MIXED PENTAGON EQUATION AND DOUBLE SHUFFLE RELATION

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Abstract. This paper is a review of the paper [F4] where a geometric interpretation of the generalized (including the regularization relation) double shuffle relation for multiple L-values is given. In precise, it is shown that Enriquez' mixed pentagon equation implies the relations.

0. Introduction

Multiple L-values $L(k_1, \cdots, k_m; \zeta_1, \cdots, \zeta_m)$ are the complex numbers defined by the following series

$$L(k_1, \cdots, k_m; \zeta_1, \cdots, \zeta_m) := \sum_{0<n_1<\cdots<n_m} \frac{\zeta_1^{n_1} \cdots \zeta_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \tag{1}$$

for m, $k_1, \ldots, k_m \in \mathbf{N}(= \mathbf{Z}_{>0})$ and $\zeta_1, \ldots, \zeta_m \in \mu_N$(:the group of N-th roots of unity in $\mathbf{C}$). They converge if and only if $(k_m, \zeta_m) \neq (1, 1)$. Multiple zeta values are regarded as a special case for $N = 1$. These values have been discussed in several papers [AK, BK, G, R] etc. Multiple L-values appear as coefficients of the cyclotomic Drinfel'd associator Φ_{KZ}^N (5) in $U\mathfrak{F}_{N+1}$: the non-commutative formal power series ring with $N + 1$ variables A and $B(a)$ ($a \in \mathbf{Z}/N\mathbf{Z}$).

The mixed pentagon equation (4) is a geometric equation introduced by Enriquez [E]. The series Φ_{KZ}^N satisfies the equation, which yields non-trivial relations among multiple L-values. The generalised double shuffle relation (the double shuffle relation and the regularization relation) is a combinatorial relation among multiple L-values. It is formulated as (6) for $h = \Phi_{KZ}^N$. It is Zhao's remark [Z] that for specific N's the generalized double shuffle relation does not provide all the possible relations among multiple L-values.

Our main theorem is an implication of the generalised double shuffle relation (6) from the mixed pentagon equation (4).

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Theorem 1. *Let $U\mathfrak{F}_{N+1}$ be the universal enveloping algebra of the free Lie algebra $\mathfrak{F}_{N+1}$ with variables A and $B(a)$ ($a \in \mathbf{Z}/N\mathbf{Z}$). Let h be a group-like element in $U\mathfrak{F}_{N+1}$ with $c_{B(0)}(h) = 0$ satisfying the mixed pentagon equation* (4) *with a group-like series $g \in U\mathfrak{F}_2$. Then h also satisfies the generalised double shuffle relation* (6).

The contents of the article are as follows: We recall the mixed pentagon equation in §1 and the generalised double shuffle relation in §2. In §3 we calculate the 0-th cohomologies of Chen's reduced bar complex for the Kummer coverings of the moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. Two variable cyclotomic multiple polylogarithms and their associated bar elements there are introduced in §4. By using them, we prove theorem 1 in §5.

1. Mixed pentagon equation

This section is to recall Enriquez' mixed pentagon equation [E].

Let us fix notations: For $n \geqslant 2$, the Lie algebra $\mathfrak{t}_n$ of infinitesimal pure braids is the completed $\mathbf{Q}$-Lie algebra with generators t^{ij} ($i \neq j$, $1 \leqslant i, j \leqslant n$) and relations $t^{ij} = t^{ji}$, $[t^{ij}, t^{ik} + t^{jk}] = 0$ and $[t^{ij}, t^{kl}] = 0$ for all distinct i, j, k, l. We note that $\mathfrak{t}_2$ is the 1-dimensional abelian Lie algebra generated by t^{12}. The element $z_n = \sum_{1\leqslant i<j\leqslant n} t^{ij}$ is central in $\mathfrak{t}_n$. Put $\mathfrak{t}_n^0$ to be the Lie subalgebra of $\mathfrak{t}_n$ with the same generators except t^{1n} and the same relations as $\mathfrak{t}_n$. Then we have $\mathfrak{t}_n = \mathfrak{t}_n^0 \oplus \mathbf{Q}\cdot z_n$. Especially when $n = 3$, $\mathfrak{t}_3^0$ is a free Lie algebra $\mathfrak{F}_2$ of rank 2 with generators $A := t^{12}$ and $B = t^{23}$. For a partially defined map $f : \{1, \dots, m\} \to \{1, \dots, n\}$, the Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_m : x \mapsto x^f = x^{f^{-1}(1),\dots,f^{-1}(n)}$ is uniquely defined by $(t^{ij})^f = \sum_{i'\in f^{-1}(i),j'\in f^{-1}(j)} t^{i'j'}$.

For a pair $(\mu, g) \in \mathbf{Q} \times \exp\mathfrak{F}_2$ the *pentagon equation* is the following equation in $\exp\mathfrak{t}_4^0$

$$g^{1,2,34}g^{12,3,4} = g^{2,3,4}g^{1,23,4}g^{1,2,3}. \tag{2}$$

and *two hexagon equations* the following two equations in $\exp\mathfrak{F}_2 = \exp\mathfrak{t}_3^0$

$$g(A, B)g(B, A) = 1 \qquad \text{and} \tag{3}$$

$$\exp\{\frac{\mu A}{2}\}g(C, A)\exp\{\frac{\mu C}{2}\}g(B, C)\exp\{\frac{\mu B}{2}\}g(A, B) = 1$$

with $C = -A - B$. These

By our notation, the equation (2) can be read as

$$g(t^{12}, t^{23} + t^{24})g(t^{13} + t^{23}, t^{34}) = g(t^{23}, t^{34})g(t^{12} + t^{13}, t^{24} + t^{34})g(t^{12}, t^{23}).$$

Remark 2. It is shown in [F2] that the two hexagon equations (3) are consequences of the pentagon equation (2).

Remark 3. The *Drinfel'd associator* $\Phi_{KZ} = \Phi_{KZ}(A,B) \in \mathbf{C}\langle\langle A,B\rangle\rangle$ is defined to be the quotient $\Phi_{KZ} = G_1(z)^{-1}G_0(z)$ where G_0 and G_1 are the solutions of the formal KZ equation

$$\frac{d}{dz}G(z) = (\frac{A}{z} + \frac{B}{z-1})G(z)$$

such that $G_0(z) \approx z^A$ when $z \to 0$ and $G_1(z) \approx (1-z)^B$ when $z \to 1$ (cf.[Dr]). The series has the following expression

$$\Phi_{KZ} = 1+\sum(-1)^m\zeta(k_1,\cdots,k_m)A^{k_m-1}B\cdots A^{k_1-1}B+(\text{regularized terms})$$

and the regularised terms are explicitly calculated to be linear combinations of multiple zeta values $\zeta(k_1,\cdots,k_m) = L(k_1,\dots,k_m;1,\dots,1)$ in [F1] proposition 3.2.3 by Le-Murakami's method [LM]. It is shown in [Dr] that the pair $(2\pi\sqrt{-1},\Phi_{KZ})$ satisfies the pentagon equation (2) and the hexagon equations (3).

For $n \geqslant 2$ and $N \geqslant 1$, the Lie algebra $\mathfrak{t}_{n,N}$ is the completed $\mathbf{Q}$-Lie algebra with generators t^{1i} $(2 \leqslant i \leqslant n)$, $t(a)^{ij}$ $(i \neq j,\ 2 \leqslant i,j \leqslant n,$ $a \in \mathbf{Z}/N\mathbf{Z})$ and relations $t(a)^{ij} = t(-a)^{ji}$, $[t(a)^{ij}, t(a+b)^{ik}+t(b)^{jk}] = 0$, $[t^{1i} + t^{1j} + \sum_{c\in\mathbf{Z}/N\mathbf{Z}} t(c)^{ij}, t(a)^{ij}] = 0$, $[t^{1i}, t^{1j} + \sum_{c\in\mathbf{Z}/N\mathbf{Z}} t(c)^{ij}] = 0$, $[t^{1i}, t(a)^{jk}] = 0$ and $[t(a)^{ij}, t(b)^{kl}] = 0$ for all a, $b \in \mathbf{Z}/N\mathbf{Z}$ and all distinct i, j, k, l $(2 \leqslant i,j,k,l \leqslant n)$. We note that $\mathfrak{t}_{n,1}$ is equal to $\mathfrak{t}_n$ for $n \geqslant 2$. We have a natural injection $\mathfrak{t}_{n-1,N} \hookrightarrow \mathfrak{t}_{n,N}$. The Lie subalgebra $\mathfrak{f}_{n,N}$ of $\mathfrak{t}_{n,N}$ generated by t^{1n} and $t(a)^{in}$ $(2 \leqslant i \leqslant n-1,\ a \in \mathbf{Z}/N\mathbf{Z})$ is free of rank $(n-2)N+1$ and forms an ideal of $\mathfrak{t}_{n,N}$. Actually it shows that $\mathfrak{t}_{n,N}$ is a semi-direct product of $\mathfrak{f}_{n,N}$ and $\mathfrak{t}_{n-1,N}$. The element $z_{n,N} = \sum_{1\leqslant i<j\leqslant n} t^{ij}$ with $t^{ij} = \sum_{a\in\mathbf{Z}/N\mathbf{Z}} t(a)^{ij}$ $(2 \leqslant i < j \leqslant n)$ is central in $\mathfrak{t}_{n,N}$. Put $\mathfrak{t}^0_{n,N}$ to be the Lie subalgebra of $\mathfrak{t}_{n,N}$ with the same generators except t^{1n}. Then we have $\mathfrak{t}_{n,N} = \mathfrak{t}^0_{n,N} \oplus \mathbf{Q}\cdot z_{n,N}$. Occasionally we regard $\mathfrak{t}^0_{n,N}$ as the quotient $\mathfrak{t}_{n,N}/\mathbf{Q}\cdot z_{n,N}$. Especially when $n = 3$, $\mathfrak{t}^0_{3,N}$ is free Lie algebra $\mathfrak{F}_{N+1}$ of rank $N+1$ with generators $A := t^{12}$ and $B(a) = t(a)^{23}$ $(a \in \mathbf{Z}/N\mathbf{Z})$.

For a partially defined map $f : \{1,\dots,m\} \to \{1,\dots,n\}$ such that $f(1) = 1$, the Lie algebra morphism $\mathfrak{t}_{n,N} \to \mathfrak{t}_{m,N} : x \mapsto x^f = x^{f^{-1}(1),\dots,f^{-1}(n)}$ is uniquely defined by $(t(a)^{ij})^f = \sum_{i'\in f^{-1}(i),j'\in f^{-1}(j)} t(a)^{i'j'}$ $(i \neq j,\ 2 \leqslant i,j \leqslant n)$ and $(t^{1j})^f = \sum_{j'\in f^{-1}(j)} t^{1j'} + \frac{1}{2}\sum_{j',j''\in f^{-1}(j)}\sum_{c\in\mathbf{Z}/N\mathbf{Z}} t(c)^{j'j''}$ $+\sum_{i'\neq 1\in f^{-1}(1),j'\in f^{-1}(j)}\sum_{c\in\mathbf{Z}/N\mathbf{Z}} t(c)^{i'j'}$ $(2 \leqslant j \leqslant n)$. Again for a partially defined map $g : \{2,\dots,m\} \to \{1,\dots,n\}$, the Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_{m,N} : x \mapsto x^g = x^{g^{-1}(1),\dots,g^{-1}(n)}$ is uniquely defined by $(t^{ij})^g = \sum_{i'\in g^{-1}(i),j'\in g^{-1}(j)} t(0)^{i'j'}$ $(i \neq j,\ 1 \leqslant i,j \leqslant n)$.

For a pair $(g,h)\in\exp\mathfrak{F}_2\times\exp\mathfrak{F}_{N+1}$, the *mixed pentagon equation* means the following equation in $\exp\mathfrak{t}^0_{4,N}$

$$h^{1,2,34}h^{12,3,4}=g^{2,3,4}h^{1,23,4}h^{1,2,3}. \tag{4}$$

By our notation, each term in the equation (4) can be read as

$$h^{1,2,34}=h(t^{12},t^{23}(0)+t^{24}(0),t^{23}(1)+t^{24}(1),\dots,t^{23}(N-1)+t^{24}(N-1)),$$
$$h^{12,3,4}=h(t^{13}+\sum_c t^{23}(c),t^{34}(0),t^{34}(1),\dots,t^{34}(N-1)),$$
$$g^{2,3,4}=g(t^{23}(0),t^{34}(0)),$$
$$h^{1,23,4}=h(t^{12}+t^{13}+\sum_c t^{23}(c),t^{24}(0)+t^{34}(0),\dots,t^{24}(N-1)+t^{34}(N-1)),$$
$$h^{1,2,3}=h(t^{12},t^{23}(0),t^{23}(1),\dots,t^{23}(N-1)).$$

Remark 4. In [E], the cyclotomic analogue $\Phi^N_{KZ}\in\exp\mathfrak{F}_{N+1}(\mathbf{C})$ of the Drinfel'd associator is introduced to be the renormalised holonomy from 0 to 1 of the KZ-like differential equation

$$\frac{d}{dz}H(z)=\Big(\frac{A}{z}+\sum_{a\in\mathbf{Z}/N\mathbf{Z}}\frac{B(a)}{z-\zeta_N^a}\Big)H(z)$$

with $\zeta_N=\exp\{\frac{2\pi\sqrt{-1}}{N}\}$, i.e., $\Phi^N_{KZ}=H_1^{-1}H_0$ where H_0 and H_1 are the solutions such that $H_0(z)\approx z^A$ when $z\to 0$ and $H_1(z)\approx(1-z)^{B(0)}$ when $z\to 1$ (cf.[E]). There appear multiple L-values (1) in each of its coefficient;

(5)
$$\Phi^N_{KZ}=1+\sum(-1)^m L(k_1,\cdots,k_m;\xi_1,\dots,\xi_m)A^{k_m-1}B(a_m)\cdots A^{k_1-1}B(a_1)$$
$$+\text{(regularized terms)}$$

with $\xi_1=\zeta_N^{a_2-a_1}$, $\dots$, $\xi_{m-1}=\zeta_N^{a_m-a_{m-1}}$ and $\xi_m=\zeta_N^{-a_m}$, where the regularised terms can be explicitly calculated to combinations of multiple L-values by the method of Le-Murakami [LM]. In [E] it is shown that the triple $(2\pi\sqrt{-1},\Phi_{KZ},\Phi^N_{KZ})$ satisfies the mixed pentagon equation (4). This is achieved by considering monodromy in the pentagon formed by the divisors $y=0$, $x=1$, the exceptional divisor of the blowing-up at $(1,1)$, $y=1$ and $x=0$ in $\mathcal{M}^{(N)}_{0,5}$ (see §3).

Remark 5. In [EF] it is proved that the mixed pentagon equation (4) implies the distribution relation for a specific case and that the octagon equation follows from the mixed pentagon equation and the special action condition for $N=2$.

2. Double shuffle relation

This section is to recall the generalised double shuffle relation in Racinet's setting [R].

Let us fix notations: Let $\mathfrak{F}_{Y_N}$ be the completed graded Lie $\mathbf{Q}$-algebra generated by $Y_{n,a}$ ($n \geqslant 1$ and $a \in \mathbf{Z}/N\mathbf{Z}$) with $\deg Y_{n,a} = n$. Put $U\mathfrak{F}_{Y_N}$ its universal enveloping algebra: the non-commutative formal series ring with free variables $Y_{n,a}$ ($n \geqslant 1$ and $a \in \mathbf{Z}/N\mathbf{Z}$). Let $\pi_Y : U\mathfrak{F}_{N+1} \to U\mathfrak{F}_{Y_N}$ be the $\mathbf{Q}$-linear map between non-commutative formal power series rings that sends all the words ending in A to zero and the word $A^{n_m-1}B(a_m)\cdots A^{n_1-1}B(a_1)$ ($n_1,\dots,n_m \geqslant 1$ and $a_1,\dots,a_m \in \mathbf{Z}/N\mathbf{Z}$) to

$$(-1)^m Y_{n_m,-a_m} Y_{n_{m-1},a_m-a_{m-1}} \cdots Y_{n_1,a_2-a_1}.$$

Define the coproduct Δ_* of $U\mathfrak{F}_{Y_N}$ by $\Delta_* Y_{n,a} = \sum_{k+l=n,b+c=a} Y_{k,b} \otimes Y_{l,c}$ ($n \geqslant 0$ and $a \in \mathbf{Z}/N\mathbf{Z}$) with $Y_{0,a} := 1$ if $a = 0$ and 0 if $a \neq 0$. For $h = \sum_{W:\mathrm{word}} c_W(h) W \in U\mathfrak{F}_{N+1}$, define the series shuffle regularization $h_* = h_{\mathrm{corr}} \cdot \pi_Y(h)$ with the correction term

$$h_{\mathrm{corr}} = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{A^{n-1}B(0)}(h) Y_{1,0}^n \right).$$

For a series $h \in \exp \mathfrak{F}_{N+1}$ the *generalised double shuffle relation* stands for the following relation in $U\mathfrak{F}_{Y_N}$

$$\Delta_*(h_*) = h_* \widehat{\otimes} h_*. \tag{6}$$

Remark 6. The series Φ_{KZ}^N (5) satisfies the generalised double shuffle relation (6) because regularised multiple L-values satisfy the double shuffle relation.

3. Bar constructions

This section gives a review of the notion of the reduced bar construction and calculates it for $\mathcal{M}_{0,4}^{(N)}$ and $\mathcal{M}_{0,5}^{(N)}$.

We recall the notion of Chen's reduced bar construction [C]. Let $(A^\bullet = \oplus_{q=0}^{\infty} A^q, d)$ be a differential graded algebra (DGA). The reduced bar complex $\bar{B}^\bullet(A)$ is the tensor algebra $\oplus_{r=0}^{\infty} (\bar{A}^\bullet)^{\otimes r}$ with $\bar{A}^\bullet = \oplus_{i=0}^{\infty} \bar{A}^i$ where $\bar{A}^0 = A^1/dA^0$ and $\bar{A}^i = A^{i+1}$ ($i > 0$). We denote $a_1 \otimes \cdots \otimes a_r$ ($a_i \in \bar{A}^\bullet$) by $[a_1|\cdots|a_r]$. The degree of elements in $\bar{B}^\bullet(A)$ is given by the total degree of $\bar{A}^\bullet$. Put $Ja = (-1)^{p-1}a$ for $a \in \bar{A}^p$. Define

$$d'[a_1|\cdots|a_k] = \sum_{i=1}^{k} (-1)^i [Ja_1|\cdots|Ja_{i-1}|da_i|a_{i+1}|\cdots|a_k]$$

and

$$d''[a_1|\cdots|a_k]=\sum_{i=1}^{k}(-1)^{i-1}[Ja_1|\cdots|Ja_{i-1}|Ja_i\cdot a_{i+1}|a_{i+2}|\cdots|a_k].$$

Then $d'+d''$ forms a differential. The differential and the shuffle product (loc.cit.) give $\bar{B}^\bullet(A)$ a structure of commutative DGA. Actually it also forms a Hopf algebra, whose coproduct Δ is given by

$$\Delta([a_1|\cdots|a_r])=\sum_{s=0}^{r}[a_1|\cdots|a_s]\otimes[a_{s+1}|\cdots|a_r].$$

For a smooth complex manifold $\mathcal{M}$, $\Omega^\bullet(\mathcal{M})$ means the de Rham complex of smooth differential forms on $\mathcal{M}$ with values in $\mathbf{C}$. We denote the 0-th cohomology of the reduced bar complex $\bar{B}^\bullet(\Omega(\mathcal{M}))$ with respect to the differential by $H^0\bar{B}(\mathcal{M})$.

Let $\mathcal{M}_{0,4}$ be the moduli space $\{(x_1,\cdots,x_4)\in(\mathbf{P}^1_{\mathbf{C}})^4|x_i\neq x_j(i\neq j)\}/PGL_2(\mathbf{C})$ of 4 different points in $\mathbf{P}^1$. It is identified with $\{z\in\mathbf{P}^1_{\mathbf{C}}|z\neq 0,1,\infty\}$ by sending $[(0,z,1,\infty)]$ to z. Denote its Kummer N-covering

$$\mathbf{G}_m\backslash\mu_N=\{z\in\mathbf{P}^1_{\mathbf{C}}|z^N\neq 0,1,\infty\}$$

by $\mathcal{M}^{(N)}_{0,4}$. The space $H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})$ is generated by

$$\omega_0:=d\log(z)\text{ and }\omega_\zeta:=d\log(z-\zeta)\quad(\zeta\in\mu_N).$$

We have an identification $H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})$ with the graded $\mathbf{C}$-linear dual of $U\mathfrak{F}_{N+1}$,

$$H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})\simeq U\mathfrak{F}^*_{N+1}\otimes\mathbf{C},$$

by $\operatorname{Exp}\Omega^{(N)}_4:=\sum X_{i_m}\cdots X_{i_1}\otimes[\omega_{i_m}|\cdots|\omega_{i_1}]\in U\mathfrak{F}_{N+1}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})$. Here the sum is taken over $m\geqslant 0$ and $i_1,\cdots,i_m\in\{0\}\cup\mu_N$ and $X_0=A$ and $X_\zeta=B(a)$ when $\zeta=\zeta_N^a$. It is easy to see that the identification is compatible with Hopf algebra structures. We note that the product $l_1\cdot l_2\in H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})$ for l_1, $l_2\in H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})$ is given by $l_1\cdot l_2(f):=\sum_i l_1(f_1^{(i)})l_2(f_2^{(i)})$ for $f\in U\mathfrak{F}_{N+1}\otimes\mathbf{C}$ with $\Delta(f)=\sum_i f_1^{(i)}\otimes f_2^{(i)}$. Occasionally we regard $H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})$ as the regular function ring of $F_{N+1}(\mathbf{C})=\{g\in U\mathfrak{F}_{N+1}\otimes\mathbf{C}|g:\text{group-like}\}=\{g\in U\mathfrak{F}_{N+1}\otimes\mathbf{C}|g(0)=1,\Delta(g)=g\otimes g\}$.

Let $\mathcal{M}_{0,5}$ be the moduli space $\{(x_1,\cdots,x_5)\in(\mathbf{P}^1_{\mathbf{C}})^5|x_i\neq x_j(i\neq j)\}/PGL_2(\mathbf{C})$ of 5 different points in $\mathbf{P}^1$. It is identified with $\{(x,y)\in\mathbf{G}_m^2|x\neq 1,y\neq 1,xy\neq 1\}$ by sending $[(0,xy,y,1,\infty)]$ to (x,y). Denote its Kummer N^2-covering

$$\{(x,y)\in\mathbf{G}_m^2|x^N\neq 1,y^N\neq 1,(xy)^N\neq 1\}$$

by $\mathcal{M}_{0,5}^{(N)}$. It is identified with $W_N/\mathbf{C}^\times$ by $(x,y)\mapsto(xy,y,1)$ where

$$W_N=\{(z_2,z_3,z_4)\in\mathbf{G}_m|z_i^N\neq z_j^N(i\neq j)\}.$$

The space $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ is a subspace of the tensor coalgebra generated by

$$\omega_{1,i}:=d\log z_i \text{ and } \omega_{i,j}(a):=d\log(z_i-\zeta_N^a z_j)\quad(2\leqslant i,j\leqslant 4, a\in\mathbf{Z}/N).$$

Proposition 7. *We have an identification*

$$H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})\simeq(U\mathbf{t}_{4,N}^0)^*\otimes\mathbf{C}.$$

Proof . By [K], $H^0\bar{B}(W_N)$ can be calculated to be the 0-th cohomology $H^0\bar{B}^\bullet(S)$ of the reduced bar complex of the Orlik-Solomon algebra $S^\bullet$. The algebra $S^\bullet$ is the (trivial-)differential graded $\mathbf{C}$-algebra $S^\bullet=\oplus_{q=0}^\infty S^q$ defined by generators

$$\omega_{1,i}=d\log z_i \text{ and } \omega_{i,j}(a)=d\log(z_i-\zeta_N^a z_j)\quad(2\leqslant i,j\leqslant 4, a\in\mathbf{Z}/N\mathbf{Z})$$

in degree 1 and relations

$$\begin{aligned}&\omega_{i,j}(a)=\omega_{j,i}(-a),\qquad \omega_{ij}(a)\wedge\{\omega_{ik}(a+b)+\omega_{jk}(b)\}=0,\\&\{\omega_{1i}+\omega_{1j}+\sum_{c\in\mathbf{Z}/N\mathbf{Z}}\omega(c)_{ij}\}\wedge\omega(a)_{ij}=0,\\&\omega_{1i}\wedge\{\omega_{1j}+\sum_{c\in\mathbf{Z}/N\mathbf{Z}}\omega(c)_{ij}\}=0,\\&\omega_{1i}\wedge\omega(a)_{jk}=0\quad\text{ and }\quad\omega(a)_{ij}\wedge\omega(b)_{kl}=0\end{aligned}$$

for all a, $b\in\mathbf{Z}/N\mathbf{Z}$ and all distinct i, j, k, l $(2\leqslant i,j,k,l\leqslant n)$. By direct calculation, the element

$$\sum_{i=2}^4 t_{1i}\otimes\omega_{1i}+\sum_{2\leqslant i<j\leqslant 4,a\in\mathbf{Z}/N\mathbf{Z}}t_{ij}(a)\otimes\omega_{ij}(a)\in(\mathbf{t}_{4,N})^{\deg=1}\otimes S^1$$

yields a Hopf algebra identification of $H^0\bar{B}(W_N)$ with $(U\mathbf{t}_{4,N})^*\otimes\mathbf{C}$ since both are quadratic.

By the long exact sequence of cohomologies induced from the $\mathbf{G}_m$-bundle $W_N\to\mathcal{M}_{0,5}^{(N)}=W_N/\mathbf{C}^\times$, we get

$$0\to H^1(\mathcal{M}_{0,5}^{(N)})\to H^1(W_N)\to H^1(\mathbf{G}_m)\to 0$$

and

$$H^i(\mathcal{M}_{0,5}^{(N)})\simeq H^i(W_N)\qquad(i\geqslant 2).$$

It yields the identification of the subspace $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ of $H^0\bar{B}(W_N)$ with $(U\mathbf{t}_{4,N}^0)^*\otimes\mathbf{C}$. □

The above identification is induced from

$$\text{Exp }\Omega_5^{(N)} := \sum t_{J_m}\cdots t_{J_1}\otimes[\omega_{J_m}|\cdots|\omega_{J_1}] \in U\mathfrak{t}^0_{4,N}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$$

where the sum is taken over $m\geqslant 0$ and $J_1,\cdots,J_m\in\{(1,i)|2\leqslant i\leqslant 4\}\cup\{(i,j,a)|2\leqslant i<j\leqslant 4, a\in\mathbf{Z}/N\mathbf{Z}\}$.

Especially the identification between degree 1 terms is given by

$$\Omega_5^{(N)} = \sum_{i=2}^4 t_{1i}d\log z_i + \sum_{2\leqslant i<j\leqslant 4}\sum_{a\in\mathbf{Z}/N\mathbf{Z}} t_{i,j}(a)d\log(z_i-\zeta_N^a z_j)$$
$$\in \mathfrak{t}^0_{4,N}\otimes H^1_{DR}(\mathcal{M}_{0,5}^{(N)}).$$

In terms of the coordinate (x,y),

$$\begin{aligned}\Omega_5^{(N)} &= t_{12}d\log(xy)+t_{13}d\log y+\sum_a t_{23}(a)d\log y(x-\zeta_N^a)\\ &\quad+\sum_a t_{24}(a)d\log(xy-\zeta_N^a)+\sum_a t_{34}(a)d\log(y-\zeta_N^a)\\ &= t_{12}d\log x+\sum_a t_{23}(a)d\log(x-\zeta_N^a)+(t_{12}+t_{13}+t_{23})d\log y\\ &\quad+\sum_a t_{34}(a)d\log(y-\zeta_N^a)+\sum_a t_{24}(a)d\log(xy-\zeta_N^a).\end{aligned}$$

It is easy to see that the identification is compatible with Hopf algebra structures. We note again that the product $l_1\cdot l_2\in H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ for $l_1,\ l_2\in H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ is given by $l_1\cdot l_2(f):=\sum_i l_1(f_1^{(i)})l_2(f_2^{(i)})$ for $f\in U\mathfrak{t}^0_{4,N}\otimes\mathbf{C}$ with $\Delta(f)=\sum_i f_1^{(i)}\otimes f_2^{(i)}$ (Δ: the coproduct of $U\mathfrak{t}^0_{4,N}$). Occasionally we also regard $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ as the regular function ring of $K_4^N(\mathbf{C})=\{g\in U\mathfrak{t}^0_{4,N}\otimes\mathbf{C}|g:\text{group-like}\}$.

By a generalization of Chen's theory [C] to the case of tangential basepoints, especially for $\mathcal{M}=\mathcal{M}_{0,4}^{(N)}$ or $\mathcal{M}_{0,5}^{(N)}$, we have an isomorphism

$$\rho: H^0\bar{B}(\mathcal{M})\simeq I_o(\mathcal{M})$$

as algebras over $\mathbf{C}$ which sends $\sum_{I=(i_m,\cdots,i_1)}c_I[\omega_{i_m}|\cdots|\omega_{i_1}]$ ($c_I\in\mathbf{C}$) to $\sum_I c_I\text{It}\int_o\omega_{i_m}\circ\cdots\circ\omega_{i_1}$. Here $\sum_I c_I\text{It}\int_o\omega_{i_m}\circ\cdots\circ\omega_{i_1}$ means the iterated integral defined by

$$(7)\quad \sum_I c_I\int_{0<t_1<\cdots<t_{m-1}<t_m<1}\omega_{i_m}(\gamma(t_m))\cdot\omega_{i_{m-1}}(\gamma(t_{m-1}))\cdots\cdot\omega_{i_1}(\gamma(t_1))$$

for all analytic paths $\gamma:(0,1)\to\mathcal{M}(\mathbf{C})$ starting from the tangential basepoint o (defined by $\frac{d}{dz}$ for $\mathcal{M}=\mathcal{M}_{0,4}^{(N)}$ and defined by $\frac{d}{dx}$ and $\frac{d}{dy}$ for $\mathcal{M}=\mathcal{M}_{0,5}^{(N)}$) at the origin in $\mathcal{M}$ (for its treatment see also [De]§15) and $I_o(\mathcal{M})$ stands for the $\mathbf{C}$-algebra generated by all such homotopy invariant iterated integrals with $m \geqslant 1$ and $\omega_{i_1}, \ldots, \omega_{i_m} \in H^1_{DR}(\mathcal{M})$.

4. Two variable cyclotomic multiple polylogarithms

We introduce cyclotomic multiple polylogarithms, $Li_{\mathbf{a}}(\bar{\zeta}(z))$ and $Li_{\mathbf{a},\mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y))$, and their associated bar elements, $l^{\bar{\zeta}}_{\mathbf{a}}$ and $l^{\bar{\zeta}(x),\bar{\eta}(y)}_{\mathbf{a},\mathbf{b}}$, which play important roles to prove our main theorems.

For a pair $(\mathbf{a}, \bar{\zeta})$ with $\mathbf{a} = (a_1, \cdots, a_k) \in \mathbf{Z}^k_{>0}$ and $\bar{\zeta} = (\zeta_1, \ldots, \zeta_k)$ with $\zeta_i \in \mu_N$: the group of roots of unity in $\mathbf{C}$ $(1 \leqslant i \leqslant k)$, its weight and its depth are defined to be $wt(\mathbf{a}, \bar{\zeta}) = a_1 + \cdots + a_k$ and $dp(\mathbf{a}, \bar{\zeta}) = k$ respectively. Put $\bar{\zeta}(x) = (\zeta_1, \ldots, \zeta_{k-1}, \zeta_k x)$. Put $z \in \mathbf{C}$ with $|z| < 1$. Consider the following complex analytic function, *one variable cyclotomic multiple polylogarithm*

$$Li_{\mathbf{a}}(\bar{\zeta}(z)) := \sum_{0<m_1<\cdots<m_k} \frac{\zeta_1^{m_1} \cdots \zeta_{k-1}^{m_{k-1}} (\zeta_k z)^{m_k}}{m_1^{a_1} \cdots m_{k-1}^{a_{k-1}} m_k^{a_k}}.$$

It satisfies the following differential equation

$$\frac{d}{dz} Li_{\mathbf{a}}(\bar{\zeta}(z)) = \begin{cases} \frac{1}{z} Li_{(a_1, \cdots, a_{k-1}, a_k - 1)}(\bar{\zeta}(z)) & \text{if } a_k \neq 1, \\ \frac{1}{\zeta_k^{-1} - z} Li_{(a_1, \cdots, a_{k-1})}(\zeta_1, \ldots, \zeta_{k-2}, \zeta_{k-1} z) & \text{if } a_k = 1, k \neq 1, \\ \frac{1}{\zeta_1^{-1} - z} & \text{if } a_k = 1, k = 1. \end{cases}$$

It gives an iterated integral starting from o, which lies on $I_o(\mathcal{M}^{(N)}_{0,4})$. Actually by the map ρ it corresponds to an element of the $\mathbf{Q}$-structure $U\mathfrak{F}^*_{N+1}$ of $V(\mathcal{M}^{(N)}_{0,4})$ denoted by $l^{\bar{\zeta}}_{\mathbf{a}}$. It is expressed as

$$l^{\bar{\zeta}}_{\mathbf{a}} = (-1)^k [\underbrace{\omega_0 | \cdots | \omega_0}_{a_k - 1} | \omega_{\zeta_k^{-1}} | \underbrace{\omega_0 | \cdots | \omega_0}_{a_{k-1} - 1} | \omega_{\zeta_k^{-1} \zeta_{k-1}^{-1}} | \omega_0 | \cdots\cdots | \omega_0 | \omega_{\zeta_k^{-1} \cdots \zeta_1^{-1}}].$$

By the standard identification $\mu \simeq \mathbf{Z}/N\mathbf{Z}$ sending $\zeta_N = \exp\{\frac{2\pi\sqrt{-1}}{N}\} \mapsto 1$, for a series $\varphi = \sum_{W:\text{word}} c_W(\varphi) W$ it is calculated by

$$l^{\bar{\zeta}}_{\mathbf{a}}(\varphi) = (-1)^k c_{A^{a_k - 1} B(-e_k) A^{a_{k-1} - 1} B(-e_k - e_{k-1}) \cdots A^{a_1 - 1} B(-e_k - \cdots - e_1)}(\varphi)$$

with $\zeta_i = \zeta_N^{e_i}$ $(e_i \in \mathbf{Z}/N\mathbf{Z})$.

For $\mathbf{a} = (a_1, \cdots, a_k) \in \mathbf{Z}^k_{>0}$, $\mathbf{b} = (b_1, \cdots, b_l) \in \mathbf{Z}^l_{>0}$, $\bar{\zeta} = (\zeta_1, \ldots, \zeta_k)$, $\bar{\eta} = (\eta_1, \ldots, \eta_l)$ with $\zeta_i, \eta_j \in \mu_N$ and $x, y \in \mathbf{C}$ with $|x| < 1$ and $|y| < 1$, consider the following complex function, the *two variables multiple polylogarithm*

$$Li_{\mathbf{a},\mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y)) := \sum_{\substack{0<m_1<\cdots<m_k \\ <n_1<\cdots<n_l}} \frac{\zeta_1^{m_1} \cdots \zeta_{k-1}^{m_{k-1}} (\zeta_k x)^{m_k} \cdot \eta_1^{n_1} \cdots \eta_{l-1}^{n_{l-1}} (\eta_l y)^{n_l}}{m_1^{a_1} \cdots m_{k-1}^{a_{k-1}} m_k^{a_k} \cdot n_1^{b_1} \cdots n_{l-1}^{b_{l-1}} n_l^{b_l}}.$$

It satisfies the following differential equations.

$$\frac{d}{dx}Li_{\mathbf{a},\mathbf{b}}(\bar\zeta(x),\bar\eta(y))$$
$$=\begin{cases}\frac{1}{x}Li_{(a_1,\cdots,a_{k-1},a_k-1),\mathbf{b}}(\bar\zeta(x),\bar\eta(y)) & \text{if } a_k\neq 1,\\ \frac{1}{\zeta_k^{-1}-x}Li_{(a_1,\cdots,a_{k-1}),\mathbf{b}}(\zeta_1,\dots,\zeta_{k-2},\zeta_{k-1}x,\bar\eta(y))-\left(\frac{1}{x}+\frac{1}{\zeta_k^{-1}-x}\right)\cdot & \\ \quad Li_{(a_1,\cdots,a_{k-1},b_1),(b_2,\cdots,b_l)}(\zeta_1,\dots\zeta_{k-1},\zeta_k\eta_1x,\eta_2,\dots,\eta_{l-1},\eta_ly) & \\ & \text{if } a_k=1,k\neq 1,l\neq 1,\\ \frac{1}{\zeta_1^{-1}-x}Li_{\mathbf{b}}(\eta(y))-\left(\frac{1}{x}+\frac{1}{\zeta_1^{-1}-x}\right)Li_{(b_1),(b_2,\cdots,b_l)}(\zeta_1\eta_1x,\eta_2,\dots,\eta_{l-1},\eta_ly) & \\ & \text{if } a_k=1,k=1,l\neq 1,\\ \frac{1}{\zeta_k^{-1}-x}Li_{(a_1,\cdots,a_{k-1}),b_1}(\zeta_1,\dots,\zeta_{k-1}x,\eta_1y)-\left(\frac{1}{x}+\frac{1}{\zeta_k^{-1}-x}\right)\cdot & \\ \quad Li_{(a_1,\cdots,a_{k-1},b_1)}(\zeta_1,\dots,\zeta_{k-1},\zeta_k\eta_1xy) & \text{if } a_k=1,k\neq 1,l=1,\\ \frac{1}{\zeta_1^{-1}-x}Li_{b_1}(\eta_1y)-\left(\frac{1}{x}+\frac{1}{\zeta_1^{-1}-x}\right)Li_{b_1}(\zeta_1\eta_1xy) & \text{if } a_k=1,k=1,l=1,\end{cases}$$

$$\frac{d}{dy}Li_{\mathbf{a},\mathbf{b}}(\bar\zeta(x),\bar\eta(y))$$
$$=\begin{cases}\frac{1}{y}Li_{\mathbf{a},(b_1,\cdots,b_{l-1},b_l-1)}(\bar\zeta(x),\bar\eta(y)) & \text{if } b_l\neq 1,\\ \frac{1}{\eta_l^{-1}-y}Li_{\mathbf{a},(b_1,\cdots,b_{l-1})}(\bar\zeta(x),\eta_1,\dots,\eta_{l-2},\eta_{l-1}y) & \text{if } b_l=1,l\neq 1,\\ \frac{1}{\eta_1^{-1}-y}Li_{\mathbf{a}}(\bar\zeta(\eta_1xy)) & \text{if } b_l=1,l=1.\end{cases}$$

By analytic continuation, the functions $Li_{\mathbf{a},\mathbf{b}}(\bar\zeta(x),\bar\eta(y))$, $Li_{\mathbf{b},\mathbf{a}}(\bar\eta(y),\bar\zeta(x))$, $Li_{\mathbf{a}}(\bar\zeta(x))$, $Li_{\mathbf{a}}(\bar\zeta(y))$ and $Li_{\mathbf{a}}(\bar\zeta(xy))$ give iterated integrals starting from o, which lie on $I_o(\mathcal{M}_{0,5}^{(N)})$. They correspond to elements of the $\mathbf{Q}$-structure $(U\mathfrak{t}^0_{4,N})^*$ of $V(\mathcal{M}_{0,5}^{(N)})$ by the map ρ denoted by $l_{\mathbf{a},\mathbf{b}}^{\bar\zeta(x),\bar\eta(y)}$, $l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}$, $l_{\mathbf{a}}^{\bar\zeta(x)}$, $l_{\mathbf{a}}^{\bar\eta(y)}$ and $l_{\mathbf{a}}^{\bar\zeta(xy)}$ respectively. Note that they are expressed as

$$\sum_{I=(i_m,\cdots,i_1)}c_I[\omega_{i_m}|\cdots|\omega_{i_1}]$$

for some $m\in\mathbf{N}$ with $c_I\in\mathbf{Q}$ and $\omega_{i_j}\in\{\frac{dx}{x},\frac{dx}{\zeta-x},\frac{dy}{y},\frac{dy}{\zeta-y},\frac{xdy+ydx}{\zeta-xy}(\zeta\in\mu_N)\}$.

5. Proof of main theorems

This section gives a proof of theorem 1.

Proof of theorem 1. Let $\mathbf{a}=(a_1,\dots,a_k)\in\mathbf{Z}^k_{>0}$, $\mathbf{b}=(b_1,\dots,b_l)\in\mathbf{Z}^l_{>0}$, $\bar\zeta=(\zeta_1,\dots,\zeta_k)$ and $\bar\eta=(\eta_1,\dots,\eta_l)$ with $\zeta_i,\eta_j\in\mu_N\subset\mathbf{C}$ $(1\leqslant i\leqslant k$ and $1\leqslant j\leqslant l)$. Put $\bar\zeta(x)=(\zeta_1,\dots,\zeta_{k-1},\zeta_kx)$ and

$\bar{\eta}(y) = (\eta_1, \ldots, \eta_{l-1}, \eta_l y)$. Recall that multiple polylogarithms satisfy the following analytic identity, the series shuffle formula in $I_o(\mathcal{M}_{0,5}^{(N)})$:

$$Li_{\mathbf{a}}(\bar{\zeta}(x)) \cdot Li_{\mathbf{b}}(\bar{\eta}(y)) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} Li_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}.$$

Here $Sh^{\leqslant}(k,l) := \cup_{N=1}^{\infty}\{\sigma : \{1, \cdots, k+l\} \to \{1, \cdots, N\} | \sigma \text{ is onto}, \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\}$, $\sigma(\mathbf{a}, \mathbf{b}) := (c_1, \cdots, c_N)$ with

$$c_i = \begin{cases} a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s,t\} \text{ with } s < t, \\ a_s & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s \leqslant k, \\ b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s > k, \end{cases}$$

and $\sigma(\bar{\zeta}(x), \bar{\eta}(y)) := (z_1, \ldots, z_N)$ with

$$z_i = \begin{cases} x_s y_{t-k} & \text{if } \sigma^{-1}(i) = \{s,t\} \text{ with } s < t, \\ x_s & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s \leqslant k, \\ y_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s > k, \end{cases}$$

for $x_i = \zeta_i$ $(i \neq k)$, $\zeta_k x$ $(i = k)$ and $y_j = \eta_j$ $(j \neq l)$, $\eta_j y$ $(j = l)$. Since ρ is an embedding of algebras, the above analytic identity immediately implies the algebraic identity, the series shuffle formula in the $\mathbf{Q}$-structure $(U\mathfrak{t}_{4,N}^0)^*$ of $V(\mathcal{M}_{0,5}^{(N)})$

$$l_{\mathbf{a}}^{\bar{\zeta}(x)} \cdot l_{\mathbf{b}}^{\bar{\eta}(y)} = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}. \tag{8}$$

Let (g, h) be a pair in theorem 1. By the group-likeness of h, i.e. $h \in \exp \mathfrak{F}_{N+1}$, the product $h^{1,23,4}h^{1,2,3}$ is group-like, i.e. belongs to $\exp \mathfrak{t}_{4,N}^0$. Hence $\Delta(h^{1,23,4}h^{1,2,3}) = (h^{1,23,4}h^{1,2,3})\widehat{\otimes}(h^{1,23,4}h^{1,2,3})$, where Δ is the standard coproduct of $U\mathfrak{t}_{4,N}^0$. Therefore

$$\begin{aligned} l_{\mathbf{a}}^{\bar{\zeta}(x)} \cdot l_{\mathbf{b}}^{\bar{\eta}(y)}(h^{1,23,4}h^{1,2,3}) &= (l_{\mathbf{a}}^{\bar{\zeta}(x)} \widehat{\otimes} l_{\mathbf{b}}^{\bar{\eta}(y)})(\Delta(h^{1,23,4}h^{1,2,3})) \\ &= l_{\mathbf{a}}^{\bar{\zeta}(x)}(h^{1,23,4}h^{1,2,3}) \cdot l_{\mathbf{b}}^{\bar{\eta}(y)}(h^{1,23,4}h^{1,2,3}). \end{aligned}$$

Evaluation of the equation (8) at the group-like element $h^{1,23,4}h^{1,2,3}$ gives the series shuffle formula

$$l_{\mathbf{a}}^{\bar{\zeta}}(h) \cdot l_{\mathbf{b}}^{\bar{\eta}}(h) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(\bar{\zeta},\bar{\eta})}(h) \tag{9}$$

for admissible pairs [1] $(\mathbf{a}, \bar{\zeta})$ and $(\mathbf{b}, \bar{\eta})$ by the results in [F4] because the group-likeness and (4) for h implies $c_0(h) = 1$ and $c_A(h) = 0$.

[1] A pair $(\mathbf{a}, \bar{\zeta})$ with $\mathbf{a} = (a_1, \cdots, a_k)$ and $\bar{\zeta} = (\zeta_1, \ldots, \zeta_k)$ is called *admissible* if $(a_k, \zeta_k) \neq (1, 1)$.

By putting $l_1^{1,S}(h) := -T$ and $l_{\mathbf{a}}^{\bar{\zeta},S}(h) := l_{\mathbf{a}}^{\bar{\zeta}}(h)$ for all admissible pairs $(\mathbf{a}, \bar{\zeta})$, the series regularized value $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$ in $\mathbf{Q}[T]$ (T: a parameter which stands for $\log z$, cf. [R]) for a non-admissible pair $(\mathbf{a}, \bar{\zeta})$ is uniquely determined in such a way (cf.[AK]) that the above series shuffle formulae remain valid for $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$ with all pairs $(\mathbf{a}, \bar{\zeta})$.

Define the integral regularized value $l_{\mathbf{a}}^{\bar{\zeta},I}(h)$ in $\mathbf{Q}[T]$ for all pairs $(\mathbf{a}, \bar{\zeta})$ by $l_{\mathbf{a}}^{\bar{\zeta},I}(h) = l_{\mathbf{a}}^{\bar{\zeta}}(e^{TB(0)}h)$. Equivalently $l_{\mathbf{a}}^{\bar{\zeta},I}(h)$ for any pair $(\mathbf{a}, \bar{\zeta})$ can be uniquely defined in such a way that the iterated integral shuffle formulae (loc.cit) remain valid for all pairs $(\mathbf{a}, \bar{\zeta})$ with $l_1^{1,I}(h) := -T$ and $l_{\mathbf{a}}^{\bar{\zeta},I}(h) := l_{\mathbf{a}}^{\bar{\zeta}}(h)$ for all admissible pairs $(\mathbf{a}, \bar{\zeta})$ because they hold for admissible pairs by the group-likeness of h (cf. loc.cit).

Let $\mathbb{L}$ be the $\mathbf{Q}$-linear map from $\mathbf{Q}[T]$ to itself defined via the generating function:

$$\mathbb{L}(\exp Tu) = \sum_{n=0}^{\infty} \mathbb{L}(T^n)\frac{u^n}{n!} = \exp\left\{ -\sum_{n=1}^{\infty} l_n^{1,I}(h)\frac{u^n}{n} \right\}.$$

Proposition 8. *Let h be an element as in theorem 1. Then the regularization relation holds, i.e.* $l_{\mathbf{a}}^{\bar{\zeta},S}(h) = \mathbb{L}\big(l_{\mathbf{a}}^{\bar{\zeta},I}(h)\big)$ *for all pairs* $(\mathbf{a}, \bar{\zeta})$*.*

Proof . We may assume that $(\mathbf{a}, \bar{\zeta})$ is non-admissible because the proposition is trivial if it is admissible. Put $1^n = (\underbrace{1, 1, \cdots, 1}_{n})$. When $\mathbf{a} = 1^n$ and $\bar{\zeta} = \bar{1}^n$, the proof is given by the same argument to [F3] as follows: By the series shuffle formulae,

$$\sum_{k=0}^{m} (-1)^k l_{k+1}^{\bar{1},S}(h) \cdot l_{1^{m-k}}^{\bar{1}^{m-k},S}(h) = (m+1) l_{1^{m+1}}^{\bar{1}^{m+1},S}(h)$$

for $m \geqslant 0$. Here we put $l_{\emptyset}^{\emptyset,S}(h) = 1$. This means

$$\sum_{k,l\geqslant 0} (-1)^k l_{k+1}^{\bar{1},S}(h) \cdot l_{1^l}^{\bar{1}^l,S}(h) u^{k+l} = \sum_{m\geqslant 0} (m+1) l_{1^{m+1}}^{\bar{1}^{m+1},S}(h) u^m.$$

Put $f(u) = \sum_{n\geqslant 0} l_{1^n}^{\bar{1}^n,S}(h)u^n$. Then the above equality can be read as

$$\sum_{k\geqslant 0} (-1)^k l_{k+1}^{\bar{1},S}(h) u^k = \frac{d}{du}\log f(u).$$

Integrating and adjusting constant terms gives

$$\sum_{n\geqslant 0} l_{1^n}^{\bar{1}^n,S}(h)u^n = \exp\left\{ -\sum_{n\geqslant 1} (-1)^n l_n^{\bar{1},S}(h)\frac{u^n}{n} \right\} = \exp\left\{ -\sum_{n\geqslant 1} (-1)^n l_n^{\bar{1},I}(h)\frac{u^n}{n} \right\}$$

because $l_n^{\bar{1},S}(h)=l_n^{\bar{1},I}(h)=l_n^1(h)$ for $n>1$ and $l_1^{\bar{1},S}(h)=l_1^{\bar{1},I}(h)=-T$. Since $l_{1^m}^{\bar{1}^m,I}(h)=\frac{(-T)^m}{m!}$, we get $l_{1^m}^{\bar{1}^m,S}(h)=\mathbb{L}\big(l_{1^m}^{\bar{1}^m,I}(h)\big)$.

When $(\mathbf{a},\bar{\zeta})$ is of the form $(\mathbf{a}'1^l,\bar{\zeta}'\bar{1}^l)$ with $(\mathbf{a}',\bar{\zeta}')$ admissible, the proof is given by the following induction on l. By (8),

$$l_{\mathbf{a}'}^{\bar{\zeta}'(x)}(h')\cdot l_{1^l}^{\bar{1}^l(y)}(h')=\sum_{\sigma\in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a}',1^l)}^{\sigma(\bar{\zeta}'(x),\bar{1}^l(y))}(h')$$

for $h'=e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}$ with $k=dp(\mathbf{a}')$. The group-likeness and (4) for h implies $c_0(h)=1$ and $c_A(h)=0$ and the group-likeness and our assumption $c_{B(0)}(h)=0$ implies $c_{B(0)^n}(h)=0$ for $n\in\mathbf{Z}_{>0}$. Hence by the results in [F4]

$$l_{\mathbf{a}'}^{\bar{\zeta}'}(h)\cdot l_{1^l}^{\bar{1}^l,I}(h)=\sum_{\sigma\in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a}',1^l)}^{\sigma(\bar{\zeta}',\bar{1}^l),I}(h).$$

Then by our induction assumption, taking the image by the map $\mathbb{L}$ gives

$$l_{\mathbf{a}'}^{\bar{\zeta}'}(h)\cdot l_{1^l}^{\bar{1}^l,S}(h)=\mathbb{L}\big(l_{\mathbf{a}'1^l}^{\bar{\zeta}'\bar{1}^l,I}(h)\big)+\sum_{\sigma\neq id\in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a}',1^l)}^{\sigma(\bar{\zeta}',\bar{1}^l),S}(h).$$

Since $l_{\mathbf{a}'}^{\bar{\zeta}',S}(h)$ and $l_{1^l}^{\bar{1}^l,S}(h)$ satisfy the series shuffle formula, $\mathbb{L}\big(l_{\mathbf{a}}^{\bar{\zeta},I}(h)\big)$ must be equal to $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$, which concludes proposition 8. □

Embed $U\mathfrak{F}_{Y_N}$ into $U\mathfrak{F}_{N+1}$ by sending $Y_{m,a}$ to $-A^{m-1}B(-a)$. Then by the above proposition,

$$\begin{aligned}
l_{\mathbf{a}}^{\bar{\zeta},S}(h)&=\mathbb{L}(l_{\mathbf{a}}^{\bar{\zeta},I}(h))=\mathbb{L}(l_{\mathbf{a}}^{\bar{\zeta}}(e^{TB(0)}h))=l_{\mathbf{a}}^{\bar{\zeta}}\left(\mathbb{L}(e^{TB(0)}\pi_Y(h))\right)\\
&=l_{\mathbf{a}}^{\bar{\zeta}}(\exp\left\{-\sum_{n=1}^{\infty}l_n^{1,I}(h)\frac{B(0)^n}{n}\right\}\cdot\pi_Y(h))\\
&=l_{\mathbf{a}}^{\bar{\zeta}}(\exp\left\{-TY_{1,0}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}c_{A^{n-1}B(0)}(h)Y_{1,0}^n\right\}\cdot\pi_Y(h))=l_{\mathbf{a}}^{\bar{\zeta}}(e^{-TY_{1,0}}h_*)
\end{aligned}$$

for all $(\mathbf{a},\bar{\zeta})$ because $l_1^1(h)=0$. As for the third equality we use $(\mathbb{L}\otimes_{\mathbf{Q}} id)\circ(id\otimes_{\mathbf{Q}} l_{\mathbf{a}}^{\bar{\zeta}})=(id\otimes_{\mathbf{Q}} l_{\mathbf{a}}^{\bar{\zeta}})\circ(\mathbb{L}\otimes_{\mathbf{Q}} id)$ on $\mathbf{Q}[T]\otimes_{\mathbf{Q}} U\mathfrak{F}_{N+1}$. All $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$'s satisfy the series shuffle formulae (9), so the $l_{\mathbf{a}}^{\bar{\zeta}}(e^{-TY_{1,0}}h_*)$'s do also. By putting $T=0$, we get that $l_{\mathbf{a}}^{\bar{\zeta}}(h_*)$'s also satisfy the series shuffle formulae for all $\mathbf{a}$. Therefore $\Delta_*(h_*)=h_*\widehat{\otimes}h_*$. This completes the proof of theorem 1. □

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