The motivic Galois group, the Grothendieck-Teichmüller group and the double shuffle group

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Let $DM(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated category of *mixed motives* over $\mathbf{Q}$ constructed by Hanamura, Levine and Voevodsky. *Tate motives* $\mathbf{Q}(n)$ $(n \in \mathbf{Z})$ are Tate objects of the category. Let $DMT(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated sub-category of $DM(\mathbf{Q})_{\mathbf{Q}}$ generated by Tate motives $\mathbf{Q}(n)$ $(n \in \mathbf{Z})$. By the work of Levine a neutral tannakian $\mathbf{Q}$-category $MT(\mathbf{Q}) = MT(\mathbf{Q})_{\mathbf{Q}}$ of *mixed Tate motives over* $\mathbf{Q}$ can be extracted by taking a heart with respect to a t-structure of $DMT(\mathbf{Q})_{\mathbf{Q}}$. Deligne and Goncharov [1] defined the full subcategory $MT(\mathbf{Z}) = MT(\mathbf{Z})_{\mathbf{Q}}$ of *unramified mixed Tate motives*, whose objects are mixed Tate motives M (an object of $MT(\mathbf{Q})$) such that for each subquotient E of M which is an extension of $\mathbf{Q}(n)$ by $\mathbf{Q}(n+1)$ for $n \in \mathbf{Z}$, the extension class of E in $Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(n), \mathbf{Q}(n+1)) = Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Q}^\times \otimes \mathbf{Q}$ is equal to $\mathbf{Z}^\times \otimes \mathbf{Q} = \{0\}$. The category $MT(\mathbf{Z})$ forms a neutral tannakian $\mathbf{Q}$-category with the fiber functor $\omega_{\mathrm{can}} : MT(\mathbf{Z}) \to Vect_{\mathbf{Q}}$ ($Vect_{\mathbf{Q}}$: the category of $\mathbf{Q}$-vector spaces) sending each motive M to $\oplus_n Hom(\mathbf{Q}(n), Gr^W_{-2n} M)$.

Definition 1. The *motivic Galois group* of unramified mixed Tate motives $MT(\mathbf{Z})$ is defined to be the pro-algebraic group $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) := \underline{Aut}^{\otimes}(MT(\mathbf{Z}) : \omega_{\mathrm{can}})$.

The action of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ on $\omega_{\mathrm{can}}(\mathbf{Q}(1)) = \mathbf{Q}$ defines a surjection $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \mathbf{G}_m$ and its kernel $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1$ is the unipotent radical of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$. In [1] §4 they constructed the *motivic fundamental group* $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ with $X = \mathbf{P}^1 \backslash \{0, 1, \infty\}$, which is an ind-object of $MT(\mathbf{Z})$. This is an affine group $MT(\mathbf{Z})$-scheme. It induces the morphism $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \underline{Aut}\underline{F_2}$ where $\underline{F_2} = \omega_{\mathrm{can}}(\pi_1^{\mathcal{M}}(X : \overrightarrow{01}))$ is the free pro-unipotent algebraic group of rank 2. Denote its restriction into the unipotent part by

$$\Psi : \mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1 \to \underline{Aut}\underline{F_2}. \tag{1}$$

This map is expected to be injective.

Let us fix notations: Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure and $U\mathfrak{F}_2 = k\langle\langle X_0, X_1 \rangle\rangle$ a non-commutative formal power series ring with two variables X_0 and X_1. Its element $\varphi = \varphi(X_0, X_1)$ is called *group-like* if it satisfies $\Delta(\varphi) = \varphi \otimes \varphi$ with $\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0$ and $\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1$ and its constant term is equal to 1. For a monic monomial W, $c_W(\varphi)$ means the coefficient of W in φ. For any k-algebra homomorphism $\iota : U\mathfrak{F}_2 \to S$ the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(X_0), \iota(X_1))$.

Definition 2 ([2])**.** The *Grothendieck-Teichmüller group* GRT_1 is defined to be the pro-unipotent algebraic variety whose set of k-valued points consists of group-like series $\varphi \in U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$ satisfying *Drinfel'd's two hexagon equations* in $U\mathfrak{F}_2$:

$$\varphi(t_{13}, t_{12})\varphi(t_{13}, t_{23})^{-1}\varphi(t_{12}, t_{23}) = 1, \tag{2}$$

$$\varphi(t_{23}, t_{13})^{-1}\varphi(t_{12}, t_{13})\varphi(t_{12}, t_{23})^{-1} = 1 \tag{3}$$

and *his pentagon equation* in $U\mathfrak{a}_4$:

(4) $\varphi(t_{12}, t_{23}+t_{24})\varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\varphi(t_{12}+t_{13}, t_{24}+t_{34})\varphi(t_{12}, t_{23}).$

Here $U\mathfrak{a}_4$ means the universal enveloping algebra of the *completed pure braid Lie algebra* $\mathfrak{a}_4$ *over* k *with 4 strings*, generated by t_{ij} ($1 \leqslant i,j \leqslant 4$) with defining relations $t_{ii} = 0$, $t_{ij} = t_{ji}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$ (i,j,k: all distinct) and $[t_{ij}, t_{kl}] = 0$ (i,j,k,l: all distinct).

By the multiplication below, GRT_1 really forms a group

(5) $$\varphi_2 \circ \varphi_1 := \varphi_1(\varphi_2 X_0 \varphi_2^{-1}, X_1) \cdot \varphi_2 = \varphi_2 \cdot \varphi_1(X_0, \varphi_2^{-1} X_1 \varphi_2).$$

The group was introduced by Drinfel'd [2] in his study of quasitriangular quasi-Hopf quantized universal enveloping algebras, certain types of quantum groups. Let $\underline{F_2}$ be the free pro-unipotent algebraic group with two generators e^{X_0} and e^{X_1} and $\underline{Aut F_2}$ be the pro-algebraic group which represents $k \mapsto Aut\underline{F_2}(k)$. By the map sending $X_0 \mapsto X_0$ and $X_1 \mapsto \varphi X_1 \varphi^{-1}$, the group GRT_1 is regarded as a subgroup of $\underline{Aut F_2}$. By geometric interpretations of the equations (2)$\sim$ (4), it is shown that $\mathrm{Im}\Psi$ is contained in GRT_1. Actually it is expected that they are isomorphic. Our first result here is on defining equations of GRT_1.

Theorem 3 ([4])**.** *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$. Suppose that φ satisfies the pentagon equation* (4)*. Then it also satisfies two hexagon equations* (2) *and* (3)*.*

This theorem claims that the pentagon equation (4) is essentially a single defining equation of the Grothendieck-Teichmüller group.

Again let us fix notations: Let $\pi_Y : k\langle\langle X_0, X_1\rangle\rangle \to k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ be the k-linear map between non-commutative formal power series rings that sends all the words ending in X_0 to zero and the word $X_0^{n_m-1}X_1 \cdots X_0^{n_1-1}X_1$ ($n_1, \dots, n_m \in \mathbf{N}$) to $(-1)^m Y_{n_m} \cdots Y_{n_1}$. Define the coproduct Δ_* on $k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ by $\Delta_* Y_n = \sum_{i=0}^n Y_i \otimes Y_{n-i}$ with $Y_0 := 1$. For $\varphi = \sum_{W:\mathrm{word}} c_W(\varphi) W \in k\langle\langle X_0, X_1\rangle\rangle$, put $\varphi_* = \exp\left(\sum_{n=1}^\infty \frac{(-1)^n}{n} c_{X_0^{n-1}X_1}(\varphi) Y_1^n\right) \cdot \pi_Y(\varphi)$. For a group-like series $\varphi \in U\mathfrak{F}_2$ the *generalised double shuffle relation* means the equality

(6) $$\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*.$$

Definition 4 ([5])**.** The *double shuffle group* DMR_0 is the pro-unipotent algebraic variety whose set of k-valued points consists of the group-like series $\varphi \in U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$ which satisfy (6).

The generalized double shuffle relation (6) arises from the generalized (regularised) double shuffle relations among multiple zeta values, which are expected to be the strongest relation among them. In [5] it is proved that DMR_0 is closed by the multiplication (5) as GRT_1. By the same way to the GRT_1-case, the group DMR_0 is regarded as a subgroup of $\underline{Aut F_2}$. It is also shown that $\mathrm{Im}\Psi$ is contained in DMR_0. Actually it is expected that they are isomorphic. And DMR_0 is also expected to be isomorphic to GRT_1. Our second result here is a relationship between them.

Theorem 5 ([3]). $GRT_1 \subset DMR_0$.

We note that this realizes the project of Deligne-Terasoma where they indicated a different approach. Their arguments concerned multiplicative convolutions whereas our methods are based on a bar construction calculus (cf. [3]).

References

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