Five Lectures on Determinants

'21 4/13

§1 Foundation of Determinants

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad i,j \in [n] := \{1,2,\cdots,n\}$$

Def 1.1 (Determinant of A)

$$\underset{\left(|A|\right)}{\det A} := \sum_{\underbrace{\sigma \in S_n}_{\text{permutation gp.}}} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

Thm 1.2 (Cramer's formula)

Consider $A\vec{x} = \vec{b}$ $\cdots (*)$ w/ $|A| \neq 0$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Solution to $(*)$ is given by

$$x_i = \frac{1}{|A|} \begin{vmatrix} a_{11} \cdots & b_1 & \cdots a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} \cdots & b_n & \cdots a_{nn} \end{vmatrix}$$

$\hat{i}$ ← i-th column is replaced with $\vec{b}$

Thm 1.3 (Laplace) $|A| \neq 0$, $B := A^{-1}$

$$b_{ij} = (A^{-1})_{ij} = (-1)^{i+j} \frac{|A^{ji}|}{|A|}$$

← submatrix of A deleting j-th row & i-th column

← Quiz 1. Prove it for $n=3$

Cor 1.4 $\tilde{a}_{ij} := (-1)^{i+j} |A^{ij}|$ ← (i,j) minor, $\tilde{A} := (\tilde{a}_{ji}) \Rightarrow A\tilde{A} = \tilde{A}A = |A|$

Relation to exterior algebra

$\Lambda^n(V)$: exterior algebra of V

$\overset{\in}{v_i, v_j}$ $\quad v_i \wedge v_j = -v_j \wedge v_i \quad (v_i \wedge v_i = 0)$

<u>Prop 1.5</u> $\quad w_i = \sum_{j=1}^{n} a_{ij} v_j$

$$w_1 \wedge \cdots \wedge w_n = |A| \, v_1 \wedge \cdots \wedge v_n$$

(Ex) $n=2$) $\quad w_1 \wedge w_2 = (a_{11} v_1 + a_{12} v_2) \wedge (a_{21} v_1 + a_{22} v_2)$

$$= a_{11} a_{22} v_1 \wedge v_2 + a_{12} a_{21} \overbrace{v_2 \wedge v_1}^{-v_1 \wedge v_2}$$

$$= \underbrace{(a_{11} a_{22} - a_{12} a_{21})}_{\sum_{\sigma \in S_2} sgn(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)}} v_1 \wedge v_2 = |A| v_1 \wedge v_2$$

<u>Thm 1.6</u> (Laplace Expansion Theorem)

$I := \{ i_1, \cdots, i_r \in [n] \mid i_1 < \cdots < i_r \}$

$J := \{ j_1, \cdots, j_r \in [n] \mid j_1 < \cdots < j_r \}$

$[n] \setminus I = \{ i_{r+1}, \cdots, i_n \mid i_{r+1} < \cdots < i_n \}$

$[n] \setminus J = \{ j_{r+1}, \cdots, j_n \mid j_{r+1} < \cdots < j_n \}$

$\underset{r \times r}{A_{I,J}} := (a_{i_p, j_q})_{1 \le p, q \le r}$, $\quad \underset{(n-r)\times(n-r)}{A_{[n]\setminus I, [n]\setminus J}} = (a_{i_p, j_q})_{r+1 \le p, q \le n}$

$$|A| = \sum_{\substack{I \subset [n] \\ \# I = r}} (-1)^{\Sigma(I) + \Sigma(J)} |A_{I,J}| \, |A_{[n]\setminus I, [n]\setminus J}|$$

$$\Sigma(I) := i_1 + \cdots + i_r$$

(Proof) $(n=4,\ r=2)$

$$w_1 \wedge w_2 \wedge w_3 \wedge w_4 = |\overset{\swarrow 4\times 4}{A}|\, v_1 \wedge v_2 \wedge v_3 \wedge v_4$$

$\parallel$ on the other hend

$$(w_1 \wedge w_2) \wedge (w_3 \wedge w_4)$$

$$w_1 \wedge w_2 = (a_{11} v_1 + a_{12} v_2 + a_{13} v_3 + a_{14} v_4) \wedge (a_{21} v_1 + a_{22} v_2 + a_{23} v_3 + a_{24} v_4)$$

$$= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} v_1 \wedge v_2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} v_1 \wedge v_3 + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} v_1 \wedge v_4$$

$$+ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} v_2 \wedge v_3 + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} v_2 \wedge v_4 + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} v_3 \wedge v_4$$

$w_3 \wedge w_4 =$ Quiz 2 (Calculate this)

$\therefore\ (w_1 \wedge w_2) \wedge (w_3 \wedge w_4)$

$$= \Big\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \Big\}$$

$v_1 \wedge v_2 \wedge v_3\ v_4$

$\parallel$ expanded by "2x2 det"

$\underset{4\times 4}{|A|}$ ...... (**)

Similarly $(n\text{-form}) = (r\text{-form}) \wedge ((n-r)\text{-form})$

$\Downarrow$

Thm 1.6 (report)

<u>Rmk 1.7</u> $r = 1$ or $n-1$ $\Rightarrow$ ordinary expansion in a row / column

Thm 1.8 (Plücker relation)

$\alpha_1, \cdots, \alpha_{r-1} \in [n]$ $\quad \alpha_1 < \cdots < \alpha_{r-1}$

$\beta_1, \cdots, \beta_{r+1} \in [n]$ $\quad \beta_1 < \cdots < \beta_{r+1}$ $\quad \alpha_i \neq \beta_j \ (\forall i, j)$

$$\sum_{k=1}^{r+1} (-1)^{k-1} \underbrace{\begin{vmatrix} a_{1\alpha_1} & \cdots & a_{1\alpha_{r-1}} & a_{1\beta_k} \\ \vdots & & \vdots & \vdots \\ a_{r\alpha_1} & \cdots & a_{r\alpha_{r-1}} & a_{r\beta_k} \end{vmatrix}}_{r \times r} \underbrace{\begin{vmatrix} a_{1\beta_1} & \cdots & \overset{\text{omit}}{a_{1\beta_k}} & \cdots & a_{1\beta_{r+1}} \\ \vdots & & \vdots & & \vdots \\ a_{r\beta_1} & \cdots & a_{r\beta_k} & \cdots & a_{r\beta_{r+1}} \end{vmatrix}}_{r \times r} = 0$$

Ex 1.9 $\quad n = 4, \ r = 2$ $\qquad \alpha = 1, \ \beta_1 = 2, \ \beta_2 = 3, \ \beta_3 = 4$

$$\underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}}_{k=1} - \underbrace{\begin{vmatrix} a_{11} & a_{13} \\ a_{23} & a_{23} \end{vmatrix} \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}}_{k=2} + \underbrace{\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}_{k=3} = 0$$

Rmk This is a defining eq. of $Gr(2,4)$ in $\mathbb{CP}_5$

$\because$ Laplace expansion (**) for $n = 4, r = 2$ with

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \end{vmatrix} = 0 \qquad \blacksquare$$

Thm 1.10 (Jacobi identity)

Thm 1.11 (Cauchy–Binet)

$\rightarrow$ Report?