Nahm transform of triply periodic instantons

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The definition of instanton

Definition

(X, g) : an oriented Riemannian 4-fold.
(V, h) : a Hermitian vector bundle on X.
A : a connection on (V, h).

1. A tuple (V, h, A) is called an instanton.
   $\overset{\text{def}}{\iff}$ The curvature $F(A)$ satisfies the ASD equation $F(A) = - * F(A)$.
2. An instanton (V, h, A) on X is called an L^2-finite instanton.
   $\overset{\text{def}}{\iff}$ $|F(A)| \in L^2(X, g)$.

Remark

If we assume that (X, g) is a compact connected Kähler surface with the Kähler form ω, then the ASD equation for (V, h, A) is equivalent to the following equations:

$$\begin{cases} \overline{\partial}_A \circ \overline{\partial}_A & = 0 \\ F(A) \wedge \omega & = 0. \end{cases} \quad \text{(The Hermite-Einstein condition)}$$

The Nahm transform

"Definition"

$\Lambda \subset \mathbb{R}^4$: a closed subgroup.
$\Lambda^* := \{\xi \in \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}) \mid \xi(\Lambda) \subset \mathbb{Z}\}$: the dual subgroup of Λ.
(It is believed that) There exists a transform between the following.

- Λ-invariant instantons on $\mathbb{R}^4$ (with some conditions)
- Λ^*-invariant instantons on $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R})$ (with some conditions)

This transform is called **the Nahm transform**.

Example

$\Lambda = \mathbb{R}^4$, $\Lambda^* = \{0\}$: the ADHM transform
$\Lambda \simeq \Lambda^* \simeq \mathbb{Z}^4$: the Fourier-Mukai transform
$\Lambda \simeq \mathbb{R}$, $\Lambda^* \simeq \mathbb{R}^3$: Nahm, Hitchin, Nakajima, etc

The Nahm transform of instantons on $\mathbb{R} \times T^3$

Today : $\Lambda \simeq \mathbb{Z}^3$, $\Lambda^* \simeq \mathbb{R} \times \mathbb{Z}^3$.

By taking the quotient spaces, we obtain $\mathbb{R}^4/\Lambda = \mathbb{R} \times T^3$ and $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R})/\Lambda^* = \hat{T}^3$, where T^3 is a 3-dimensional torus and $\hat{T}^3$ is the dual torus.

From

- L^2-finite instantons on $\mathbb{R} \times T^3$,

we will construct

- Dirac-type singular monopoles on the dual torus $\hat{T}^3$.

Previous studies

Charbonneau constructed this Nahm transform for the case that instantons are of rank 2 and satisfy a certain genericity.

The Kobayashi-Hitchin correspondence

The Kobayashi-Hitchin Correspondence is the relation between (the moduli spaces of) some algebraic geometric objects and (hyper-)Kähler geometric objects.

Fact (Donaldson, Uhlenbeck-Yau)

(X, ω) : a connected compact Kähler surface.
Then, the following objects correspond each other unique up to gauge transformations.

- (irreducible) instantons on X
- (poly-)stable holomorphic vector bundles of degree 0 on X

Considering similar relation with both L^2-finite instantons on $\mathbb{R} \times T^3$ and Dirac-type singular monopoles on $\hat{T}^3$, we obtain

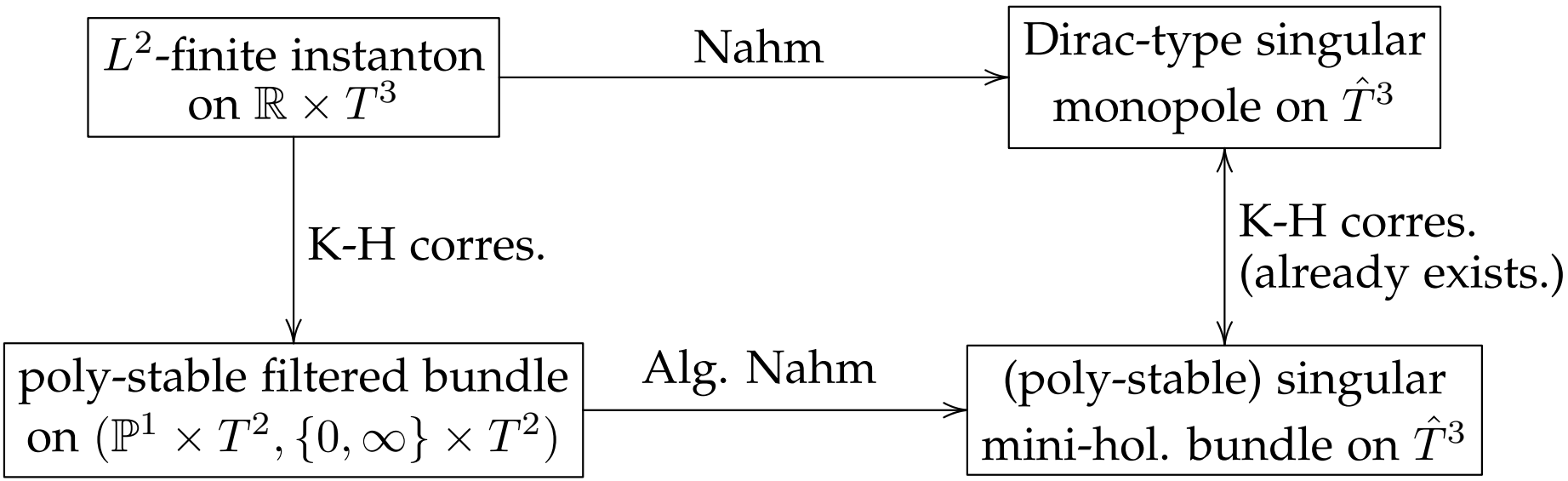
- stable filtered bundle on $\mathbb{P}^1 \times T^2$
- stable singular mini-holomorphic bundle on $\hat{T}^3$

respectively.

The algebraic Nahm transform

As a variant of the Fourier-Mukai transform, we will construct singular mini-holomorphic bundle on $\hat{T}^3$ from stable filtered bundle on $\mathbb{P}^1 \times T^2$. we call this construction **the algebraic Nahm transform**.

Hence we obtain the following commutative diagram.



The Fourier-Mukai Transform

Definition

$T^4 := \mathbb{C}^2/\Lambda$: 4-dimensional torus.
$\hat{T}^4 \simeq \mathrm{Pic}^0(T^4)$: the dual torus of T^4.

1. The Poincaré bundle $\mathcal{L}$ on $T^4 \times \hat{T}^4$ is characterized by the following.
   - $\forall \xi \in \hat{T}^4,\ \mathcal{L}|_{T^4\times\{\xi\}} \simeq L_{-\xi} := (\underline{\mathbb{C}}, \underline{h}, d - 2\pi\sqrt{-1}\langle \xi, dx\rangle)$.
   - $\forall z \in T^4,\ \mathcal{L}|_{\{z\}\times\hat{T}^4} \simeq L_x := (\underline{\mathbb{C}}, \underline{h}, d + 2\pi\sqrt{-1}\langle x, d\xi\rangle)$.
2. The Fourier-Mukai transform $\hat{\bullet} : D^b(\mathrm{Coh}(T^4)) \to D^b(\mathrm{Coh}(\hat{T}^4))$ is defined by $\hat{V}^\cdot := Rp_{2*}(p_1^* V^\cdot \otimes \mathcal{L})$, where p_i is the projection from $T^4 \times \hat{T}^4$ to the i-th component.
3. The inverse transform $\check{\bullet} : D^b(\mathrm{Coh}(\hat{T}^4)) \to D^b(\mathrm{Coh}(T^4))$ is defined by $\check{W}^\cdot := Rp_{1*}(p_2^* W^\cdot \otimes \mathcal{L}^{-1})$.

Theorem (Mukai)

We have isomorphisms $\hat{\hat{\bullet}} \simeq \mathrm{Id}_{D^b(\mathrm{Coh}(\hat{T}^4))}[-2]$ and $\check{\check{\bullet}} \simeq \mathrm{Id}_{D^b(\mathrm{Coh}(T^4))}[-2]$.

The FM transform of irreducible instantons

(V, h, A) : an irreducible instanton on T^4 of rank $r > 1$.

Proposition

- $H^i(\hat{V}) = 0$ for $i = 0, 2$.
- $H^1(\hat{V})$ is a locally free sheaf of rank $\int_{T^4} \mathrm{ch}_2(V) = (8\pi^2)^{-1}||F(A)||^2_{L^2}$.

Under the identifications $S^+ \simeq \Omega^{0,0}_{T^4} \oplus \Omega^{0,2}_{T^4}$ and $S^- \simeq \Omega^{0,1}_{T^4}$, the Dirac operator $\not{\partial}^{\pm}_A : \Gamma(T^4, V \otimes S^{\pm}) \to \Gamma(T^4, V \otimes S^{\mp})$ can be written as $\not{\partial}^{\pm}_A = \sqrt{2}(\overline{\partial}_A + \overline{\partial}^*_A)$.

Theorem

For $\xi \in \hat{T}^4$, let (V, h, A_ξ) denote the twisted instanton $(V, h, A) \otimes L_{-\xi}$. Then we have

$$H^1(\hat{V}) \simeq \coprod_{\xi \in \hat{T}^4} \mathrm{Ker}(\not{\partial}^-_A) \subset \underline{L^2(T^4, V \otimes S^-)}.$$

Moreover, $H^1(\hat{V})$ become an irreducible instanton by the induced metric.

Model solutions of the Nahm equations

Definition

The Nahm equation is the following $\mathfrak{u}(r)$-valued ordinary differential euqation:

$$\frac{\partial A_i}{\partial t} + [\alpha, A_i] = -\mathrm{sgn}(ijk)[A_j, A_k] \ \ ((ijk) \text{ is any permuation of } (123)).$$

This equation is the 3-dimensional reduction of the ASD equation.

For commuting matrices $\Gamma = (\Gamma_i)_{i=1,2,3} \subset \mathfrak{u}(r)$ and $N = (N_i)_{i=1,2,3} \subset \mathrm{C}(\Gamma)$ satisfying the relations $N_i = [N_j, N_k]$ for any even permutation (ijk) of (123), the tuple of $\alpha = 0$ and $A_i = \Gamma_i + N_i/t$ forms a solution of the Nahm equation.

Definition

A pair of tuples $(\Gamma = (\Gamma_i), N = (N_i))$ is called a model solution of the Nahm equation.

Asymptotic behavior of L^2-finite instantons

Theorem (Main Result (1))

(V, h, A) : an L^2-finite instanton on $(0, \infty) \times T^3$ of rank r.
$\exists\, R \gg 0, \exists\, \delta > 0,\ \exists\, \sigma : (V, h)|_{(R,\infty)\times T^3} \simeq (\underline{\mathbb{C}^r}, \underline{h})$: a trivialization, $\exists\, (\Gamma, N)$: a model solution of the Nahm equation $\quad s.t. \quad$ we have the following for the connection form $\alpha dt + \sum_i A_i dx^i$ of A with respect to σ.

1. σ is a temporal gauge of A, *i.e.* $\alpha = 0$.
2. The following estimate holds at $t \to \infty$ for $1 \leq i, j \leq 3$:

$$|A_i - (\Gamma_i + N_i/t)| = O(t^{-(1+\delta)}).$$
$$|\,[A_i, \Gamma_j]\,| = O(-\exp(-\delta t)).$$

Corollary

For any L^2-finite instanton on $\mathbb{R} \times T^3$, there exist model solutions $(\Gamma_\pm, N_\pm)$ which approximate (V, h, A) at $t \to \pm\infty$.

Sketch of proof of main result (1) I

Theorem (Morgan-Mrowka-Ruberman)

(V, h, A) : an L^2-finite instanton on $(0, \infty) \times T^3$.

$\exists\, R \gg 0, \exists\, \delta > 0,\ \exists\, \sigma : (V, h)|_{(R,\infty)\times T^3} \simeq (\underline{\mathbb{C}^r}, \underline{h})$: a trivialization, $\exists\, (\tilde{\alpha}, \tilde{A}_i)$: a solution of the Nahm equation on (R, ∞) $\quad s.t.$ for the connection form $\alpha dt + \sum_i A_i dx^i$ of A with respect to σ, we have

$$|\alpha - \tilde{\alpha}| + \sum_{i=1}^{3} |A_i - \tilde{A}_i| = O(\exp(-\delta t)).$$

Briefly speaking, Any L^2-finite instanton on $(0, \infty) \times T^3$ can be approximated by a solution of the Nahm equation.

Sketch of proof of main result (1) II

Theorem (Biquard)

$(\alpha(t), A_i(t))$: a solution of the Nahm equation on $(0, \infty)$.

$\exists\,(\Gamma, N)$: a model solution of the Nahm equation, $\exists\, g : (0,\infty) \to U(r)$: a gauge transformation $s.t.$ we have the following.

- g is a temporal gauge, *i.e.* $g^{-1}(\partial_t + \alpha)g = 0$.
- $\exists\,\delta > 0 \quad s.t.$ for $1 \leq i, j \leq 3$ we have

$$|g^{-1}A_i g - (\Gamma_i + N_i/t)| = O(t^{-(1+\delta)}).$$
$$|\,[g^{-1}A_i g, \Gamma_j]\,| = O(-\exp(-\delta t)).$$

This Biquard's result can be summarized as "any solution of the Nahm equation on $(0, \infty)$ can be approximated by a model solution".

Singularity set

(V, h, A) : an L^2-finite instanton on $\mathbb{R} \times T^3$.
$(\Gamma_\pm, N_\pm)$: the model solutions which approximate (V, h, A).

Definition

Let $\widetilde{\mathrm{Spec}(\Gamma_\pm)} \subset \mathbb{R}^3$ denote the simultaneous eigenvalues of $((2\pi\sqrt{-1})^{-1}\Gamma_{\pm,i})$.

- We define $\mathrm{Spec}(\Gamma_\pm) := q(\widetilde{\mathrm{Spec}(\Gamma_\pm)})$, where q is the quotient map $\mathbb{R}^3 \to \hat{T}^3$.
- We define the singularity set $\mathrm{Sing}(V, h, A) \subset \hat{T}^3$ to be $\mathrm{Sing}(V, h, A) := \mathrm{Spec}(\Gamma_+) \cup \mathrm{Spec}(\Gamma_-)$.

Remark

Take $\xi \in \hat{T}^3$. For the twisted instanton $(V, h, A_\xi) = (V, h, A) \otimes L_{-\xi}$, we have $\mathrm{Sing}(V, h, A_\xi) = \mathrm{Sing}(V, h, A) - \xi$.

Propaties of Dirac operators and harmonic spinors

(V, h, A) : an L^2-finite instanton on $\mathbb{R} \times T^3$.
From Main result (1), we obtain the following corollaries.

Corollary

$\forall\, \xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$, the Dirac operator

$$\not{\partial}^-_{A_\xi} : L^2(\mathbb{R} \times T^3, V \times S^-) \to L^2(\mathbb{R} \times T^3, V \times S^+)$$

is a surjective closed Fredholm operator of index $(8\pi^2)^{-1}||F(A)||^2_{L^2}$.

Corollary

$\exists\, C > 0, \forall\, \xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A),\ \forall\, f \in \mathrm{Ker}(\not{\partial}^-_{A_\xi})$
$s.t.$

- $|f|$ decays exponentially at $t \to \pm\infty$.
- We have $||tf||_{L^2} \leq C \cdot \mathrm{dist}(\xi, \mathrm{Sing}(V, h, A))^{-1}||f||_{L^2}$.

The definition of monopole

Definition

(Y, g) : an oriented Riemannian 3-fold.
(V, h, A) : a Hermitian vector bundle with a connection on Y.
Φ : a skew-Hermitian endomorphism of (V, h).

A tuple (V, h, A, Φ) is a monopole.
$\overset{\text{def}}{\iff}$ It satitfies the Bogomolny equation $F(A) = *\nabla_A(\Phi)$.

Remark

Let $\pi : \mathbb{R} \times Y \to Y$ be the projection.

A tuple (V, h, A, Φ) is called a monopole on Y
$\iff$ $(\pi^*V, \pi^*h, \pi^*A + \pi^*\Phi dt)$ is an instanton on $\mathbb{R} \times Y$.

Dirac type singularities of monopoles

Definition

(Y, g) : an oriented Riemannian 3-fold.
$p \in Y$.
(V, h, A, Φ) : a monopole of rank r / $Y \setminus \{p\}$.

The point p is called a Dirac-type singularity with weight $\vec{k} = (k_i) \in \mathbb{Z}^r$ if the following hold:

1. $\exists B \subset Y$: a nbd. of p , $\exists V|_{B\setminus\{p\}} = \bigoplus_{i=1}^{r} L_i$:a unitary decomposition into line bundles $\{L_i\}$ $s.t.$ $\int_{\partial B} c_1(L_i) = k_i$.
2. In above decomposition, we have the following estimates:

$$\begin{cases} \Phi = \dfrac{\sqrt{-1}}{2R} \displaystyle\sum_{i=1}^{r} k_i \cdot Id_{L_i} + O(1) \\ \nabla_A(R\Phi) = O(1). \end{cases}$$

Main Result (2) Construction of monopoles I

(V, h, A) : an L^2-finite instanton on $\mathbb{R} \times T^3$.

Construction

Let $\mathcal{V}$ denote the product Hermitian vector bundle $\left(L^2(\mathbb{R} \times T^3, V \otimes S^-), ||\cdot||_{L^2}\right)$ on $\hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$.

- We take a Hermitian subbundle $(\hat{V}, \hat{h})$ to be

$$(\hat{V}_\xi, \hat{h}_\xi) := \mathrm{Ker}(\not{\partial}^-_{A_\xi}) \cap L^2(\mathbb{R} \times T^3, V \otimes S^-)$$

  for any $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$.
- Using the orthogonal projection $P : \mathcal{V} \to \hat{V}$ and the trivial connection d on $\mathcal{V}$, we take a connection $\hat{A}$ to be

$$\hat{A} := Pd.$$

Main Result (2) Construction of monopoles II

Construction

- We take a skew-Hermitian endomorphism $\Phi : \hat{V} \to \hat{V}$ to be

$$\hat{\Phi}(f) := P(2\pi\sqrt{-1}tf)$$

for $f \in \hat{V}_\xi = \mathrm{Ker}(\not{\partial}^-_{A_\xi})$.

Theorem (Main Result (2))

- The tuple $(\hat{V}, \hat{h}, \hat{A}, \Phi)$ is well-defined and a monopole on $\hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$.
- Each point of $\mathrm{Sing}(V, h, A)$ is a Dirac-type singularity of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$.

Sketch of proof of main result (2)

- Well-definedness of $(\hat{V}, \hat{h}, \hat{A})$ follows from Fredholmness of the Dirac operator $\not{\partial}^-_{A_\xi}$ for $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$.
- One of $\hat{\Phi}$ follows from exponentially decay of harmonic spinors.
- the last assertion is an easy consequence of the following theorem.

Theorem (Mochizuki-Y.)

$U \subset \mathbb{R}^3$: a nbd. of $0 \in \mathbb{R}^3$.
(V, h, A, Φ) : a monopole on $U \setminus \{0\}$.

Then the following are equivalent :

- The point 0 is a Dirac-type singularity of (V, h, A, Φ).
- The estimate $|\Phi(x)| = O(|x|^{-1})$ is satisfied.

An assumption of the torus T^3

For simplicity, we assume the following condition.

Assumption

The 3-dimensional torus T^3 is isometric to the product of a circle $S^1 = \mathbb{R} \times \mathbb{Z}$ and a 2-dimensional torus T^2.

Under this assumption, we take holomorphic coordinates

$$\tau = t + \sqrt{-1}x^1 \quad \in \mathbb{R} \times S^1$$
$$w = x^2 + \sqrt{-1}x^3 \in T^2$$

on $\mathbb{R} \times T^3 = (\mathbb{R} \times S^1) \times T^2 \simeq \mathbb{C}^* \times T^2$.

Remark

For general T^3, we fix a 2-dimensional subtorus $T^2 \subset T^3$ and regard $\mathbb{R} \times T^3$ as a principal T^2-bundle on $\mathbb{R} \times S^1 \simeq \mathbb{C}^*$. Then, $\hat{T}^3$ is regarded as a principal S^1-bundle on $\hat{T}^2$.

Prolongation of instantons I

$\Delta \subset \mathbb{C}$: the unit disk, and put $\Delta^* := \Delta \setminus \{0\}$.
X : a complex manifold.
$i : \Delta^* \times X \to \Delta \times X$: the inclusion map.

Definition

$(V, \overline{\partial}_V, h)$: a holomorphic Hermitian vector bundle on $\Delta^* \times X$.

For $a \in \mathbb{R}$, we take an $\mathcal{O}_{X \times \Delta}$-submodule $P_a V$ of $i_* V$ as follows:

$f \in \Gamma(U, i_* V)$ belongs to $\Gamma(U, P_a V)$
$\iff \forall p \in U \cap (X \times \{0\})$, we have $|f|_h = O(|z|^{-a-\varepsilon})$ $(\forall \varepsilon > 0)$ on a nbd. of p.

The family $P_* V := \{P_a V\}_{a \in \mathbb{R}}$ is called the prolongation of V.

For an L^2-finite instanton (V, h, A) on $\mathbb{R} \times T^3 \simeq \mathbb{C}^* \times T^2$, we take the prolongation $P_{**} V = \{P_{ab} V\}_{a,b \in \mathbb{R}}$ on $\mathbb{P}^1 \times T^2$. Here the first and second indices represent the prolongation over $\{0\} \times T^2$ and $\{\infty\} \times T^2$ resprectively.

Prolongation of instantons II

The prolongation $P_{**}V = \{P_{ab}V\}_{a,b\in\mathbb{R}}$ has the following propaties.

- Each $P_{ab}V$ is a locally free sheaf on $\mathbb{P}^1 \times T^2$.
- If $a' < a$ and $b' < b$, then we have $P_{a'b'}V \subset P_{ab}V$.
- $\forall\, a, b \in \mathbb{R},\ \exists\, \varepsilon > 0\ \ s.t.$ we have $P_{(a+\varepsilon)b}V \simeq P_{a(b+\varepsilon)}V \simeq P_{ab}V$.
- We set $P_{<a,b}V := \bigcup_{a'<a} P_{a'b}V$ and ${}^0Gr(P_{**}V) := P_{ab}V/P_{<a,b}V$. Then ${}^0Gr_a(P_{**}V) := P_{ab}V/P_{<a,b}V$ is a locally free sheaf on $\{0\} \times T^2$. A similar condition holds for $\{\infty\} \times T^2 \subset \mathbb{P}^1 \times T^2$.
- We have $P_{(a+1)b}V \simeq P_{ab}V(\{0\} \times T^2)$ and $P_{a(b+1)}V \simeq P_{ab}V(\{\infty\} \times T^2)$. In particular, we have $P_{(a-1)(b+1)}V \simeq P_{ab}V$.

Definition

A family of holomorphic bundles satisfying the above conditions is called a filtered bundle on $(\mathbb{P}^1 \times T^2, \{0,\infty\} \times T^2)$.

Propaties of the prolongation $P_{**}V$

If (V, h, A) is irreducible and of rank $r > 1$, then the prolongation $P_{**}V$ satisfies a kind of stability. From this, we obtain the following propaties.

Proposition

$p : \mathbb{P}^1 \times T^2 \to T^2$: the projection.

For any $a \in \mathbb{R}$ and any $F \in \mathrm{Pic}^0(T^2)$, we have

$$H^0(\mathbb{P}^1 \times T^2, P_{-aa}V \otimes F) = H^2(\mathbb{P}^1 \times T^2, P_{-a<a<}V \otimes F) = 0.$$

Proposition

For any $a \in \mathbb{R}$, the gradations ${}^0\mathrm{Gr}_a(P_{**}V)$ and ${}^\infty\mathrm{Gr}_a(P_{**}V)$ are semi-stable bundles on T^2 of degree 0.

Gradations of the prolongation $P_{**}V$

Moreover, for the prolongation $P_{**}V$, the gradation can be explicitly represented by the model solutions which approximate (V, h, A).

Proposition

$(\Gamma_\pm, N_\pm)$: the model solutions that approximate (V, h, A) to $t \to \pm\infty$.
$X^a_\pm \subset \mathbb{C}^r$: the eigenspace of $\Gamma_{\pm,1}$ of eigenvalue $2\pi\sqrt{-1}a \in \sqrt{-1}\mathbb{R}$.
$B_\pm := (\sum_{i=2}^3 (\Gamma_{\pm,i} + N_{\pm,i})dx^i)^{(0,1)} \in \Omega^{0,1}_{T^2}$.

Then, we have isomorphisms

$$
\begin{aligned}
{}^0\mathrm{Gr}_a(P_{**}V) &\simeq (\underline{X^a_-},\ \overline{\partial}_{T^2} + B_-|_{X^a_-}), \\
{}^\infty\mathrm{Gr}_a(P_{**}V) &\simeq (\underline{X^a_+},\ \overline{\partial}_{T^2} + B_+|_{X^a_+}).
\end{aligned}
$$

Mini-holomorphic structure of monopoles

(Σ, g_Σ): a compact Riemann surface. $I \subset \mathbb{R}_s$: an open interval.
(V, h, A, Φ): a monopole on $I \times \Sigma$.

Definition

We take a decomposition $\nabla_A = \nabla_{A,s}\, ds + \partial_A + \overline{\partial}_A$.

- For $s \in I$, we define a holomorphic bundle V^s as $V^s := (V, \overline{\partial}_A)|_{\{s\}\times\Sigma}$.
- Let $\Psi_{s,s'} : V^s \to V^{s'}$ denote the parallel transport by a differential operator $d_{A,s} := \nabla_{A,s} - \sqrt{-1}\Phi$. We call $\Psi_{s,s'}$ the scattering map.

Proposition

We have the commutativity condition $[d_{A,s}, \overline{\partial}_A] = 0$. In particular, $\Psi_{s,s'} : V^s \to V^{s'}$ is a holomorphic isomorphism.

Definition

A tuple $(d_{A,s}, \overline{\partial}_A)$ that satisfies the commutativity condition $[d_{A,s}, \overline{\partial}_A] = 0$ is called a mini-holomorphic structure of V.

Mini-hol. structures on Dirac-type singularity I

(V, h, A, Φ) : a monopole on $I \times \Sigma \setminus \{(0, p)\}$.
(s_0, p) : a Dirac-type singularity of (V, h, A, Φ) of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$.

Proposition

Assume that $[-\varepsilon, \varepsilon] \subset I$.
$\exists \boldsymbol{v}_\pm = (v_{\pm,i})$: a local frame of $V^{\pm\varepsilon}$ on a nbd. of p $\quad s.t.$

$$\Psi_{-\varepsilon,\varepsilon}(\boldsymbol{v}_-) = \boldsymbol{v}_+ \cdot \mathrm{diag}(z^{k_i}),$$

where z is a local chart with $z(p) = 0$, and $\mathrm{diag}(c_i)$ means the diagonal matrix whose (i, i)-th entry is c_i.

By this proposition, we obtain a mini-holomorphic bundle $(V, d_{A,s}, \overline{\partial}_A)$ on $S^1 \times \Sigma$ with meromorphic singularities from a Dirac-type singular monopole (V, h, A, Φ) on $S^1 \times \Sigma$.

Mini-hol. structures on Dirac-type singularity II

A stability condition of singular mini-holomorphic bundles is defined by Charbonneau and Hurtubise, and they proved the following theorem.

Theorem (Charbonneau-Hurtubise)

$\forall (V, d_{A,s}, \overline{\partial}_A)$: stable mini-hol. bundle with meromorphic singularities,
$\exists (V, h, A, \Phi)$: a Dirac-type singular monopole
$s.t.$ the underlying mini-hol. bundle of (V, h, A, Φ) is isomorphic with $(V, d_{A,s}, \overline{\partial}_A)$, and the monopole (V, h, A, Φ) is unique up to suitable gauge transformations.

The algebraic Nahm transform I

As a variant of the Fourier-Mukai transform, we introduce the algebraic Nahm transform, which construct a mini-holomorphic bundle on $S^1 \times \hat{T}^2$ from a stable filtered bundle on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$.

Proposition

$P_{**}V$: a stable filtered bundle on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$.
$\mathcal{L}$: the Poincaré bundle on $T^2 \times \hat{T}^2$.

We take coherent sheaves $\{\mathrm{AN}_s(P_{**}V)\}_{s\in\mathbb{R}}$ on $\hat{T}^2$ to be

$$\mathrm{AN}_s(P_{**}V) := R^1 p_3(p_{12}^*(P_{-ss}V \times p_{23}^*\mathcal{L})),$$

where p_{ij} is the projection from $\mathbb{P}^1 \times T^2 \times \hat{T}^2$ to the product of the i-th and j-th components.
Then, $\mathrm{AN}_s(P_{**}V)$ is a locally-free sheaf for any $s \in \mathbb{R}$.

We have an isomorphism $\mathrm{AN}_s(P_{**}V) \simeq \mathrm{AN}_{s+1}(P_{**}V)$.
Hence we regard the parameter s as a element of $S^1 = \mathbb{R}/\mathbb{Z}$.

The algebraic Nahm transform II

By the cohomology vanishing of $P_{**}V$ and semi-stability of the gradation module of $P_{**}V$, $\{AN_s(P_{**}V)\}_{s\in S^1}$ forms a mini-holomorphic bundle $(\mathrm{AN}(P_{**}V), d_{\mathrm{AN},s}, \overline{\partial}_{\mathrm{AN}})$ on $\hat{T}^3$ with meromorphic singularities.

Definition

We call $(\mathrm{AN}(P_{**}V), d_{\mathrm{AN},s}, \overline{\partial}_{\mathrm{AN}})$ the algebraic Nahm transform of $P_{**}V$.

Then we have the following theorem.

Theorem

(V, h, A) : an irreducible L^2-finite instanton on $\mathbb{R}\times T^3$.
$P_{**}V$: the prolongation of (V, h, A).
$(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$: the Nahm transform of (V, h, A).
$(\mathrm{AN}(P_{**}V), d_{\mathrm{AN},s}, \overline{\partial}_{\mathrm{AN}})$: the algebraic Nahm transform of $P_{**}V$.

Then the algebraic Nahm transform $(\mathrm{AN}(P_{**}V), d_{\mathrm{AN},s}, \overline{\partial}_{\mathrm{AN}})$ is isomorphic with the underlying mini-holomorphic bundle of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$.

Main Result (3) Correspondence between weights I

By the algebraic Nahm transform, we can calculate weight of Dirac-type singularities of the transformed monopole $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ from the gradations ${}^0\mathrm{Gr}_*(P_{**}V)$ and ${}^\infty\mathrm{Gr}_*(P_{**}V)$.

Moreover, we showed that the gradations can be explicitly represented by the model solutions $(\Gamma_\pm, N_\pm)$.

Hence we can combine weight of singularities of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ with $(\Gamma_\pm, N_\pm)$. We make a notational preparation.

Notation

We take a representation $\rho_\pm : \mathfrak{su}(2) \to \mathfrak{u}(r)$ to be $\rho_\pm(e_i) = N_{\pm,i}$, where $\{e_i\}$ is a basis of $\mathfrak{su}(2)$ satisfying the relation $[e_i, e_j] = \mathrm{sgn}(ijk)e_k$. Since $N_{\pm,i} \in C(\Gamma_\pm)$, we obtain the following decomposition:

$$\rho_\pm = \bigoplus_{\xi \in \mathrm{Sing}(V,h,A)} \rho_{\pm,\xi}.$$

Let $w_{\pm,\xi}$ denote dimensions of irreducible representations contained by $\rho_{\pm,\xi}$.

Main Result (3) Correspondence between weights II

Theorem (Main result (3))

(V, h, A) : an irreducible L^2-finite instanton of rank r on $\mathbb{R} \times T^3$.
$(\Gamma_\pm, N_\pm)$: the model solutions which approximate (V, h, A).
$k_{\pm,\xi}$: the positive (negative) part of weight of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ at $\xi \in \mathrm{Sing}(V, h, A)$.

For any $\xi \in \mathrm{Sing}(V, h, A)$,
$k_{\pm,\xi}$ agrees with $\pm w_{\pm,\xi}$ under a suitable permutation.