theta_I} + e^{-\theta_I} & \dots & 1 \\ k_1 & & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & & (-k_N) \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} & \dots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & & (-k_N)^{N-1} \end{array} \right| \end{aligned}$$

$\Delta(a_1, \dots, a_n)$
 van der Monde det
 $\prod_{i < j} (a_i - a_j)$

e^{linear}
 not contribute to

$$= \underbrace{\Delta(k_1, \dots, k_I, (-k_{I+1}), \dots, (-k_N))}_{\Delta_{I,1}} e^{\theta_I} + \underbrace{\Delta(k_1, \dots, k_{I-1}, (-k_I), \dots, (-k_N))}_{\Delta_{I,2}} e^{-\theta_I}$$

$$U = \partial_x^2 \log T = 2k_I^2 \cosh^{-2}(k_I x - 4k_I^3 t + \frac{1}{2} \log \frac{\Delta_{I,1}}{\Delta_{I,2}})$$

(I-th) 1-soliton \square with the $\xrightarrow{\text{position shift}}$ phase shift