The motivic Galois group, the Grothendieck-Teichmüller group and the double shuffle group

Hidekazu Furusho

Let $DM(Q)_Q$ be the triangulated category of mixed motives over $Q$ constructed by Hanamura, Levine and Voevodsky. Tate motives $Q(n)$ ($n \in Z$) are Tate objects of the category. Let $DMT(Q)_Q$ be the triangulated sub-category of $DM(Q)_Q$ generated by Tate motives $Q(n)$ ($n \in Z$). By the work of Levine a neutral tannakian $Q$-category $MT(Q)_Q$ of mixed Tate motives over $Q$ can be extracted by taking a heart with respect to a $t$-structure of $DMT(Q)_Q$. Deligne and Goncharov [1] defined the full subcategory $MT(Z)_Q$ of unramified mixed Tate motives, whose objects are mixed Tate motives $M$ (an object of $MT(Q)_Q$) such that for each subquotient $E$ of $M$ which is an extension of $Q(n)$ by $Q(n+1)$ for $n \in Z$, the extension class of $E$ in $Ext^1_{MT(Q)}(Q(n), Q(n+1)) = Ext^1_{MT(Q)}(Q(0), Q(1)) = Q \otimes Q$ is equal to $Z \otimes Q = \{0\}$. The category $MT(Z)_Q$ forms a neutral tannakian $Q$-category with the fiber functor $\omega_{can} : MT(Z)_Q \rightarrow Vect_Q (Vect_Q :$ the category of $Q$-vector spaces) sending each motive $M$ to $\oplus_n Hom(Q(n), Gr^{W,n}_2 M)$.

**Definition 1.** The motivic Galois group of unramified mixed Tate motives $MT(Z)_Q$ is defined to be the pro-algebraic group $Gal^M(Z) := \text{Aut}^{\otimes}(MT(Z) : \omega_{can})$.

The action of $Gal^M(Z)$ on $\omega_{can}(Q(1)) = Q$ defines a surjection $Gal^M(Z) \rightarrow G_{un}$ and its kernel $Gal^M(Z)_1$ is the unipotent radical of $Gal^M(Z)$. In [1] §4 they constructed the motivic fundamental group $\pi^M_1(X : \overline{01})$ with $X = P^1 \setminus \{0, 1, \infty\}$, which is an ind-object of $MT(Z)_Q$. This is an affine group $MT(Z)_Q$-scheme. It induces the morphism $Gal^M(Z) \rightarrow Aut_{F_2}$ where $F_2 = \omega_{can}(\pi_1^M(X : \overline{01}))$ is the free pro-unipotent algebraic group of rank $2$. Denote its restriction into the unipotent part by

$$\Psi : Gal^M(Z)_1 \rightarrow Aut_{F_2}.$$ 

This map is expected to be injective.

Let us fix notations: Let $k$ be a field of characteristic $0$, $\bar{k}$ its algebraic closure and $U\mathfrak{S}_2 = k\langle X_0, X_1 \rangle$ a non-commutative formal power series ring with two variables $X_0$ and $X_1$. Its element $\varphi = \varphi(X_0, X_1)$ is called group-like if it satisfies $\Delta(\varphi) = \varphi \otimes \varphi$ with $\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0$ and $\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1$ and its constant term is equal to $1$. For a monic monomial $W$, $c_W(\varphi)$ means the coefficient of $W$ in $\varphi$. For any $k$-algebra homomorphism $\iota : U\mathfrak{S}_2 \rightarrow S$ the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(X_0), \iota(X_1))$.

**Definition 2.** The Grothendieck-Teichmüller group $GRT_1$ is defined to be the pro-unipotent algebraic variety whose set of $k$-valued points consists of group-like series $\varphi \in U\mathfrak{S}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$ satisfying Drinfel’d’s two hexagon equations in $U\mathfrak{S}_2$:

$$\varphi(t_{13}, t_{12})\varphi(t_{13}, t_{23})^{-1}\varphi(t_{12}, t_{23}) = 1,$$

$$(2) \quad \varphi(t_{23}, t_{13})^{-1}\varphi(t_{12}, t_{13})\varphi(t_{12}, t_{23})^{-1} = 1$$
and his pentagon equation in $Ua_4$:

\begin{align}
(4) \varphi(t_{12},t_{23} + t_{24})\varphi(t_{13} + t_{24},t_{34}) = \varphi(t_{21},t_{34})\varphi(t_{12} + t_{13},t_{24} + t_{34})\varphi(t_{12},t_{23}).
\end{align}

Here $Ua_4$ means the universal enveloping algebra of the completed pure braid Lie algebra $a_4$ over $k$ with 4 strings, generated by $t_{ij}$ ($1 \leq i,j \leq 4$) with defining relations $t_{ii} = 0$, $t_{ij} = t_{ji}$, $[t_{ij},t_{ik} + t_{jk}] = 0$ ($i,j,k$: all distinct) and $[t_{ij},t_{kl}] = 0$ ($i,j,k,l$: all distinct).

By the multiplication below, $GRT_1$ really forms a group

\begin{align}
(5) \quad \varphi_2 \circ \varphi_1 := \varphi_1(\varphi_2X_0\varphi_2^{-1},X_1) \cdot \varphi_2 = \varphi_2 \cdot \varphi_1(X_0,\varphi_2^{-1}X_1\varphi_2).
\end{align}

The group was introduced by Drinfel'd [2] in his study of quasitriangular quasi-Hopf quantized universal enveloping algebras, certain types of quantum groups. Let $F_2$ be the free pro-unipotent algebraic group with two generators $e^{X_0}$ and $e^{X_1}$ and $AutF_2$ be the pro-algebraic group which represents $k \mapsto AutF_2(k)$. By the map sending $X_0 \mapsto X_0$ and $X_1 \mapsto \varphi X_1 \varphi^{-1}$, the group $GRT_1$ is regarded as a subgroup of $AutF_2$. By geometric interpretations of the equations (2)~(4), it is shown that $\text{Im}\Psi$ is contained in $GRT_1$. Actually it is expected that they are isomorphic. Our first result here is on defining equations of $GRT_1$.

**Theorem 3** ([4]). Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{H}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$. Suppose that $\varphi$ satisfies the pentagon equation (4). Then it also satisfies two hexagon equations (2) and (3).

This theorem claims that the pentagon equation (4) is essentially a single defining equation of the Grothendieck-Teichmüller group.

Again let us fix notations: Let $\pi_Y : k\langle\langle X_0, X_1 \rangle\rangle \rightarrow k\langle\langle Y_1, Y_2, \ldots \rangle\rangle$ be the $k$-linear map between non-commutative formal power series rings that sends all the words ending in $X_0$ to zero and the word $X_0^{n_m-1}X_1 \cdots X_0^{n_1-1}X_1 (n_1, \ldots , n_m \in \mathbb{N})$ to $(-1)^m Y_{n_m} \cdots Y_{n_1}$. Define the coproduct $\Delta_*$ on $k\langle\langle Y_1, Y_2, \ldots \rangle\rangle$ by $\Delta_* Y_n = \sum_{i=0}^{\infty} Y_i \otimes Y_{n-i}$ with $Y_0 := 1$. For $\varphi = \sum_{\text{W.word}} c_W(\varphi)W \in k\langle\langle X_0, X_1 \rangle\rangle$, put $\varphi_* = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} c_{X_0^{n-1}X_1}(\varphi)Y_1^n\right) \cdot \pi_Y(\varphi)$. For a group-like series $\varphi \in U\mathfrak{H}_2$ the generalised double shuffle relation means the equality

\begin{align}
(6) \quad \Delta_*(\varphi_*) = \varphi_* \circ \varphi_*.
\end{align}

**Definition 4** ([5]). The double shuffle group $DMR_0$ is the pro-unipotent algebraic variety whose set of $k$-valued points consists of the group-like series $\varphi \in U\mathfrak{H}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$ which satisfy (6).

The generalized double shuffle relation (6) arises from the generalized (regularised) double shuffle relations among multiple zeta values, which are expected to be the strongest relation among them. In [5] it is proved that $DMR_0$ is closed by the multiplication (5) as $GRT_1$. By the same way to the $GRT_1$-case, the group $DMR_0$ is regarded as a subgroup of $AutF_2$. It is also shown that $\text{Im}\Psi$ is contained in $DMR_0$. Actually it is expected that they are isomorphic. And $DMR_0$ is also expected to be isomorphic to $GRT_1$. Our second result here is a relationship between them.
Theorem 5 ([3]). $GRT_1 \subset DMR_0$.

We note that this realizes the project of Deligne-Terasoma where they indicated a different approach. Their arguments concerned multiplicative convolutions whereas our methods are based on a bar construction calculus (cf. [3]).

References