

# Five Lectures on Determinants

1

## §1 Foundation of Determinants

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad i, j \in [n] := \{1, 2, \dots, n\}$$

### Def 1.1 (Determinant of A)

$$\det A := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

( " )  $|A|$  — permutation gp.

### Thm 1.2 (Cramer's formula)

Consider  $A \vec{x} = \vec{b}$  ... (\*) w/  $|A| \neq 0$ ,  $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Solution to (\*) is given by

$$x_i = \frac{1}{|A|} \begin{vmatrix} a_{11} \cdots b_1 \cdots a_{1n} \\ \vdots & \vdots & \vdots \\ a_{ni} \cdots \overset{i}{\underset{\sim}{b_n}} \cdots a_{nn} \end{vmatrix}$$

←  $i$ -th column is replaced with  $\vec{b}$

### Thm 1.3 (Laplace)

$$|A| \neq 0, B := A^{-1}$$

$$b_{ij} = (A^{-1})_{ij} = (-1)^{i+j} \frac{|A^{ij}|}{|A|} \quad \begin{matrix} \leftarrow \text{submatrix of } A \text{ deleting } j\text{-th row \&} \\ \text{ } \end{matrix} \quad \begin{matrix} \leftarrow \text{Quiz 1. Prove it for } n=3 \end{matrix}$$

(i,j) minor

$$\text{Cor 1.4} \quad \tilde{a}_{ij} := (-1)^{i+j} |A^{ij}|, \quad \tilde{A} := (\tilde{a}_{ji}) \Rightarrow \tilde{A}\tilde{A} = \tilde{A}\tilde{A} = |A|$$

## Relation to exterior algebra

$\Lambda^n(V)$ : exterior algebra of  $V$

$$v_i, v_j \quad v_i \wedge v_j = -v_j \wedge v_i \quad (v_i \wedge v_i = 0)$$

Prop 1.5  $w_i = \sum_{j=1}^n a_{ij} v_j$

$$w_1 \wedge \dots \wedge w_n = |A| v_1 \wedge \dots \wedge v_n$$

$$\begin{aligned} (\text{Ex}) \quad n=2 \quad w_1 \wedge w_2 &= (a_{11} v_1 + a_{12} v_2) \wedge (a_{21} v_1 + a_{22} v_2) \\ &= a_{11} a_{22} v_1 \wedge v_2 + a_{12} a_{21} \underbrace{v_2 \wedge v_1}_{-v_1 \wedge v_2} \\ &= (a_{11} a_{22} - a_{12} a_{21}) v_1 \wedge v_2 = |A| v_1 \wedge v_2 \\ &\quad \sum_{\sigma \in S_2} \text{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \end{aligned}$$

## Thm 1.6 (Laplace Expansion Theorem)

$$I = \{i_1, \dots, i_r \in [n] \mid i_1 < \dots < i_r\}$$

$$J = \{j_1, \dots, j_r \in [n] \mid j_1 < \dots < j_r\}$$

$$[n] \setminus I = \{i_{r+1}, \dots, i_n \mid i_{r+1} < \dots < i_n\}$$

$$[n] \setminus J = \{j_{r+1}, \dots, j_n \mid j_{r+1} < \dots < j_n\}$$

$$A_{I,J} := (a_{ip, jq})_{1 \leq p, q \leq r}, \quad A_{[n] \setminus I, [n] \setminus J} = (a_{ip, jq})_{r+1 \leq p, q \leq n} \quad (n-r) \times (n-r)$$

$$|A| = \sum_{\substack{I \subset [n] \\ \#I=r}} (-1)^{\sum(I) + \sum(J)} |A_{I,J}| |A_{[n] \setminus I, [n] \setminus J}| : \text{for a given } J$$

Sum is taken over all  $I$

$$= \sum_{\substack{J \subset [n] \\ \#J=r}} (-1)^{\sum(I) + \sum(J)} |A_{I,J}| |A_{[n] \setminus I, [n] \setminus J}| : \text{for a given } I$$

$$(\sum(I) := i_1 + \dots + i_r)$$

(Proof) ( $n=4, r=2$ )

[3]

$$W_1 \wedge W_2 \wedge W_3 \wedge W_4 = |A| V_1 \wedge V_2 \wedge V_3 \wedge V_4$$

II on the other hand

$$(W_1 \wedge W_2) \wedge (W_3 \wedge W_4)$$

(cf. Report Prob 1 (2))

$$\begin{aligned} W_1 \wedge W_2 &= (a_{11}V_1 + a_{12}V_2 + a_{13}V_3 + a_{14}V_4) \wedge (a_{21}V_1 + a_{22}V_2 + a_{23}V_3 + a_{24}V_4) \\ &= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} V_1 \wedge V_2 + \begin{vmatrix} a_{11} & 0_{13} \\ a_{21} & a_{23} \end{vmatrix} V_1 \wedge V_3 + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} V_1 \wedge V_4 \\ &\quad + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} V_2 \wedge V_3 + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} V_2 \wedge V_4 + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} V_3 \wedge V_4 \end{aligned}$$

$W_3 \wedge W_4 = \text{Quiz 2 (Calculate this)}$

$$\therefore (W_1 \wedge W_2) \wedge (W_3 \wedge W_4)$$

$$\begin{aligned} &= \left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \right. \\ &\quad \left. + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \right\} \\ &\quad |A| \text{ "expanded by " } 2 \times 2 \text{ det"} \\ &\quad \text{---} \quad (***) \\ &\quad V_1 \wedge V_2 \wedge V_3 \wedge V_4 \end{aligned}$$

Similarly ( $n$ -form) = ( $r$ -form)  $\wedge$  ( $(n-r)$ -form)

↓  
Thm 1.6 (report)

Rmk 1.7  $r=1$  or  $n-1 \Rightarrow$  ordinary expansion in a <sup>row</sup>  
<sup>column</sup>

4

Thm 1.8 (Plücker relation)

$$\alpha_1, \dots, \alpha_{r-1} \in [n]$$

$$\alpha_1 < \dots < \alpha_{r-1}$$

$$\beta_1, \dots, \beta_{r+1} \in [n]$$

$$\beta_1 < \dots < \beta_{r+1}$$

$$\alpha_i \neq \beta_j \quad (\forall i, j)$$

$$\sum_{k=1}^{r+1} (-1)^{k-1} \begin{vmatrix} \alpha_1 \alpha_1 \dots \alpha_1 \alpha_{r-1} & \alpha_1 \beta_k \\ \vdots & \vdots \\ \alpha_r \alpha_1 \dots \alpha_r \alpha_{r-1} & \alpha_r \beta_k \end{vmatrix} \begin{vmatrix} \alpha_1 \beta_1 \dots \overset{\check{\alpha_1}}{\alpha_1} \beta_k \dots \alpha_1 \beta_{r+1} \\ \vdots \\ \alpha_r \beta_1 \dots \alpha_r \beta_k \dots \alpha_r \beta_{r+1} \end{vmatrix} = 0$$

$r \times r \qquad r \times r$

$$\underline{\text{Ex 1.9}} \quad n=4, r=2 \quad \alpha_1=1, \alpha_2=2, \alpha_3=3, \alpha_4=4$$

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \begin{vmatrix} \alpha_{13} & \alpha_{14} \\ \alpha_{23} & \alpha_{24} \end{vmatrix} - \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{23} & \alpha_{23} \end{vmatrix} \begin{vmatrix} \alpha_{12} & \alpha_{14} \\ \alpha_{22} & \alpha_{24} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{14} \\ \alpha_{21} & \alpha_{24} \end{vmatrix} \begin{vmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{22} & \alpha_{23} \end{vmatrix} \approx 0$$

$k=1 \qquad k=2 \qquad k=3$

Rmk This is a defining eq. of  $\text{Gr}(2,4)$  in  $\mathbb{C}\mathbb{P}_5$

⦿ Laplace expansion (\*\*) for  $n=4, r=2$  with

$$|A| = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{vmatrix} = 0$$

Thm 1.10 (Jacobi identity)Thm 1.11 (Cauchy-Binet)

→ Report?