Formal Adventures in Convex and Conical Spaces

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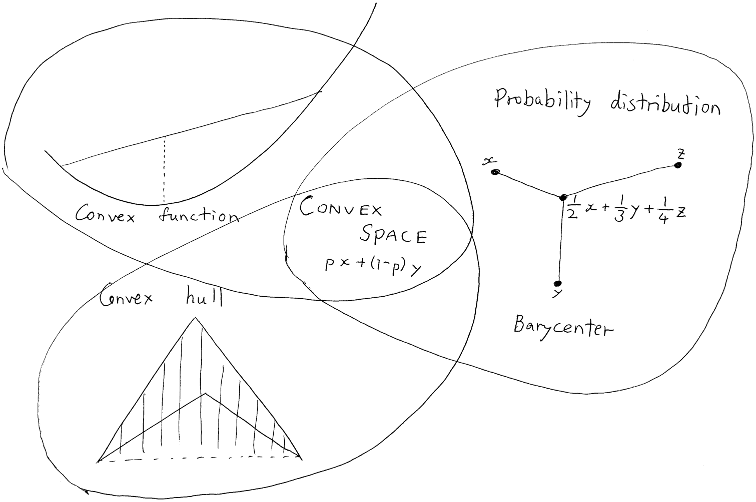
July 28, 2020

Background


Convex function
Convex hull
Convex Space
$px + (1-p)y$
Probability distribution
x
z
y
$\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z$
Barycenter

Overview

This work builds a hub to connect many interesting domains of formalization: probability, analysis, program semantics, etc.
Applications:

- Convex functions
- Barycenter
- Convex hulls
- Semantics of probabilistic programs

Contribution: we formalized different interfaces and lemmas connecting them, and defined derived constructions for applications.

Convex combination and convex space

Convexity notions such as

- Convex functions:
  $f(px + (1-p)y) \leq pf(x) + (1-p)f(y)$
- Convex sets: $x, y \in X \Rightarrow px + (1-p)y \in X$

are based on convex combinations in vector spaces:

$$x \triangleleft_p \triangleright y = px + (1-p)y$$

Many authors axiomatized convex combinations, capturing essentially the same concept:

- Barycentric calculus [Stone, 1949]
- Semiconvex algebra [Flood, 1981]
- Abstract probabilistic domain [Jones and Plotkin, 1989]
- Convex Spaces [Fritz, 2015]

Convex space

- Carrier set X
- Convex combination operations $(\_ \triangleleft_p \triangleright \_) : X \times X \to X$ indexed by $p \in [0, 1]$
- Some appropriate laws

Laws come from a geometric intuition: convex combination defines a barycenter of points.

Barycenter = n-ary combination

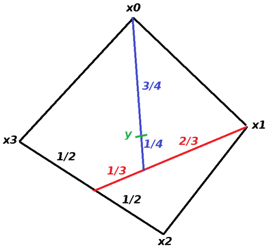
Let

- d : a finite distribution ($\sum_i d_i = 1$)
- x : sequence of points in a convex space X

The barycenter of n points x_i with weights d_i is recursively defined:

$$\underset{i<n}{\diamondsuit} d_i x_i = \begin{cases} x_0 & \text{if } d_0 = 1 \\ x_0 \triangleleft d_0 \triangleright \left(\underset{i<n-1}{\diamondsuit} d'_i x_{i+1} \right) & \text{otherwise} \end{cases}$$

where d' is a new distribution: $d'_i = d_{i+1}/(1-d_0)$



$$y = \frac{1}{4}x_0 + \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3$$

Hull = set of barycenters

Let A be a convex space. For a subset X of A, its hull is

$$\left\{ \underset{i<n-1}{\diamond} d_i x_i \;\middle|\; \begin{array}{l} \quad (n \in \mathbb{N}) \\ \wedge\, (d : n\text{-point distribution}) \\ \wedge\, (x : \text{sequence of } n \text{ points in } X) \end{array} \right\}$$

Convex space

- Carrier set X
- Convex combination operations $(\_ \triangleleft_p \triangleright \_) : X \times X \to X$ indexed by $p \in [0,1]$
- Unit law: $x \triangleleft_1 \triangleright y = x$
- Idempotence law: $x \triangleleft_p \triangleright x = x$
- Skewed commutativity law: $x \triangleleft_{1-p} \triangleright y = y \triangleleft_p \triangleright x$
- Quasi-associativity law:
  $x \triangleleft_p \triangleright (y \triangleleft_q \triangleright z) = (x \triangleleft_r \triangleright y) \triangleleft_s \triangleright z$, where
  $s = 1 - (1-p)(1-q)$ and $r = \begin{cases} p/s & \text{if } s \neq 0 \\ 0 & \text{otherwise} \end{cases}$
  Note: r is irrelevant to the value of $(x \triangleleft_r \triangleright y) \triangleleft_s \triangleright z$ if $s = 0$.

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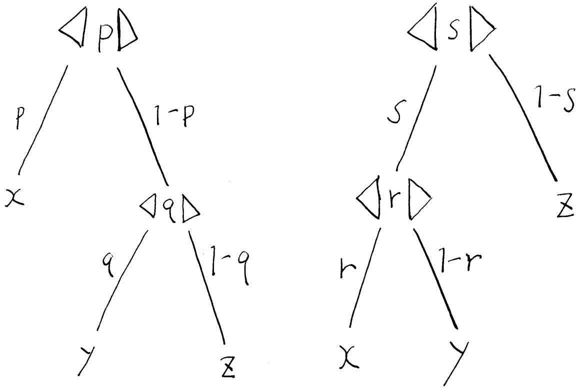
A hell of a quasi-associativity



- $s = 1 - (1-p)(1-q)$ and $r = \frac{p}{s}$
- Zero introduces a special case due to the division.
- Recursive barycenter is even more difficult to formally reason about.

The solution: conical spaces [Flood, 1981]

A conical space is a semimodule over the semiring $\mathbb{R}_{\geq 0}$:

- Carrier set X
- Zero $0 : X$.
- Addition operation $\_ + \_ : X \times X \to X$
- Scaling operations $c\_ : X \to X$ indexed by $c \in \mathbb{R}_{\geq 0}$
- Associativity law for addition: $x + (y + z) = (x + y) + z$
- Commutativity law for addition: $x + y = y + x$
- Associativity law for scaling: $c(dx) = (cd)x$
- Left-distributivity law: $(c + d)x = cx + dx$
- Right-distributivity law: $c(x + y) = cx + cy$
- Zero law for addition: $0 + x = x$
- Zero law for scaling: $0x = 0$
- One law for scaling: $1x = x$

Embedding convex spaces into cones [Flood, 1981]

A conical space is canonically made from convex space X:

$$S_X := (\mathbb{R}_{>0} \times X) \cup \{0\}$$

with an accompanying embedding:

$$\begin{array}{rccc} \iota : & X & \rightarrowtail & S_X \\ & x & \mapsto & (1, x) \end{array}$$

Addition is defined for $a, b \in S_X$,

$$a + b := \begin{cases} (r + q, x \triangleleft_{r/(r+q)} \triangleright y) & \text{if } a = (r, x) \text{ and } b = (q, y) \\ a & \text{if } b = 0 \\ b & \text{if } a = 0 \end{cases}$$

Scalar multiplication is for $a \in S_X$ and $p \in \mathbb{R}_{\geq 0}$,

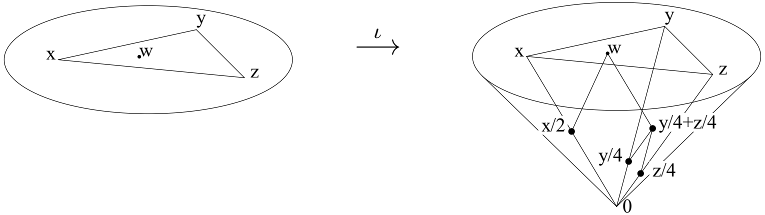
$$pa := \begin{cases} (pq, x) & \text{if } p > 0 \text{ and } a = (p, x) \\ 0 & \text{otherwise} \end{cases}$$

Key lemma for barycenter computation

Lemma (S1_convn)

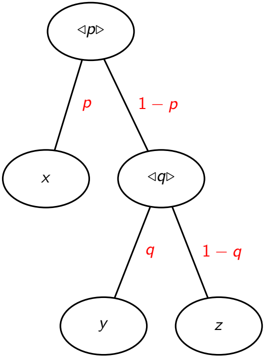
$$\iota(\underset{i<n}{\diamondsuit} d_i x_i) = \sum_{i<n} d_i \iota(x_i)$$

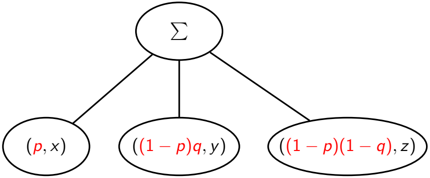
Example ($\iota(w) = \frac{1}{2}\iota(x) + \frac{1}{4}\iota(y) + \frac{1}{4}\iota(z)$)



Welcome to ~~quasi~~-associativity paradise!

The embedding moves weights from edges to leaves:





Proof of a typical property: entropic identity

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Lemma convACA (a b c d : T) p q :
  (a <|q|> b) <|p|> (c <|q|> d) = (a <|p|> c) <|q|> (b <|p|> d).
apply S1_inj; rewrite ![in LHS]S1_conv !convptE.
rewrite !scalept_addpt !scalept_comp //.
rewrite !(mulRC p) !(mulRC p.~) addptA addptC
rewrite (addptC (scalept (q*p) _)) !addptA -addptA
rewrite !(addptC (scalept (_.~ * _.~) _)) {1}addptC.
by rewrite !S1_conv !convptE !scalept_addpt !scalept_comp.
```

Injectivity of ι:

$\iota((a \triangleleft_q \triangleright b) \triangleleft_p \triangleright (c \triangleleft_q \triangleright d)) = \iota((a \triangleleft_p \triangleright c) \triangleleft_q \triangleright (b \triangleleft_p \triangleright d))$

Key lemma:

$\mathrm{LHS} = ((pq)\iota(a) + (p\overline{q})\iota(b)) + ((\overline{p}q)\iota(c) + (\overline{p}\overline{q})\iota(d))$

$\mathrm{RHS} = ((qp)\iota(a) + (q\overline{p})\iota(c)) + ((\overline{q}p)\iota(b) + (\overline{q}\overline{p})\iota(d))$

Associativity and Commutativity:

$\mathrm{LHS} = \mathrm{RHS}$

Multiary axiomatizations

Multiary axiomatizations take the barycenter operation ($\lhd\!\rhd_{i<n} d_i x_i$) as primitive.
Multiary is preferred in many work (e.g. [Bonchi et al., 2017]).
However,

- Different authors use different axiomatizations. How do they compare?
- Their equivalence to the binary axiomatization is often assumed without an explicit proof, citing a lemma by Stone [Stone, 1949, Lemma 4].

We want to prove that these axiomatizations are really equivalent.
This also ensures that our whole formalization is correctly done.

Two multiary axiomatizations

Standard:

- Projection law: if $d_j = 1$, $\underset{i<n}{\diamond} d_i x_i = x_j$.
- Barycenter law: $\underset{i<n}{\diamond} d_i \left(\underset{j<m}{\diamond} e_{i,j} x_j \right) = \underset{j<m}{\diamond} \left(\sum_{i<n} d_i e_{i,j} \right) x_j$.

Beaulieu:

- Partition law: $\underset{i \in I}{\diamond} \lambda_i x_i = \underset{j \in J}{\diamond} \rho_j \left(\underset{k \in K_j}{\diamond} \frac{\lambda_k}{\rho_j} x_k \right)$ where $\{K_j \mid j \in J\}$ is a partition of I, and $\rho_j = \sum_{k \in K_j} \lambda_k \neq 0$.
- Idempotence law: $\underset{i \in I}{\diamond} \lambda_i A_i = A$ if $A_i = A$ for all $\lambda_i > 0$.

Standard ⇔ Beaulieu

Sketch of the proof

Equivalence between Standard and Beaulieu:

- Barycenter ⊢ Partition
- Projection, Barycenter ⊢ Idempotence
- Idempotence ⊢ Projection
- Idempotence, Partition ⊢ Barycenter (difficult)

Binary ⇔ Standard

Bi-interpretability between Binary (convex space) and Standard:

[Binary ⊨ Standard]

- $\diamond_{i<n} d_i x_i := \begin{cases} x_0 & \text{if } d_0 = 1 \\ x_0 \triangleleft d_0 \triangleright (\diamond_{i<n-1} d'_i x_{i+1}) & \text{otherwise} \end{cases}$
  where $d'_i = d_{i+1}/(1 - d_0)$
- Prove Standard axioms for this definition

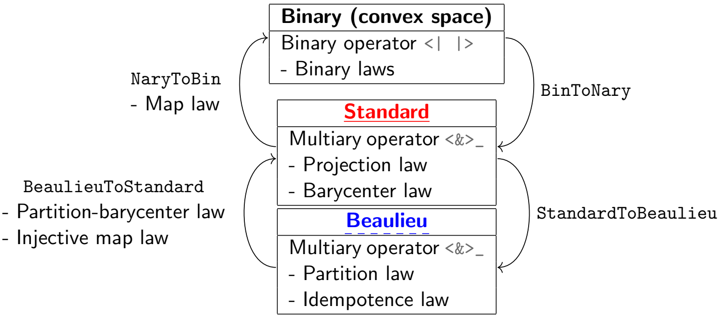
[Standard ⊨ Binary]

- $x_0 \triangleleft p \triangleright x_1 := \diamond_{i<2} d_i x_i$ where $d_0 = p$ and $d_1 = 1 - p$
- Prove Binary axioms for this definition

[Interpretations are inverse to each other]

- $\left\{ \begin{matrix} \text{Binary } \triangleleft_p \triangleright \\ \text{Standard } \diamond_i \end{matrix} \right\} \xrightarrow{\text{def}} \left\{ \begin{matrix} \diamond_i \\ \triangleleft_p \triangleright \end{matrix} \right\} \xrightarrow{\text{def}} \left\{ \begin{matrix} \triangleleft_p \triangleright' \\ \diamond'_i \end{matrix} \right\}$
- Prove $\triangleleft_p \triangleright = \triangleleft_p \triangleright'$ and $\diamond_i = \diamond'_i$

Overview of the equivalences



Conclusion

We formalized:

- Convex spaces with several interfaces
- Equivalence lemmas connecting them
- Further useful constructions such as convex sets and hulls

On-going work includes:

- Convexity of information theoretic functions [Infotheo, 2020]
- Category of convex spaces and probabilistic programming [Affeldt et al., 2019]

Future work:

- What is the true difference between conical and convex spaces?
- Ordered convex spaces with monotonicity
- Topological convex spaces
- Affine Lie algebras

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