p-adic KZ equation and p-adic multiple zeta value

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p MZV p-adic multiple zeta value

p MPL p-adic multiple polylog

p KZ equation p-adic Knizhnik-Zamolodchikov

① $m, k_1, \dots, k_m \in \mathbb{N} = \mathbb{Z}_{>0}$

$$\zeta(k_1,\dots,k_m) = \sum_{\substack{0<n_1<\dots<n_m \\ n_i\in\mathbb{N}}} \cancel{\frac{1}{n_1^{k_1}\cdots n_m^{k_m}}} \in \mathbb{R}$$

$\in \mathbb{Q}_p$

$z=1$

: **MZV** $(k_m>1)$

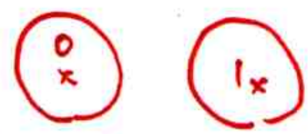
- $\underline{m=1}$ $\zeta(k)$: Riemann zeta value
- motive, knot, quantumzation, high energy physics, etc

② $$Li_{k_1\cdots k_m}(z) = \sum_{0<n_1<\dots<n_m} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}} \text{ : MPL}$$

$\boxed{z\in\mathbb{C}}$ converges on $|z|<1$

$\boxed{z\in\mathbb{C}_p}$ converges on $|z|_p<1$

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$



$\{|z|_p<1\}\cap\{|z-1|_p<1\}=\emptyset$

We need

$\rightsquigarrow$ analytic continuation

③ $\boxed{p \neq \infty}$

$$
\begin{cases}
\dfrac{d}{dz} Li_{k_1\cdots k_m}(z) = \begin{cases} \dfrac{1}{z} Li_{k_1\cdots k_m-1}(z) & (k_m \neq 1) \\[2ex] \dfrac{1}{1-z} Li_{k_1\cdots k_{m-1}}(z) & (k_m = 1) \end{cases} \\[4ex]
\dfrac{d}{dz} Li_1(z) = \dfrac{1}{1-z}
\end{cases}
$$

Fix $a \in \mathbb{C}_p$

④ $\boxed{p \neq \infty}$ Coleman's p-adic integration ('82)

$$Li_1^a(z) = \int_0^z \frac{dt}{1-t} = -\log^a(1-z)$$

$$\rightsquigarrow Li_2^a(z) = \int_0^z Li_1^a(t) \frac{dt}{t} \rightsquigarrow \rightsquigarrow \cdots$$

$\rightsquigarrow$ ana. conn of p MPL^a

to $\cancel{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}}$ $\mathbb{P}^1(\mathbb{C}_p) \setminus \{1,\infty\}$

$\rightsquigarrow$ $\cancel{\aleph_0}$-branches ✗

$\boxed{Q}$ $\lim\limits_{z \to 1} Li_{k_1\cdots k_m}(z) = ?$

⑤ $\boxed{p \neq \infty}$ Fix $a \in \mathbb{C}_p$

Th $\lim\limits_{\substack{z \to 1 \\ z \in \mathbb{C}_p}}{}' \, Li^{a}_{k_1 \cdots k_m}(z)$ converges if $k_m > 1$

Def $\zeta_p(k_1, \ldots, k_m) := \lim\limits_{z \to 1}{}' \, Li^{a}_{k_1 \cdots k_m}(z) \in \mathbb{C}_p$

⑥ **Th** It is independent of $a \in \mathbb{C}_p$.

⑦ $\boxed{p = \infty}$ $\zeta_p(k_1, \ldots, k_m) \in \mathbb{R} \subset \mathbb{C}$

$\in \mathbb{Q}_p \subset \mathbb{C}_p$

⑧ $\boxed{p \neq \infty}$ $z \in \mathbb{C}_p$. $G(z) \in \mathbb{C}_p\langle\langle A, B \rangle\rangle$

$$\frac{d}{dz} G(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right) G(z)$$ p-adic KZ equation

Fix $a \in \mathbb{C}_p$

$$G_0^a(z) = 1 + \sum (-1)^m Li^a_{k_1 \cdots k_m}(z) A^{k_m-1} B \cdots A^{k_1-1} B + \cdots$$

$z \longleftrightarrow 1-z$, $A \longleftrightarrow B$

$$G_1^a(z) = 1 + \sum (-1)^m Li^a_{k_1 \cdots k_m}(1-z) B^{k_m-1} A \cdots B^{k_1-1} A + \cdots$$

def $\Phi^p_{KZ} := G_1^a(z)^{-1} G_0^a(z)$ p-adic Drinfeld associator

Prop independent from z

$\because$ $$\frac{d}{dz} \Phi_{KZ} = -G_1^{-1} \frac{dG_1}{dz} G_1^{-1} G_0 + G_1^{-1} \frac{dG_0}{dz}$$

$$= -G_1^{-1}\left(\frac{A}{z} + \frac{B}{z-1}\right) G_0 + G_1^{-1}\left(\frac{A}{z} + \frac{B}{z-1}\right) G_0 = 0$$

Prop free from $a \in \mathbb{C}_p$