MIXED PENTAGON EQUATION AND DOUBLE SHUFFLE RELATION

HIDEKAZU FURUSHO

Abstract. This paper is a review of the paper [F4] where a geometric interpretation of the generalized (including the regularization relation) double shuffle relation for multiple $L$-values is given. In precise, it is shown that Enriquez’ mixed pentagon equation implies the relations.

0. Introduction

Multiple $L$-values $L(k_1, \ldots, k_m; \zeta_1, \ldots, \zeta_m)$ are the complex numbers defined by the following series

$$L(k_1, \ldots, k_m; \zeta_1, \ldots, \zeta_m) := \sum_{0<n_1<\cdots<n_m} \frac{\zeta_1^{n_1} \cdots \zeta_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}$$

for $m$, $k_1, \ldots, k_m \in \mathbb{N}(= \mathbb{Z}_{>0})$ and $\zeta_1, \ldots, \zeta_m \in \mu_N$ (the group of $N$-th roots of unity in $\mathbb{C}$). They converge if and only if $(k_m, \zeta_m) \neq (1, 1)$. Multiple zeta values are regarded as a special case for $N = 1$. These values have been discussed in several papers [AK, BK, G, R] etc. Multiple $L$-values appear as coefficients of the cyclotomic Drinfel’d associator $\Phi_N^{KZ}$ (5) in $U_{N+1}$: the non-commutative formal power series ring with $N+1$ variables $A$ and $B(a)$ ($a \in \mathbb{Z}/N\mathbb{Z}$).

The mixed pentagon equation (4) is a geometric equation introduced by Enriquez [E]. The series $\Phi_N^{KZ}$ satisfies the equation, which yields non-trivial relations among multiple $L$-values. The generalised double shuffle relation (the double shuffle relation and the regularization relation) is a combinatorial relation among multiple $L$-values. It is formulated as (6) for $h = \Phi_N^{KZ}$. It is Zhao’s remark [Z] that for specific $N$’s the generalized double shuffle relation does not provide all the possible relations among multiple $L$-values.

Our main theorem is an implication of the generalised double shuffle relation (6) from the mixed pentagon equation (4).

Date: September 16, 2012.

This article is for the RIMS-kokyuroku of the conference ‘Various Aspects of Multiple Zeta Values’ held during 6th-9th September, 2010 in RIMS, Kyoto.
Theorem 1. Let $U\mathfrak{g}_{N+1}$ be the universal enveloping algebra of the free Lie algebra $\mathfrak{g}_{N+1}$ with variables $A$ and $B(a)$ ($a \in \mathbb{Z}/N\mathbb{Z}$). Let $h$ be a group-like element in $U\mathfrak{g}_{N+1}$ with $c_{B(0)}(h) = 0$ satisfying the mixed pentagon equation (4) with a group-like series $g \in U\mathfrak{g}_2$. Then $h$ also satisfies the generalised double shuffle relation (6).

The contents of the article are as follows: We recall the mixed pentagon equation in §1 and the generalised double shuffle relation in §2. In §3 we calculate the 0-th cohomologies of Chen’s reduced bar complex for the Kummer coverings of the moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. Two variable cyclotomic multiple polylogarithms and their associated bar elements there are introduced in §4. By using them, we prove theorem 1 in §5.

1. Mixed pentagon equation

This section is to recall Enriquez’ mixed pentagon equation [E].

Let us fix notations: For $n \geq 2$, the Lie algebra $t_n$ of infinitesimal pure braids is the completed $\mathbb{Q}$-Lie algebra with generators $t^{ij}$ ($i \neq j$, $1 \leq i, j \leq n$) and relations $t^{ij} = t^{ji}$, $[t^{ij}, t^{ik} + t^{jk}] = 0$ and $[t^{ij}, t^{kl}] = 0$ for all distinct $i, j, k, l$. We note that $t_2$ is the 1-dimensional abelian Lie algebra generated by $t^{12}$. The element $z_n = \sum_{1 \leq i < j \leq n} t^{ij}$ is central in $t_n$. Put $t_n^0$ to be the Lie subalgebra of $t_n$ with the same generators except $t^{1n}$ and the same relations as $t_n$. Then we have $t_n = t_n^0 \oplus \mathbb{Q} \cdot z_n$. Especially when $n = 3$, $t_3^0$ is a free Lie algebra $\mathfrak{g}_2$ of rank 2 with generators $A := t^{12}$ and $B = t^{23}$. For a partially defined map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$, the Lie algebra morphism $t_n \to t_m : x \mapsto x^f = x^{f^{-1}(1)} \cdots x^{f^{-1}(n)}$ is uniquely defined by $(t^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t^{i'j'}$.

For a pair $(\mu, g) \in \mathbb{Q} \times \exp\mathfrak{g}_2$ the pentagon equation is the following equation in $\exp t_3^0$

\begin{equation}
(2) \quad g^{1,2,3,4}g^{12,3,4} = g^{2,3,4}g^{12,3,4}g^{1,2,3}.
\end{equation}

and two hexagon equations the following two equations in $\exp\mathfrak{g}_2 = \exp t_3^0$

\begin{equation}
(3) \quad g(A, B)g(B, A) = 1 \quad \text{and} \quad \exp\frac{\mu A}{2}g(C, A)\exp\frac{\mu C}{2}g(B, C)\exp\frac{\mu B}{2}g(A, B) = 1
\end{equation}

with $C = -A - B$. These

By our notation, the equation (2) can be read as

\[ g(t^{12}, t^{23} + t^{24})g(t^{13} + t^{23}, t^{24}) = g(t^{23}, t^{34})g(t^{12} + t^{13}, t^{24} + t^{34})g(t^{12}, t^{23}). \]

Remark 2. It is shown in [F2] that the two hexagon equations (3) are consequences of the pentagon equation (2).
Remark 3. The Drinfel’d associator $\Phi_{KZ} = \Phi_{KZ}(A, B) \in C((A, B))$ is defined to be the quotient $\Phi_{KZ} = G_1(z)^{-1}G_0(z)$ where $G_0$ and $G_1$ are the solutions of the formal KZ equation

$$\frac{d}{dz}G(z) = \left(\frac{A}{z} + \frac{B}{z - 1}\right)G(z)$$

such that $G_0(z) \approx z^A$ when $z \to 0$ and $G_1(z) \approx (1 - z)^B$ when $z \to 1$ (cf. [Dr]). The series has the following expression

$$\Phi_{KZ} = 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m)A^{k_m - 1}B \cdots A^{k_1 - 1}B + \text{(regularized terms)}$$

and the regularised terms are explicitly calculated to be linear combinations of multiple zeta values $\zeta(k_1, \ldots, k_m) = L(k_1, \ldots, k_m; 1, \ldots, 1)$ in [F1] proposition 3.2.3 by Le-Murakami’s method [LM]. It is shown in [Dr] that the pair $(2\pi\sqrt{-1}, \Phi_{KZ})$ satisfies the pentagon equation (2) and the hexagon equations (3).

For $n \geq 2$ and $N \geq 1$, the Lie algebra $t_{n,N}$ is the completed $\mathbb{Q}$-Lie algebra with generators $t^{ij}$ $(2 \leq i \leq n)$, $t(a)_{ij}$ $(i \neq j, 2 \leq i, j \leq n)$, $a \in \mathbb{Z}/N\mathbb{Z}$ and relations $t(a)_{ij} = t(-a)_{ji}$, $[t(a)_{ij}, t(a)_{kl}] = 0$, $[t^{ij} + t^{jk} + \sum c \in \mathbb{Z}/N\mathbb{Z} t(c)_{ij}, t(a)_{ij}] = 0$, $[t^{ij}, t(a)_{jk}] \neq 0$ and $[t(a)_{ij}, t(b)_{kl}] = 0$ for all $a, b \in \mathbb{Z}/N\mathbb{Z}$ and all distinct $i, j, k, l$ $(2 \leq i, j, k, l \leq n)$. We note that $t_{n,1}$ is equal to $t_n$ for $n \geq 2$. We have a natural injection $t_{n-1,N} \hookrightarrow t_{n,N}$. The Lie subalgebra $f_{n,N}$ of $t_{n,N}$ generated by $t_1^n$ and $t(a)^m$ $(2 \leq i \leq n - 1, a \in \mathbb{Z}/N\mathbb{Z})$ is free of rank $(n - 2)N + 1$ and forms an ideal of $t_{n,N}$. Actually it shows that $t_{n,N}$ is a semi-direct product of $f_{n,N}$ and $t_{n-1,N}$. The element $z_{n,N} = \sum_{1 \leq i < j \leq n} t^{ij}$ with $t^{ij} = \sum a \in \mathbb{Z}/N\mathbb{Z} t(a)_{ij}$ $(2 \leq i < j \leq n)$ is central in $t_{n,N}$. Put $t^0_{n,N}$ to be the Lie subalgebra of $t_{n,N}$ with the same generators except $t^{1n}$. Then we have $t_{n,N} = t^0_{n,N} \oplus \mathbb{Q} \cdot z_{n,N}$. Occasionally we regard $t^0_{n,N}$ as the quotient $t_{n,N}/\mathbb{Q} \cdot z_{n,N}$. Especially when $n = 3$, $t^0_{3,N}$ is free Lie algebra $\mathfrak{g}_{N+1}$ of rank $N + 1$ with generators $A := t^{12}$ and $B(a) = t(a)^{23}$ $(a \in \mathbb{Z}/N\mathbb{Z})$.

For a partially defined map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $f(1) = 1$, the Lie algebra morphism $t_{m,N} \to t_{m,N} : x \mapsto x^f = x^{f_l^{-1}(1) \cdots f_l^{-1}(n)}$ is uniquely defined by $(t(a)_{ij})^f = \sum_{f_l^{-1}(i), f_l^{-1}(j)} t(a)_{ij}^f$ $(i \neq j, 2 \leq i, j \leq n)$ and $(t^{ij})^f = \sum_{f_l^{-1}(i), f_l^{-1}(j)} t^{ij} + \frac{1}{2} \sum_{f_l^{-1}(i), f_l^{-1}(j)} \sum_{c \in \mathbb{Z}/N\mathbb{Z}} t(c)^{ij}$.

Again for a partially defined map $g : \{2, \ldots, m\} \to \{1, \ldots, n\}$, the Lie algebra morphism $t_n \to t_{m,N} : x \mapsto x^g = x^{g_l^{-1}(1) \cdots g_l^{-1}(n)}$ is uniquely defined by $(t^{ij})^g = \sum_{f_l^{-1}(i), f_l^{-1}(j)} t(0)^{ij}$ $(i \neq j, 1 \leq i, j \leq n)$.
For a pair \((g, h) \in \exp \mathcal{F}_2 \times \exp \mathcal{F}_{N+1}\), the \textit{mixed pentagon equation} means the following equation in \(\exp t_{1, N}^3\)

\[ h^{1,2,3,4}h^{12,3,4} = g^{2,3,4}h^{1,23,4}h^{1,2,3}. \]

By our notation, each term in the equation (4) can be read as

\[ h^{1,2,3,4} = h(t^{12}, t^{23}(0) + t^{24}(0), t^{23}(1) + t^{24}(1), \ldots, t^{23}(N - 1) + t^{24}(N - 1)), \]
\[ h^{12,3,4} = h(t^{13} + \sum c t^{23}(c), t^{34}(0), t^{34}(1), \ldots, t^{34}(N - 1)), \]
\[ g^{2,3,4} = g(t^{23}(0), t^{34}(0)), \]
\[ h^{1,23,4} = h(t^{12} + t^{13} + \sum c t^{23}(c), t^{24}(0) + t^{34}(0), \ldots, t^{24}(N - 1) + t^{34}(N - 1)), \]
\[ h^{1,2,3} = h(t^{12}, t^{23}(0), t^{23}(1), \ldots, t^{23}(N - 1)). \]

**Remark 4.** In [E], the cyclotomic analogue \(\Phi_{KZ}^N \in \exp \mathcal{F}_{N+1}(C)\) of the Drinfel’d associator is introduced to be the renormalised holonomy from 0 to 1 of the KZ-like differential equation

\[
\frac{d}{dz} H(z) = \left(\frac{A}{z} + \sum_{a \in \mathbb{Z}_N^*} \frac{B(a)}{z - \zeta_N^a}\right) H(z)
\]

with \(\zeta_N = \exp\left\{\frac{2\pi \sqrt{-1}}{N}\right\}\), i.e., \(\Phi_{KZ}^N = H_1^{-1}H_0\) where \(H_0\) and \(H_1\) are the solutions such that \(H_0(z) \approx z^4\) when \(z \to 0\) and \(H_1(z) \approx (1 - z)^{B(0)}\) when \(z \to 1\) (cf. [E]). There appear multiple \(L\)-values (1) in each of its coefficient;

\[
\Phi_{KZ}^N = 1 + \sum (-1)^m L(k_1, \ldots, k_m; \xi_1, \ldots, \xi_m) A^{k_m-1} B(a_m) \cdots A^{k_1-1} B(a_1) + \text{(regularized terms)}
\]

with \(\xi_1 = \zeta_N^{a_2-a_1}, \ldots, \xi_{m-1} = \zeta_N^{a_m-a_{m-1}}\) and \(\xi_m = \zeta_N^{a_m}\), where the regularised terms can be explicitly calculated to combinations of multiple \(L\)-values by the method of Le-Murakami [LM]. In [E] it is shown that the triple \((2\pi \sqrt{-1}, \Phi_{KZ}, \Phi_{KZ}^N)\) satisfies the mixed pentagon equation (4). This is achieved by considering monodromy in the pentagon formed by the divisors \(y = 0, x = 1\), the exceptional divisor of the blowing-up at \((1, 1)\), \(y = 1\) and \(x = 0\) in \(\mathcal{M}_{0,5}(N)\) (see §3).

**Remark 5.** In [EF] it is proved that the mixed pentagon equation (4) implies the distribution relation for a specific case and that the octagon equation follows from the mixed pentagon equation and the special action condition for \(N = 2\).
2. Double shuffle relation

This section is to recall the generalised double shuffle relation in Racinet’s setting [R].

Let us fix notations: Let $\mathfrak{F}_{N}$ be the completed graded Lie $Q$-algebra generated by $Y_{n,a} (n \geq 1$ and $a \in \mathbb{Z}/N\mathbb{Z})$ with $\deg Y_{n,a} = n$. Put $U\mathfrak{F}_{N}$ its universal enveloping algebra: the non-commutative formal series ring with free variables $Y_{n,a} (n \geq 1$ and $a \in \mathbb{Z}/N\mathbb{Z})$. Let $\pi_{Y} : U\mathfrak{F}_{N+1} \to U\mathfrak{F}_{N}$ be the $Q$-linear map between non-commutative formal power series rings that sends all the words ending in $A$ to zero and the word $A^{m-1}B(a_{m}) \cdots A^{n-1}B(a_{1}) (n_{1}, \ldots, n_{m} \geq 1$ and $a_{1}, \ldots, a_{m} \in \mathbb{Z}/N\mathbb{Z})$ to

$$(-1)^{m}Y_{n_{m},-a_{m}}Y_{n_{m-1},a_{m}-a_{m-1}} \cdots Y_{a_{2}-a_{1}}.$$ Define the coproduct $\Delta_{a}$ of $U\mathfrak{F}_{N}$ by $\Delta_{a}Y_{n,a} = \sum_{k+l=n,b+c=a} Y_{k,b} \otimes Y_{l,c}$ ($n \geq 0$ and $a \in \mathbb{Z}/N\mathbb{Z}$) with $Y_{0,a} := 1$ if $a = 0$ and $0$ if $a \neq 0$. For $h = \sum_{\text{word } c_{W}(h)W} \in U\mathfrak{F}_{N+1}$, define the series shuffle regularization $h_{s} = h_{\text{corr}} \cdot \pi_{Y}(h)$ with the correction term

$$h_{\text{corr}} = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} c_{A^{n-1}B(0)}(h)Y_{1,0}^{n} \right).$$

For a series $h \in \exp \mathfrak{F}_{N+1}$ the generalised double shuffle relation stands for the following relation in $U\mathfrak{F}_{N}$

$$\Delta_{a}(h_{s}) = h_{s} \otimes h_{s}. \tag{6}$$

**Remark 6.** The series $\Phi_{KZ}^{N}$ (5) satisfies the generalised double shuffle relation (6) because regularised multiple $L$-values satisfy the double shuffle relation.

3. Bar constructions

This section gives a review of the notion of the reduced bar construction and calculates it for $\mathcal{M}_{1,4}^{(N)}$ and $\mathcal{M}_{0,5}^{(N)}$.

We recall the notion of Chen’s reduced bar construction [C]. Let $(A^{*} = \oplus_{q=0}^{\infty} A^{q})$ be a differential graded algebra (DGA). The reduced bar complex $\tilde{B}^{*}(A)$ is the tensor algebra $\oplus_{r=0}^{\infty} (\tilde{A}^{*})^{\otimes r}$ with $\tilde{A}^{*} = \oplus_{i=0}^{\infty} \tilde{A}^{i}$ where $\tilde{A}^{0} = A^{1}/dA^{0}$ and $\tilde{A}^{i} = A^{i+1} (i > 0)$. We denote $a_{1} \otimes \cdots \otimes a_{r} (a_{i} \in A^{*})$ by $[a_{1}] \cdots [a_{r}]$. The degree of elements in $\tilde{B}^{*}(A)$ is given by the total degree of $\tilde{A}^{*}$. Put $Ja = (-1)^{p-1}a$ for $a \in A^{p}$. Define

$$d'[a_{1}] \cdots [a_{k}] = \sum_{i=1}^{k} (-1)^{i} [Ja_{1}] \cdots [Ja_{i-1}] [da_{i}a_{i+1}] \cdots [a_{k}].$$
and
\[ d''[a_1 \cdots a_k] = \sum_{i=1}^{k} (-1)^{i-1} [Ja_1 \cdots Ja_{i-1}]Ja_i \cdot a_{i+1}a_{i+2} \cdots a_k. \]

Then \( d' + d'' \) forms a differential. The differential and the shuffle product (loc.cit.) give \( \bar{B}^*(A) \) a structure of commutative DGA. Actually it also forms a Hopf algebra, whose coproduct \( \Delta \) is given by
\[
\Delta([a_1 \cdots a_r]) = \sum_{s=0}^{r} [a_1 \cdots [a_s | \cdots | a_r] \otimes [a_{s+1} \cdots | a_r].
\]

For a smooth complex manifold \( M \), \( \Omega^*(M) \) means the de Rham complex of smooth differential forms on \( M \) with values in \( \mathbb{C} \). We denote the 0-th cohomology of the reduced bar complex \( \bar{B}^*(\Omega(M)) \) with respect to the differential by \( H^0 \bar{B}(M) \).

Let \( M_{0,4} \) be the moduli space \( \{ (x_1, \cdots, x_4) \in (\mathbb{P}^1)^4 | x_i \neq x_j (i \neq j) \}/\text{PGL}_2(\mathbb{C}) \) of 4 different points in \( \mathbb{P}^1 \). It is identified with \( \{ z \in \mathbb{P}^1 | z \neq 0, 1, \infty \} \) by sending \( [(0, z, 1, \infty)] \) to \( z \). Denote its Kummer \( N \)-covering \( G_m \backslash \mu_N = \{ z \in \mathbb{P}^1 | z^N \neq 0, 1, \infty \} \) by \( M_{0,4}^{(N)} \). The space \( H^0 \bar{B}(M_{0,4}^{(N)}) \) is generated by
\[
\omega_0 := d \log(z) \text{ and } \omega_\zeta := d \log(z - \zeta) \quad (\zeta \in \mu_N).
\]

We have an identification \( H^0 \bar{B}(M_{0,4}^{(N)}) \) with the graded \( \mathbb{C} \)-linear dual of \( U\mathfrak{F}_{N+1}^* \),
\[
H^0 \bar{B}(M_{0,4}^{(N)}) \simeq U\mathfrak{F}_{N+1}^* \otimes \mathbb{C},
\]
by \( \text{Exp} \Omega_i^{(N)} := \sum X_{i_0} \cdots X_{i_m} \otimes [\omega_{i_0} \cdots | \omega_{i_m}] \in U\mathfrak{F}_{N+1} \otimes \mathbb{Q}H^0 \bar{B}(M_{0,4}^{(N)}) \).

Here the sum is taken over \( m \geq 0 \) and \( i_1, \cdots, i_m \in \{ 0 \} \cup \mu_N \) and \( X_0 = A \) and \( X_\zeta = B(a) \) when \( \zeta = \zeta_N^i \). It is easy to see that the identification is compatible with Hopf algebra structures. We note that the product \( l_1 \cdot l_2 \in H^0 \bar{B}(M_{0,4}^{(N)}) \) for \( l_1, l_2 \in H^0 \bar{B}(M_{0,4}^{(N)}) \) is given by
\[
l_1 \cdot l_2(f) := \sum_i l_1(f^{(i)})l_2(f^{(i)}) \text{ for } f \in U\mathfrak{F}_{N+1} \otimes \mathbb{C} \text{ with } \Delta(f) = \sum_i f^{(i)} \otimes f^{(i)}. \]

Occasionally we regard \( H^0 \bar{B}(M_{0,4}^{(N)}) \) as the regular function ring of \( F_{N+1}(\mathbb{C}) = \{ g \in U\mathfrak{F}_{N+1} \otimes \mathbb{C} | g : \text{group-like} \} = \{ g \in U\mathfrak{F}_{N+1} \otimes \mathbb{C} | g(0) = 1, \Delta(g) = g \otimes g \} \).

Let \( M_{0,5} \) be the moduli space \( \{ (x_1, \cdots, x_5) \in (\mathbb{P}^1)^5 | x_i \neq x_j (i \neq j) \}/\text{PGL}_2(\mathbb{C}) \) of 5 different points in \( \mathbb{P}^1 \). It is identified with \( \{ (x, y) \in G_m^2 | x \neq 1, y \neq 1, xy \neq 1 \} \) by sending \( [(0, xy, y, 1, \infty)] \) to \( (x, y) \). Denote its Kummer \( N^2 \)-covering
\[
\{(x, y) \in G_m^2 | x^N \neq 1, y^N \neq 1, (xy)^N \neq 1 \}.
\]
by $\mathcal{M}^{(N)}_{0,5}$. It is identified with $W_N/C^\infty$ by $(x, y) \mapsto (xy, y, 1)$ where

$$W_N = \{(z_2, z_3, z_4) \in G_m | z_i^N \neq z_j^N (i \neq j)\}.$$  

The space $H^0\tilde{B}(\mathcal{M}^{(N)}_{0,5})$ is a subspace of the tensor coalgebra generated by

$$\omega_{1,i} := d\log z_i \text{ and } \omega_{i,j}(a) := d\log(z_i - \zeta_N^a z_j) \quad (2 \leq i, j \leq 4, a \in \mathbb{Z}/N).$$

**Proposition 7.** We have an identification

$$H^0\tilde{B}(\mathcal{M}^{(N)}_{0,5}) \simeq (Ut^0_{4,N})^* \otimes \mathbb{C}.$$  

**Proof.** By [K], $H^0\tilde{B}(W_N)$ can be calculated to be the 0-th cohomology $H^0\tilde{B}^*(S)$ of the reduced bar complex of the Orlik-Solomon algebra $S^*$. The algebra $S^*$ is the (trivial-)differential graded $\mathbb{C}$-algebra $S^* = \bigoplus_{n=0}^{\infty} S^n$ defined by generators

$$\omega_{1,i} = d\log z_i \text{ and } \omega_{i,j}(a) = d\log(z_i - \zeta_N^a z_j) \quad (2 \leq i, j \leq 4, a \in \mathbb{Z}/N\mathbb{Z})$$

in degree 1 and relations

$$\omega_{i,j}(a) = \omega_{j,i}(-a), \quad \omega_{ij}(a) \wedge \{\omega_{ik}(a + b) + \omega_{jk}(b)\} = 0,$$

$$\{\omega_{1i} + \omega_{1j} + \sum_{c \in \mathbb{Z}/N\mathbb{Z}} \omega(c)_{ij}\} \wedge \omega(a)_{ij} = 0,$$

$$\omega_{1i} \wedge \{\omega_{1j} + \sum_{c \in \mathbb{Z}/N\mathbb{Z}} \omega(c)_{ij}\} = 0,$$

$$\omega_{1i} \wedge \omega(a)_{jk} = 0 \quad \text{and} \quad \omega(a)_{ij} \wedge \omega(b)_{kl} = 0$$

for all $a, b \in \mathbb{Z}/N\mathbb{Z}$ and all distinct $i, j, k, l \ (2 \leq i, j, k, l \leq n)$. By direct calculation, the element

$$\sum_{i=2}^{4} t_{1i} \otimes \omega_{1i} + \sum_{2 \leq i < j \leq 4, a \in \mathbb{Z}/N\mathbb{Z}} t_{ij}(a) \otimes \omega_{ij}(a) \in (t_{4,N})_{\deg=1} \otimes S^1$$

yields a Hopf algebra identification of $H^0\tilde{B}(W_N)$ with $(Ut^0_{4,N})^* \otimes \mathbb{C}$ since both are quadratic.

By the long exact sequence of cohomologies induced from the $G_m$-bundle $W_N \to \mathcal{M}^{(N)}_{0,5} = W_N/C^\infty$, we get

$$0 \to H^1(\mathcal{M}^{(N)}_{0,5}) \to H^1(W_N) \to H^1(G_m) \to 0$$

and

$$H^i(\mathcal{M}^{(N)}_{0,5}) \simeq H^i(W_N) \quad (i \geq 2).$$

It yields the identification of the subspace $H^0\tilde{B}(\mathcal{M}^{(N)}_{0,5})$ of $H^0\tilde{B}(W_N)$ with $(Ut^0_{4,N})^* \otimes \mathbb{C}$. \qed
The above identification is induced from
\[ \text{Exp } \Omega_5^{(N)} := \sum_{i=2}^4 t_{1i} d \log z_i + \sum_{2 \leq i < j \leq 4} \sum_{a \in \mathbb{Z}/\mathbb{N}} t_{ij}(a) d \log(z_i - \zeta_N^a z_j) \]
where the sum is taken over \( m \geq 0 \) and \( J_1, \cdots, J_m \in \{(i, i) | 2 \leq i \leq 4\} \cup \{(i, j, a) | 2 \leq i < j \leq 4, a \in \mathbb{Z}/\mathbb{N}\} \).

Especially the identification between degree 1 terms is given by
\[ \Omega_5^{(N)} = \sum_{i=2}^4 t_{1i} d \log z_i + \sum_{2 \leq i < j \leq 4} \sum_{a \in \mathbb{Z}/\mathbb{N}} t_{ij}(a) d \log(z_i - \zeta_N^a z_j) \]
\[ \in U_{1,N}^0 \otimes H^1_{DR}(\mathcal{M}_L^{(N)}). \]

In terms of the coordinate \((x, y)\),
\[ \Omega_5^{(N)} = t_{12} d \log(xy) + t_{13} d \log y + \sum_a t_{23}(a) d \log(x - \zeta_N^a) \]
\[ + \sum_a t_{24}(a) d \log(xy - \zeta_N^a) + \sum_a t_{34}(a) d \log(y - \zeta_N^a) \]
\[ = t_{12} d \log x + \sum_a t_{23}(a) d \log(x - \zeta_N^a) + (t_{12} + t_{13} + t_{23}) d \log y \]
\[ + \sum_a t_{34}(a) d \log(y - \zeta_N^a) \]
\[ + \sum_a t_{24}(a) d \log(xy - \zeta_N^a). \]
It is easy to see that the identification is compatible with Hopf algebra structures. We note again that the product \( l_1 \cdot l_2 \in H^0 \bar{B}(\mathcal{M}_L^{(N)}) \) for \( l_1, l_2 \in H^0 \bar{B}(\mathcal{M}_L^{(N)}) \) is given by \( l_1 \cdot l_2(f) := \sum_i l_1(f^{(i)}_1) l_2(f^{(i)}_2) \) for \( f \in U_{1,N}^0 \otimes \mathbb{C} \) with \( \Delta(f) = \sum_i f^{(i)}_1 \otimes f^{(i)}_2 \) (\( \Delta \): the coproduct of \( U_{1,N}^0 \)).

Occasionally we also regard \( H^0 \bar{B}(\mathcal{M}_L^{(N)}) \) as the regular function ring of \( K^1_{CH}(\mathcal{C}) = \{g \in U_{1,N}^0 \otimes \mathbb{C} | g \text{ : group-like}\} \).

By a generalization of Chen’s theory \([\mathcal{C}]\) to the case of tangential basepoints, especially for \( \mathcal{M} = \mathcal{M}_L^{(N)} \) or \( \mathcal{M}_L^{(N)} \), we have an isomorphism
\[ \rho : H^0 \bar{B}(\mathcal{M}) \cong I_0(\mathcal{M}) \]
as algebras over \( \mathbb{C} \) which sends \( \sum_{I \subseteq \{1, \cdots, n\}} c_I \omega_{i_1} \cdots \omega_{i_m} \) \((c_I \in \mathbb{C})\) to \( \sum_I c_I \int_{\gamma} \omega_{i_1} \cdots \omega_{i_m} \). Here \( \sum_I c_I \int_{\gamma} \omega_{i_1} \cdots \omega_{i_m} \) means the iterated integral defined by

\[ \int_{0 < t_1 < \cdots < t_{m-1} < t_m < 1} \omega_{i_m} (\gamma(t_m)) \cdot \omega_{i_{m-1}} (\gamma(t_{m-1})) \cdots \omega_{i_1} (\gamma(t_1)) \]
for all analytic paths \( \gamma : (0, 1) \to \mathcal{M}(\mathcal{C}) \) starting from the tangential basepoint \( o \) (defined by \( \frac{d}{dx} \) for \( \mathcal{M} = \mathcal{M}_L^{(N)} \) and defined by \( \frac{d}{dx} \) and \( \frac{d}{dy} \) for \( \mathcal{M} = \mathcal{M}_L^{(N)} \) at the origin in \( \mathcal{M} \) (for its treatment see also \([De]\)§15).
and \( I_o(M) \) stands for the \( \mathbb{C} \)-algebra generated by all such homotopy invariant iterated integrals with \( m \geq 1 \) and \( \omega_{i_1}, \ldots, \omega_{i_m} \in H^1_{DR}(M) \).

4. Two variable cyclotomic multiple polylogarithms

We introduce cyclotomic multiple polylogarithms, \( Li_a(\zeta(z)) \) and \( Li_{a,b}(\zeta(x), \eta(y)) \), and their associated bar elements, \( \bar{L}_a^\zeta \) and \( \bar{L}_{a,b}^{(x,y)} \), which play important roles to prove our main theorems.

For a pair \((a, \zeta)\) with \( a = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k \) and \( \zeta = (\zeta_1, \ldots, \zeta_k) \), we have that \( \zeta_i \in \mu_N^{\zeta} \) with the group of roots of unity in \( \mathbb{C} \) with weight \( w(t, \zeta) = 1 \) and \( dp(a, \zeta) = k \) respectively. Let \( \zeta(x) = (\zeta_1, \ldots, \zeta_{k-1}, \zeta x) \). Consider the following complex analytic function, one variable cyclotomic multiple polylogarithm

\[
Li_a(\zeta(z)) := \sum_{0 < m_1 < \cdots < m_k} z^{m_1} \cdots z^{m_k-1} \frac{(\zeta m_k)_{m_k}}{m_1 \cdots m_{k-1} m_k^{m_k}}.
\]

It satisfies the following differential equation

\[
\frac{d}{dz} Li_a(\zeta(z)) = \begin{cases} 
\frac{1}{\zeta_i} Li_a(\zeta(z)) & \text{if } a_k \neq 1, \\
\frac{1}{\zeta_{k-1}} Li_{a-1, \zeta_{k-1}}(\zeta_1, \ldots, \zeta_{k-2}, \zeta_{k-1} z) & \text{if } a_k = 1, k \neq 1, \\
\frac{1}{\zeta_{k-1}} Li_{a-1, \zeta_{k-1}}(\zeta_1, \ldots, \zeta_{k-2}, \zeta_{k-1} z) & \text{if } a_k = 1, k = 1.
\end{cases}
\]

It gives an iterated integral starting from \( o \), which lies on \( I_o(M^{(N)}_{0,4}) \). Actually by the map \( \rho \) it corresponds to an element of the \( \mathbb{Q} \)-structure \( U^{3N+1}_N \) of \( V(M^{(N)}_{0,4}) \) denoted by \( \bar{L}_a^\zeta \). It is expressed as

\[
\bar{L}_a^\zeta = (-1)^k \frac{1}{\zeta_{k-1}} \frac{\prod \omega_i}{\zeta_i} \cdots \frac{\prod \omega_0}{\zeta_1} \frac{\prod \omega_0}{\zeta_1} \cdots \frac{\prod \omega_0}{\zeta_{k-1}} \frac{\prod \omega_0}{\zeta_{k-1}} \cdots \frac{\prod \omega_0}{\zeta_{k-1}}.
\]

By the standard identification \( \mu \sim \mathbb{Z}/N\mathbb{Z} \), sending \( \zeta_N = \exp\left(\frac{2\pi i}{N}\right) \mapsto 1 \), for a series \( \varphi = \sum_{W: \text{word}} c_W(\varphi) W \) it is calculated by

\[
\bar{L}_a^\zeta(\varphi) = (-1)^k e^{A^{\ast k} B(-e_k) A^{\ast k-1} B(-e_k-\cdots-e_{k-1}) \cdots A^{\ast 1} B(-e_k-\cdots-e_{k-1}) \cdots A^{\ast 1} B(-e_k-\cdots-e_{k-1})}(\varphi)
\]

with \( \zeta_i = \zeta_i^{e_i} \in \mathbb{Z}/N\mathbb{Z} \).

For \( a = (a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k \), \( b = (b_1, \ldots, b_l) \in \mathbb{Z}_{\geq 0}^l \), \( \zeta = (\zeta_1, \ldots, \zeta_k) \), \( \eta = (\eta_1, \ldots, \eta_l) \) with \( \zeta_i, \eta_j \in \mu_N \) and \( x, y \in \mathbb{C} \) with \( |x| < 1 \) and \( |y| < 1 \), consider the following complex function, the two variables multiple polylogarithm

\[
Li_{a,b}(\zeta(x), \eta(y)) := \sum_{0 < m_1 < \cdots < m_k} \frac{\prod \omega_i}{\zeta_1 m_k^{m_k}} \frac{\prod \omega_i}{\zeta_1 m_k^{m_k}} \cdots \frac{\prod \omega_i}{\zeta_1 m_k^{m_k}} \frac{\prod \omega_i}{\zeta_1 m_k^{m_k}}.
\]
It satisfies the following differential equations.

\[
d \frac{d}{dx} L_i a, b (\zeta(x), \bar{\eta}(y)) \]

\[
= \begin{cases} 
\frac{1}{a_i} L_k (a_1, \ldots, a_{k-1}, a_k, b) (\zeta(x), \bar{\eta}(y)) & \text{if } a_k \neq 1, \\
\frac{1}{a_i} L_k (a_1, \ldots, a_{k-1}, b) (\zeta_1, \ldots, \zeta_{k-2}, \zeta_{k-1} x, \bar{\eta}(y)) - \left( \frac{1}{x} + \frac{1}{a_i} \right) \cdot L_k (a_1, \ldots, a_{k-1}, b) (\zeta_1, \ldots, \zeta_{k-1}, \zeta_k \eta x, \eta_2, \ldots, \eta_{l-1}, \eta_l y) & \text{if } a_k = 1, k \neq 1, l \neq 1, \\
\frac{1}{a_i} L_k (a_1, \ldots, a_{k-1}, b) (\zeta_1, \ldots, \zeta_{k-1}, \zeta_k \eta x, \eta_2, \ldots, \eta_{l-1}, \eta_l y) & \text{if } a_k = 1, k = 1, l \neq 1, \\
\frac{1}{a_i} L_k (a_1, \ldots, a_{k-1}, b) (\zeta_1, \ldots, \zeta_{k-1}, \zeta_k \eta x y) & \text{if } a_k = 1, k \neq 1, l = 1, \\
\frac{1}{a_i} L_k (a_1, \ldots, a_{k-1}, b) (\zeta_1, \ldots, \zeta_{k-1}, \zeta_k \eta x) & \text{if } a_k = 1, k = 1, l = 1,
\end{cases}
\]

\[
d \frac{d}{dy} L_i a, b (\zeta(x), \bar{\eta}(y)) \]

\[
= \begin{cases} 
\frac{1}{y} L_i a, b (a_1, \ldots, a_{l-1}, b_l) (\zeta(x), \bar{\eta}(y)) & \text{if } b_l \neq 1, \\
\frac{1}{y} \eta_l^{-1} y L_i a, b (a_1, \ldots, a_{l-1}, b_l) (\zeta(x), \eta_1, \ldots, \eta_{l-1}, \eta_l y) & \text{if } b_l = 1, l \neq 1, \\
\frac{1}{y} \eta_l^{-1} y L_i a, b (\zeta(x), \eta_1, \ldots, \eta_{l-1}) & \text{if } b_l = 1, l = 1.
\end{cases}
\]

By analytic continuation, the functions \(L_i a, b (\zeta(x), \bar{\eta}(y)), L_i b, a (\bar{\eta}(y), \zeta(x)), L_i a (\zeta(x)), L_i a (\bar{\eta}(y))\) and \(L_i a (\zeta(xy))\) give iterated integrals starting from \(\omega\), which lie on \(L_\omega (\mathcal{M}^{(N)}_{0, 5})\). They correspond to elements of the 

\(Q\)-structure \((U_{4, N}^0)^*\) of \(V (\mathcal{M}^{(N)}_{0, 5})\) by the map \(\rho\) denoted by \(L_i a, b (\zeta(x), \bar{\eta}(y)), L_i b, a (\bar{\eta}(y), \zeta(x)), L_i a (\zeta(x)), L_i a (\bar{\eta}(y))\) and \(L_i a (\zeta(xy))\) respectively. Note that they are expressed as

\[
\sum_{I = (i_1, \ldots, i_l)} c_I [\omega_{i_1}] \cdots [\omega_{i_l}]
\]

for some \(m \in \mathbb{N}\) with \(c_I \in \mathbb{Q}\) and \(\omega_{i_j} \in \{dx, \xi-x, dy, \xi-y, xdy+gydx, \zeta \in \mu_N\}\).

5. Proof of main theorems

This section gives a proof of theorem 1.

**Proof of theorem 1.** Let \(a = (a_1, \ldots, a_k) \in \mathbb{Z}^k_{0, 5, 0}, b = (b_1, \ldots, b_l) \in \mathbb{Z}^l_{0, 6} \), \(\zeta = (\zeta_1, \ldots, \zeta_k)\) and \(\bar{\eta} = (\eta_1, \ldots, \eta_l)\) with \(\zeta_i, \eta_j \in \mu_N \subset \mathbb{C}\) \((1 \leq i \leq k\) and \(1 \leq j \leq l)\). Put \(\zeta(x) = (\zeta_1, \ldots, \zeta_{k-1}, \zeta_k x)\) and...
\( \bar{\eta}(y) = (\eta_1, \ldots, \eta_{k-1}, \eta y) \). Recall that multiple polylogarithms satisfy the following analytic identity, the series shuffle formula in \( L_o(\mathcal{M}^{(N)}) \):

\[
L_{i_a}(\bar{\zeta}(x)) \cdot L_{i_b}(\bar{\eta}(y)) = \sum_{\sigma \in Sh^\setminus(k,l)} L_i^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}.
\]

Here \( Sh^\setminus(k,l) := \cup_{N=1}^\infty \{ \sigma : \{1, \ldots, k+l\} \rightarrow \{1, \ldots, N\} \mid \sigma \text{ is onto}, \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l) \} \), \( \sigma(a,b) := (c_1, \ldots, c_N) \) with

\[
c_i = \begin{cases} 
  a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s, t\} \text{ with } s < t, \\
  a_s & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s \leq k, \\
  b_{t-k} & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s > k,
\end{cases}
\]

and \( \sigma(\bar{\zeta}(x),\bar{\eta}(y)) := (z_1, \ldots, z_N) \) with

\[
z_i = \begin{cases} 
  x_s y_{t-k} & \text{if } \sigma^{-1}(i) = \{s, t\} \text{ with } s < t, \\
  x_s & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s \leq k, \\
  y_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s > k,
\end{cases}
\]

for \( x_i = \zeta_i (i \neq k), \zeta_k x \ (i = k) \) and \( y_j = \eta_j \ (j \neq l), \eta_l y \ (j = l) \). Since \( \rho \) is an embedding of algebras, the above analytic identity immediately implies the algebraic identity, the series shuffle formula in the \( \mathbb{Q} \)-structure \( (Ut_4^0)^{\ast} \) of \( V(\mathcal{M}^{(N)}) \)

\[
\bar{\zeta}(x) \cdot \bar{\eta}(y) = \sum_{\sigma \in Sh^\setminus(k,l)} t_i^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}.
\]

Let \((g,h)\) be a pair in theorem 1. By the group-likeness of \( h \), i.e. \( h \in \exp \mathfrak{g}_{N+1} \), the product \( h^{1,23,4} h^{1,2,3} \) is group-like, i.e. belongs to \( \exp t_4^0 \). Hence \( \Delta(h^{1,23,4} h^{1,2,3}) = (h^{1,23,4} h^{1,2,3}) \otimes (h^{1,23,4} h^{1,2,3}) \), where \( \Delta \) is the standard coproduct of \( Ut_4^0 \). Therefore

\[
l_{i_a}^{\bar{\zeta}(x)} \cdot l_{i_b}^{\bar{\eta}(y)}(h^{1,23,4} h^{1,2,3}) = (l_{i_a}^{\bar{\zeta}(x)} \otimes l_{i_b}^{\bar{\eta}(y)})(\Delta(h^{1,23,4} h^{1,2,3}))
\]

\[
= l_{i_a}^{\bar{\zeta}(x)}(h^{1,23,4} h^{1,2,3}) \cdot l_{i_b}^{\bar{\eta}(y)}(h^{1,23,4} h^{1,2,3}).
\]

Evaluation of the equation (8) at the group-like element \( h^{1,23,4} h^{1,2,3} \) gives the series shuffle formula

\[
l_{i_a}^{\bar{\zeta}}(h) \cdot l_{i_b}^{\bar{\eta}}(h) = \sum_{\sigma \in Sh^\setminus(k,l)} t_i^{\sigma(\bar{\zeta},\bar{\eta})}(h)
\]

for admissible pairs \(^\text{1}\) \((a,\bar{\zeta})\) and \((b,\bar{\eta})\) by the results in [F4] because the group-likeness and (4) for \( h \) implies \( c_0(h) = 1 \) and \( c_A(h) = 0 \).

\(^1\) A pair \((a,\bar{\zeta})\) with \( a = (a_1, \ldots, a_k) \) and \( \bar{\zeta} = (\zeta_1, \ldots, \zeta_k) \) is called admissible if \((a_k, \zeta_k) \neq (1,1)\).
By putting \( l_1^1(h) := -T \) and \( \tilde{h}_a^S(h) := \tilde{I}_a^S(h) \) for all admissible pairs \((a, \tilde{\zeta})\), the series regularized value \( \tilde{I}_a^S(h) \) in \( \mathbb{Q}[T] \) \((T:\text{ a parameter which stands for } \log z, \text{ cf. }[R])\) for a non-admissible pair \((a, \tilde{\zeta})\) is uniquely determined in such a way \((\text{cf.}[AK])\) that the above series shuffle formulae remain valid for \( \tilde{I}_a^S(h) \) with all pairs \((a, \tilde{\zeta})\).

Define the integral regularized value \( \tilde{h}_a^I(h) \) in \( \mathbb{Q}[T] \) for all pairs \((a, \tilde{\zeta})\) by \( \tilde{h}_a^I(h) = \tilde{I}_a^I(e^{TB(\bar{0})}h) \). Equivalently \( \tilde{I}_a^I(h) \) for any pair \((a, \tilde{\zeta})\) can be uniquely defined in such a way that the iterated integral shuffle formulae \((\text{loc.cit})\) remain valid for all pairs \((a, \tilde{\zeta})\) with \( l_1^I(h) := -T \) and \( \tilde{h}_a^I(h) := \tilde{I}_a^I(h) \) for all admissible pairs \((a, \tilde{\zeta})\) because they hold for admissible pairs by the group-likeness of \( h \) \((\text{cf. loc.cit})\).

Let \( \mathbb{L} \) be the \( \mathbb{Q} \)-linear map from \( \mathbb{Q}[T] \) to itself defined via the generating function:

\[
\mathbb{L}(\exp Tu) = \sum_{n=0}^{\infty} \mathbb{L}(T^n) \frac{u^n}{n!} = \exp \left\{ -\sum_{n=1}^{\infty} l_n^I(h) \frac{u^n}{n} \right\}.
\]

**Proposition 8.** Let \( h \) be an element as in theorem 1. Then the regularization relation holds, i.e. \( \tilde{I}_a^S(h) = \mathbb{L}(\tilde{h}_a^I(h)) \) for all pairs \((a, \tilde{\zeta})\).

**Proof.** We may assume that \((a, \tilde{\zeta})\) is non-admissible because the proposition is trivial if it is admissible. Put \( 1^n = (1, 1, \ldots, 1) \). When \( a = 1^n \) and \( \tilde{\zeta} = \bar{1}^n \), the proof is given by the same argument to [F3] as follows: By the series shuffle formulae,

\[
\sum_{k=0}^{m} (-1)^k l_{k+1}^1(h) \cdot \tilde{I}_{1m-k}^S(h) = (m+1) l_{1m+1}^S(h)
\]

for \( m \geq 0 \). Here we put \( l_0^S(h) = 1 \). This means

\[
\sum_{k,l \geq 0} (-1)^k l_{k+1}^1(h) \cdot \tilde{I}_{1}^S(h) u^{k+l} = \sum_{m \geq 0} (m+1) l_{1m+1}^S(h) u^m.
\]

Put \( f(u) = \sum_{n \geq 0} \tilde{I}_{1n}^S(h) u^n \). Then the above equality can be read as

\[
\sum_{k \geq 0} (-1)^k l_{k+1}^1(h) u^k = \frac{d}{du} \log f(u).
\]

Integrating and adjusting constant terms gives

\[
\sum_{n \geq 0} l_{1n}^S(h) u^n = \exp \left\{ -\sum_{n \geq 1} (-1)^n \tilde{I}_{1n}^S(h) \frac{u^n}{n} \right\} = \exp \left\{ -\sum_{n \geq 1} (-1)^n l_{1n}^I(h) \frac{u^n}{n} \right\}
\]
because $t^{i, S}_n(h) = t^{i, I}_n(h) = t^{i}_n(h)$ for $n > 1$ and $t^{1, S}_1(h) = t^{1, I}_1(h) = -T$. Since $t^{i, I}_m(h) = \left(\frac{-T}{m}\right)^m$, we get $t^{i, S}_m(h) = \mathbb{L}(t^{i, I}_m(h))$.

When $(a, \tilde{\zeta})$ is of the form $(a'1^l, \tilde{\zeta}'1^l)$ with $(a', \tilde{\zeta}')$ admissible, the proof is given by the following induction on $l$. By (8),

$$l^{\tilde{\zeta}'(x)}_{a'}(h') \cdot l^{i(y)}_{1^l}(h') = \sum_{\sigma \in S_{\leq (k, l)}} l^{\sigma(\tilde{\zeta}'(x), i(y))}_{\sigma(a', 1^l)}(h')$$

for $h' = e^{T(t^{23}(0) + t^{24}(0) + t^{24}(4))} h^{1,23,4} h^{1,2,3}$ with $k = dp(a')$. The group-likeness and (4) for $h$ implies $c_0(h) = 1$ and $c_A(h) = 0$ and the group-likeness and our assumption $c_B(0)(h) = 0$ implies $c_B(0)^n(h) = 0$ for $n \in \mathbb{Z}_{>0}$. Hence by the results in [F4]

$$l^{\tilde{\zeta}}_{a'}(h) \cdot l^{i,l}_{1^l}(h) = \sum_{\sigma \in S_{\leq (k, l)}} l^{\sigma(\tilde{\zeta}'l), l}_{\sigma(a', 1^l)}(h).$$

Then by our induction assumption, taking the image by the map $\mathbb{L}$ gives

$$l^{\tilde{\zeta}}_{a'}(h) \cdot l^{i,l}_{1^l}(h) = \mathbb{L}(l^{\tilde{\zeta}'l}_{a'}(h)) + \sum_{\sigma \neq id \in S_{\leq (k, l)}} l^{\sigma(\tilde{\zeta}'l), l}_{\sigma(a', 1^l)}(h).$$

Since $l^{\tilde{\zeta}, S}_{a'}(h)$ and $l^{i, S}_{1^l}(h)$ satisfy the series shuffle formula, $\mathbb{L}(l^{\tilde{\zeta}'l}_{a'}(h))$ must be equal to $l^{\tilde{\zeta}, S}_{a'}(h)$, which concludes proposition 8.

Embed $U\mathfrak{F}_{Y_2}$ into $U\mathfrak{F}_{N+1}$ by sending $Y_{m,n}$ to $-A^{m-1}B(-a)$. Then by the above proposition,

$$l^{\tilde{\zeta}, S}_{a'}(h) = \mathbb{L}(l^{\tilde{\zeta}'l}_{a'}(h)) = \mathbb{L}(l^{\tilde{\zeta}}_{a'}(e^{TB(0)})) = l^{\tilde{\zeta}}_{a'}(\mathbb{L}(e^{TB(0)}))$$

$$= \mathbb{L}^{\tilde{\zeta}}_{a'}(\exp \left\{ -\sum_{n=1}^{\infty} \frac{l^{i,l}_{n}(h)B(0)^n}{n} \right\} \pi_Y(h))$$

$$= \mathbb{L}^{\tilde{\zeta}}_{a'}(\exp \left\{ -TY_{1,0} + \sum_{n=1}^{\infty} \frac{(-1)^n c_{A^{n-1}B(0)(h)}Y^n_{1,0}}{n} \right\} \pi_Y(h)) = l^{\tilde{\zeta}}_{a'}(e^{-TY_{1,0}h})$$

for all $(a, \tilde{\zeta})$ because $l^{i}_{1}(h) = 0$. As for the third equality we use $(\mathbb{L} \otimes \mathbb{Q} id) \circ (id \otimes \mathbb{Q} l^{\tilde{\zeta}}_{a'}) = (id \otimes \mathbb{Q} l^{\tilde{\zeta}}_{a'}) \circ (\mathbb{L} \otimes \mathbb{Q} id)$ on $\mathbb{Q}[T] \otimes \mathbb{Q} U\mathfrak{F}_{N+1}$. All $l^{\tilde{\zeta}, S}_{a'}(h)$'s satisfy the series shuffle formulae (9), so the $l^{\tilde{\zeta}}_{a'}(e^{-TY_{1,0}h})$'s do also. By putting $T = 0$, we get that $l^{\tilde{\zeta}}_{a'}(h)$'s also satisfy the series shuffle formulae for all $a$. Therefore $\Delta_s(h_s) = h_s \hat{\otimes} h_s$. This completes the proof of theorem 1.
HIDEKAZU FURUSHO

References


Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan

E-mail address: furusho@math.nagoya-u.ac.jp