'21 4/20

§2 KdV Equation and Wronskian

2.0 Introduction (→ Slide)

2.1 Definition & Properties

Def 2.1 (KdV eq.) $u = u(t, x)$: defined on (1+1)-dim space-time (time, space)

$$u_t + u_{xxx} + 6uu_x = 0 \quad \cdots ①$$

partial derivatives (u_t, u_{xxx}); non-linear term (hard to solve) (uu_x)

Rmk 2.2 Three coefficients are arbitrary

i.e. $① \Longleftrightarrow \tilde{u}_{\tilde{t}} + A u_{xxx} + B u u_x = 0$

$$\tilde{u} = \frac{6A}{B} u, \quad \tilde{t} = \frac{1}{A} t$$

Def 2.3 For $D(t,x)$, $F(t,x)$, $F(t, x \to \pm\infty) = 0$,

$D_t + F_x = 0$ is called conservation law.

D: conserved density, F: flux

Prop 2.4 $I := \int_{-\infty}^{\infty} D\, dx$ is conserved $\left(\text{i.e. } \frac{dI}{dt} = 0\right)$

(∵) $\frac{dI}{dt} = \int_{-\infty}^{\infty} D_t\, dx = -\int_{-\infty}^{\infty} F_x\, dx = -F\Big|_{x=-\infty}^{x=+\infty} = 0$ ∎

Thm 2.5 KdV eq has ∞ conserved densities

e.g. $D_1 = u$, $D_2 = u^2$, $D_3 = u^3 - \frac{1}{2}u_x^2$, $\cdots$

(∵) See, e.g. [D], [MJD]

2.2 Hirota trf. & Wronskian solution (τ-fcn)

Def 2.6 (Hirota trf.) $u = 2\partial_x^2 \log \tau(t,x)$ $\cdots$ ②

Prop 2.7 KdV eq. is transformed by ② to

$$\tau\tau_{xt} - \tau_x\tau_t + 3\tau_{xx}^2 - 4\tau_x\tau_{xxx} + \tau\tau_{xxxx} = 0 \quad \cdots ③$$

(Hirota's) bilinear eq.

☺ Quiz 1 : Prove this. $\left(① \xrightarrow{②} (\cdots)_x = 0 \xrightarrow[\text{integ. const.}]{x\text{-integ. } \& \ C=0} ③ \right)$

Def 2.8 (Wronskian)

$$\mathrm{Wr}(f_1,\cdots,f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$$

f_i : C^∞-fcns of x

$f_i^{(k)}$: k-th x-derivative of f_i

Thm 2.9 General solutions of ③ are the following Wronskians:

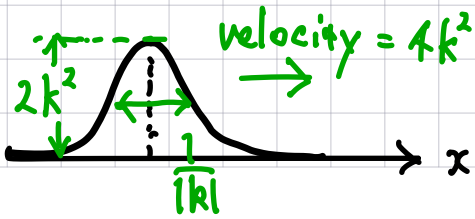
$$\tau = \mathrm{Wr}(f_1,\cdots,f_N) \qquad f_{i,t} = 4 f_{i,xxx}$$

N-soliton solutions are given by the choice:

$$f_i = e^{\theta_i} + a_i e^{-\theta_i}, \quad \theta_i := k_i x - 4k_i^3 t \qquad (a_i > 0)$$

Ex 1-soliton solution : $u = 2\partial_x^2 \log\left(e^{kx-4k^3t} + a e^{-kx+4k^3t}\right)$

$$= 2k^2 \cosh^{-2}\left(kx - 4k^3 t - \tfrac{1}{2}\log a\right)$$



Asymptotic behavior of the N-soliton solution

For simplicity, set $a_i \equiv 1$ & $0 < k_1 < k_2 < \cdots < k_n$

Consider the comoving frame where $X_I = x - 4k_I^2 t$ is finite.

$$\frac{\theta_i}{k_i} = x - 4k_i^2 t = \underbrace{x - 4k_I^2 t}_{\text{finite}} - 4(k_i^2 - k_I^2)t$$

$$f_i = e^{\theta_i} + e^{-\theta_i} \xrightarrow{t\to+\infty} \begin{cases} e^{\theta_i} & i < I \\ e^{\theta_I} + e^{-\theta_I} & i = I \\ e^{-\theta_i} & i > I \end{cases}$$

($t \to -\infty$ case is similar)

$$\tau \xrightarrow{t\to\infty} \begin{vmatrix} e^{\theta_1} & \cdots & e^{\theta_I} + e^{-\theta_I} & \cdots & e^{-\theta_N} \\ k_1 e^{\theta_1} & \cdots & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & \cdots & (-k_N) e^{-\theta_N} \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} e^{\theta_1} & \cdots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & \cdots & (-k_N^{N-1}) e^{-\theta_N} \end{vmatrix}$$

$\Delta(a_1, \cdots, a_n)$ = van der Monde det = $\prod_{i<j} (a_i - a_j)$

$$= \begin{vmatrix} 1 & \cdots & e^{\theta_I} + e^{-\theta_I} & \cdots & 1 \\ k_1 & & k_I e^{\theta_I} + (-k_I) e^{-\theta_I} & & (-k_N) \\ \vdots & & \vdots & & \vdots \\ k_1^{N-1} & \cdots & k_I^{N-1} e^{\theta_I} + (-k_I)^{N-1} e^{-\theta_I} & & (-k_N)^{N-1} \end{vmatrix} \underbrace{e^{\text{linear}}}_{\text{not contribute } u}$$

$$= \underbrace{\Delta(k_1, \cdots, k_I, (-k_{I+1}), \cdots (-k_N))}_{\Delta_{I,1}} e^{\theta_I} + \underbrace{\Delta(k_1, \cdots, k_{I-1}, (-k_I), \cdots, (-k_N))}_{\Delta_{I,2}} e^{-\theta_I}$$

$$u = \partial_x^2 \log \tau = 2k_I^2 \cosh^{-2}\left(k_I x - 4k_I^3 t + \frac{1}{2}\log\frac{\Delta_{I,1}}{\Delta_{I,2}}\right)$$

(I-th) 1-soliton ! with the position shift (phase shift)

2.3 Lax form & KdV hierarchy [MJD][D]

Def 2.10 (Lax form)

Consider the linear system:

$$\begin{cases} P\psi = 0 \\ \partial_t \psi = B\psi \end{cases} \qquad \text{Lax pair} \left(\begin{array}{l} P := \partial_x^2 + u \\ B := -4\partial_x^3 - 6u\partial_x - 3u_x \end{array} \right.$$

The compatible condition $[\partial_t - B, P] = 0 \quad \cdots$ ④

give rise to the KdV eq. ($\Leftrightarrow$ $u_t = [B, P]$)

evolution eq. or flow eq.

☺ Quiz 2 Show this ↑

(a little bit difficult)

④ means $(\partial_t - B)(P\psi) - P(\partial_t - B)\psi = 0$
(for ${}^\forall\psi$)

Thm 2.11 There exist ∞ flow eqs.

$$\frac{\partial u}{\partial t_{2n+1}} = [B_{2n+1}, P]$$

now $u = u(x, t_3, t_5, t_7, \cdots)$

such that $\partial t_{2m+1} \partial t_{2n+1} u - \partial t_{2n+1} \partial t_{2m+1} u = 0 \quad ({}^\forall m, n)$

(commuting flows)

This is called the KdV hierarchy

$u_{t_3} = [B_3, P] \Rightarrow$ (3rd) KdV eq ($t_3 \equiv t$, $B_3 \equiv B$ in ④)

$u_{t_5} = [B_5, P] \Rightarrow$ (5th) 〃 (exist)

$u_{t_7} = [B_7, P] \Rightarrow$ (7th) 〃 (〃)

⋮ (〃)

infinite commuting flows

$\Downarrow$

$u = u(x, t_3, t_5, t_7, \cdots)$: defined on ∞-dim "space-time"

- u has common conserved density: $\dfrac{\partial}{\partial t_{2n+1}} \tilde{D}_k = 0$

conserved density in Thm 2.5

- KdV hierarchy eqs. are transformed to bilinear eqs.

3rd KdV $\longrightarrow$ bilinear eq. ③

5th KdV $\longrightarrow$ another bilinear eq.

$\vdots \qquad u = 2\partial_x^2 \log \tau \qquad \vdots$

- $\tau(x, t_3, t_5, t_7, \cdots)$ has common Wronskian solutions

$\tau = \mathrm{Wr}(f_1, \cdots, f_N) \qquad f_k = f_k(x, t_3, t_5, t_7, \cdots)$

$\partial_{t_{2n+1}} f_k = 4 \partial_x^{2n+1} f_k$