Grothendieck-Teichmüller theory III, IV

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[reference] p-adic multiple zeta values I,

H. Furusho, to appear to Inv Math.

$$Z^p_w = \langle \zeta(\mathbb{k}) \mid \mathrm{wt}\,\mathbb{k} = w \rangle_{\mathbb{Q}} \subset \overset{\mathbb{Q}_p}{\cancel{\mathbb{R}}} \ , \quad Z^p_0 = \mathbb{Q}$$

$$Z^p_\bullet = \bigoplus_{w=0}^{\infty} Z^p_w \ : \text{graded } \mathbb{Q}\text{-vector space}$$

<u>Property</u> $Z^p_\bullet$ becomes a graded $\mathbb{Q}$-algebra

(i.e. $Z^p_a \cdot Z^p_b \subset Z^p_{a+b}$)

<u>pf 1</u> ((series shuffle product formula)) (Besser = F)

$$\zeta_p(a) \cdot \zeta_p(b) = \sum_m \frac{1}{m^a} \sum_n \frac{1}{n^b}$$

$$= \left(\sum_{m<n} + \sum_{m=n} + \sum_{m>n} \right) \frac{1}{m^a n^b}$$

$$= \zeta_p(a,b) + \zeta_p(a+b) + \zeta_p(b,a)$$

<u>pf 2</u> ((iterated integral shuffle product formula)) (F)

$$\zeta_p(a) \cdot \zeta_p(b) = \int_0^1 \frac{dt_a}{t_a} \int_0^{t_a} \frac{dt_{a-1}}{t_{a-1}} \cdots \int_0^{t_2} \frac{dt_1}{1-t_1} \times \int_0^1 \frac{ds_b}{s_b} \int_0^{s_b} \cdots \int_0^{s_3} \frac{ds_2}{s_2} \int_0^{s_2} \frac{ds_1}{1-s_1}$$

$$= \sum_{i=0}^{a-1} \binom{b-1+i}{i} \zeta_p(a-i, b+i) + \sum_{j=0}^{b-1} \binom{a-1+j}{j} \zeta_p(b-j, a+j)$$

$$Z_0^p = \langle 1 \rangle_{\mathbb{Q}}$$

$$Z_1^p = 0$$

$$Z_2^p = \langle \underset{0}{\cancel{\pi^2}} \rangle_{\mathbb{Q}}$$

$$Z_3^p = \langle \zeta_p(3) \rangle_{\mathbb{Q}}$$

$$Z_4^p = \langle \underset{0}{\cancel{\pi^4}} \rangle_{\mathbb{Q}}$$

$$Z_5^p = \langle \zeta_p(5) , \cancel{\pi^2 \zeta(3)} \rangle_{\mathbb{Q}}$$

$$Z_6^p = \langle \zeta_p(3)^2 , \cancel{\pi^6} \rangle_{\mathbb{Q}}$$

$$Z_7^p = \langle \zeta_p(7) , \cancel{\pi^2 \zeta(5)} , \cancel{\pi^4 \zeta(3)} \rangle_{\mathbb{Q}}$$

$$Z_8^p = \langle \zeta_p(3,5) , \zeta_p(3)\zeta_p(5) , \cancel{\pi^2 \zeta(3)^2} , \cancel{\pi^8} \rangle_{\mathbb{Q}}$$

① $\mathbb{k} = (k_1, \dots, k_m) \quad m, k_i \in \mathbb{N}$

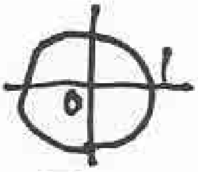
$\mathrm{wt}\,\mathbb{k} = k_1 + \cdots + k_m$: weight

$$\zeta(\mathbb{k}) = \sum_{\substack{0<m_1<\cdots<m_m \\ m_i \in \mathbb{N}}} \frac{1}{m_1^{k_1} \cdots m_m^{k_m}} : \text{MZV}$$

But this never converge on $\mathbb{Q}_p$!!

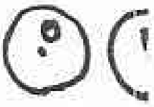
② $$\mathcal{L}i_{k_1 \cdots k_m}(z) = \sum_{0<m_1<\cdots<m_m} \frac{z^{m_m}}{m_1^{k_1} \cdots m_m^{k_m}} : p\text{-adic MPL}$$

$\boxed{z \in \mathbb{C}}$: converges on $|z| < 1$



$\boxed{z \in \mathbb{C}_p}$: converges on $|z|_p < 1 \Longleftrightarrow z \in m_{\mathbb{C}_p}$

$z = 1 \Longrightarrow z \in 1 + m_{\mathbb{C}_p}$



⤳ We need to give an analytic continuation of p-adic MPL

⤳ ④

③ $\boxed{p \neq \infty}$

$$\frac{d}{dz} \mathcal{L}i_{k_1 \cdots k_m}(z) = \begin{cases} \frac{1}{z} \mathcal{L}i_{k_1 \cdots k_{m-1}\, k_m - 1}(z) & (k_m \neq 1) \\ \frac{1}{1-z} \mathcal{L}i_{k_1 \cdots k_{m-1}}(z) & (k_m = 1) \end{cases}$$

$$\frac{d}{dz} \mathcal{L}i_1(z) = \frac{1}{1-z}$$

④ analytic continuation of MPL

$\boxed{p=\infty}$ $\mathcal{L}i_1(z)=\int_0^z \frac{dt}{1-t}=-\log(1-z)$

$\rightsquigarrow \mathcal{L}i_2(z)=\int_0^z \frac{\mathcal{L}i_1(t)}{t}dt \rightsquigarrow \cdots\cdots$

$\rightsquigarrow$ We can give an analytic continuation of MPL to $\overbrace{\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\}}$

$\rightsquigarrow \aleph_0$ branches

$\boxed{p\neq\infty}$ Coleman's p-adic iterated integration theory ('82)

Remark ① Frobenius action plays an important role to construct this integration theory.

② This integration theory is attached to each branch of p-adic logarithm.

$$\begin{array}{l}\log^a=\int\frac{dt}{t}:\mathbb{C}_p^\times \longrightarrow \mathbb{C}_p \\ \qquad\qquad\quad \rotatebox{90}{$\in$} \qquad\quad \rotatebox{90}{$\in$} \\ \qquad\qquad\quad p \longmapsto a\end{array} \quad \text{s.t} \begin{cases}\cdot \text{ locally analytic group hom} \\ \cdot\ -\sum_{n=1}^{\infty}\frac{(1-z)^n}{n}\end{cases}$$

- From now on, we fix a branch of p-adic logarithm:

$\mathcal{L}i_1^a(z)=-\log^a(1-z) \rightsquigarrow \mathcal{L}i_2^a(z) \rightsquigarrow \cdots\cdots$

$\rightsquigarrow$ We can give an analytic continuation of p-adic MPL^a to $\mathbb{P}^1(\mathbb{C}_p)-\{1,\infty\}$

$\rightsquigarrow \aleph$ branches (attached to $a\in\mathbb{C}_p$)

⑤ **Th** $\lim'_{\substack{z\to 1 \\ z\in\mathbb{C}_p}} \mathcal{L}i^{a}_{k_1,\cdots,k_m}(z)$ converges for $k_m > 1$.

$\Big($ $\lim'$ means the limit value of the sequence $\{z_n\}_{n=1}^{\infty}$

with $z_n \to 1$ and a finite absolute ramification ind

$e\left(\mathbb{Q}_p(z_1, z_2, \cdots\cdots)/\mathbb{Q}_p\right) < \infty$.

dfn (p-adic MZV)

$$\zeta_p(k_1,\cdots,k_m) = \lim'_{z\to 1} \mathcal{L}i^{a}_{k_1,\cdots,k_m}(z) \text{ if it converges.}$$

⑥ **Th** This definition is independent of $a\in\mathbb{C}_p$.

$$\because\ \mathcal{L}i^{a}(1-\varepsilon) = f_0(\varepsilon) + f_1(\varepsilon)\log^a\varepsilon + f_2(\varepsilon)(\log^a\varepsilon)^2 + \cdots\cdots$$

$$f_0(\varepsilon), f_1(\varepsilon), f_2(\varepsilon), \cdots\cdots \in \mathbb{C}_p[[\varepsilon]]$$

⑦ **Prop** $\zeta_p(k_1,\cdots\cdots,k_m) \in \mathbb{Q}_p \subset \mathbb{C}_p$.

⑧ **Th** $\mathcal{Z}^{p}_{\bullet}$ becomes a graded $\mathbb{Q}$-algebra

⑨ **Example** · $\zeta_p(n) = \frac{p^n}{p^n-1} L_p(n, \omega^{1-n})$ (Coleman)

· $\zeta_p(2k) = 0$

· $\zeta_p(1.2) = \zeta_p(3)$

· $\zeta_p(2.1) = -2\zeta_p(3)$

⑩ p-adic KZ equation (Knizhnik-Zamolodchikov equation) of $\mathbb{P}^1\backslash\{0,1,\infty\}$ is the following differential equation

$$dG(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right) G(z)\, dz$$

$$G(z) \in \mathbb{C}_p\langle\langle A.B\rangle\rangle$$

⑪ fundamental solution of p-adic KZ equation

$$G_0^p(z) = 1 + \int_0^z \frac{dt}{t} A + \int_0^z \frac{dt}{t-1} B + \int_0^z \frac{dt}{t} \int_0^t \frac{dt'}{t'-1} AB$$

$$+ \int_0^z \frac{dt}{t-1} \int_0^t \frac{dt'}{t'} BA + \cdots\cdots$$

$$= 1 + \log z A - \mathcal{L}i_1(z) B + \frac{(\log z)^2}{2} A^2$$

$$- \mathcal{L}i_2(z) AB + \cdots\cdots\cdots$$

$$+ (-1)^m \mathcal{L}i_{k_1 \cdots k_m}(z) A^{k_m-1} B \cdots\cdots A^{k_1-1} B + \cdots\cdots$$

⑫ p-adic Drinfel'd associator

$$\Phi^{p}_{KZ}(A,B) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathbb{C}_p}}{}^{p}\ \varepsilon^{-B} \cdot G^{p}_{0}(1-\varepsilon) \in \mathbb{C}_p\langle\langle A.B\rangle\rangle$$

$$\text{"} = G^{p}_{0}(1) \text{"}$$

$$\Phi^{p}_{KZ}(A.B) = 1 - \zeta_p(2)AB + \zeta_p(2)BA$$

$$- \zeta_p(3)A^2B + 2\zeta_p(3)ABA + \zeta_p(1.2)AB^2$$

$$- \zeta_p(3)BA^2 - 2\zeta_p(1.2)BAB + \zeta_p(1.2)B^2A + \cdots\cdots$$

$$\cdots\cdots$$

$$+ (-1)^m \zeta_p(k_1, \cdots, k_m) A^{k_m-1}B \cdots\cdots A^{k_1-1}B + \cdots\cdots$$

Ch3. π_1 $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$

$$\begin{tikzcd}
 & G_{\mathbb{Q}} \ni \sigma & & & \\
\text{étale} & \circlearrowright \ \pi_1^{\ell\,et}(X_{\overline{\mathbb{Q}}}:0,1)(\mathbb{Q}_\ell) \arrow[d, phantom, "\wr|"] & \ni & b, b^\sigma \arrow[d, leftrightarrow] & \\
\text{Betti} & \substack{\mathbb{Q}_\ell \times \\ \mathbb{C} \times} \ \pi_1^{Be}(X(\mathbb{C}):0,1)(\mathbb{Q}) \arrow[d, phantom, "\wr|"] & \ni & b & \pi_1^{top}(X(\mathbb{R}):0, \arrow[d, equal] \\
\text{de Rham} & \substack{\mathbb{C} \times \\ \mathbb{Q}_p \times} \ \pi_1^{DR}(X_{\mathbb{Q}}:0,1)(\mathbb{Q}) \arrow[d, phantom, "\wr|"] & \ni & d & \{b\} \ \text{Deligne} \\
\text{crystal} & \phi \circlearrowright \ \pi_1^{p.rig}(X_{\mathbb{F}_p}:0,1)(\mathbb{Q}_p) & \ni & c & \text{Besser-Vologodsk}
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_1^{\ell.et}(X_{\overline{\mathbb{Q}}}:0)(\mathbb{Q}_\ell) \arrow[r, hook] & \mathbb{Q}_\ell\langle\langle A.B\rangle\rangle \\
f_\sigma = b^{-1}\cdot b^\sigma \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & : \Phi^\ell_\sigma(A.B) \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

: ℓ-adic Ihara associator

$$\begin{tikzcd}
\pi_1^{DR}(X_{\mathbb{Q}}:0)(\mathbb{C}) \arrow[r, hook] & \mathbb{C}\langle\langle A.B\rangle\rangle \\
d^{-1}b \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \Phi_{KZ}(A.B) \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

: Drinfel'd associator

$$\begin{tikzcd}
\pi_1^{p.rig}(X_{\mathbb{F}_p}:0)(\mathbb{Q}_p) \arrow[r, hook] & \mathbb{Q}_p\langle\langle A.B\rangle\rangle \\
d^{-1}c \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \Phi^p_{KZ}(A.B) \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

: p-adic Drinfel'd associator

associator	meta-abelian quotient	$A^{m-1}B$	general
$\mathbb{Q}_\ell\langle\langle A,B\rangle\rangle$ $\rotatebox{90}{$\in$}$ $\Phi^\ell_\sigma(A,B) \longmapsto$ ℓ-adic Ihara associator	$\mathbb{Q}_\ell[[a,b]]$ $\rotatebox{90}{$\in$}$ $F^\ell_\sigma(a,b)$ universal power series of Jacobi sums	$\frac{\kappa^{\text{Soulé}}_{\ell,m}(\sigma)}{(\ell^{m-1}-1)\cdot(m-1)!}$ Soulé element	MSE (by Anderso - Ihara the
$\mathbb{C}\langle\langle A,B\rangle\rangle$ $\rotatebox{90}{$\in$}$ $\Phi_{KZ}(A,B) \longmapsto$ Drinfel'd associator	$\mathbb{C}[[a,b]]$ $\rotatebox{90}{$\in$}$ $\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-a-b)}$ gamma function	$-\zeta(m)$ Riemann zeta value	MZV (by Le-Murakami formula)
$\mathbb{Q}_p\langle\langle A,B\rangle\rangle$ $\rotatebox{90}{$\in$}$ $\Phi^p_{KZ}(A,B) \longmapsto$ p-adic Drinfel'd associator	$\mathbb{Q}_p[[a,b]]$ $\rotatebox{90}{$\in$}$ $\prod_{k=1}^{\infty}\frac{\Gamma_p(1+p^ka)\Gamma_p(1+p^kb)}{\Gamma_p(1+p^ka+p^kb)}$ p-adic gamma function (by Morita)	$\frac{p^m}{1-p^m}L_p(m,\omega^{1-m})$ p-adic L-value	$_p$MZV (by F)

- $$\begin{tikzcd} \phi \circlearrowright \pi_1^{prig}(X_{\mathbb{F}_p}, 0)(\mathbb{Q}_p) \arrow[r, hook] & \phi \circlearrowright \mathbb{Q}_p\langle\langle A.B\rangle\rangle \\ \rotatebox{90}{$\in$} \\ d^{-1}C \arrow[r, mapsto] & \Phi^p_{KZ}(A.B) \\ d^{-1}\phi(d) \arrow[r, mapsto] & \Phi^p_{De}(A.B) \end{tikzcd}$$

$\Phi^p_{KZ}(A.B)$: p-adic Drinfel'd associator $\rightarrow$ p-adic MZV (à la

$\Phi^p_{De}(A.B)$: p-adic Deligne associator $\rightarrow$ p-adic MZV (à la Deligne

- $$\Phi^p_{KZ}(A.B) = 1 - \zeta_p(2)AB + \cdots + (-1)^m \zeta_p(k_1,\cdots,k_m) A^{k_1-1}B\cdots A^{k_m-1}B + $$
$$\nparallel \qquad\qquad\qquad \nparallel$$
$$\Phi^p_{De}(A.B) = 1 - \zeta_p^{De}(2)AB + \cdots + (-1)^m \zeta_p^{De}(k_1,\cdots,k_m) A^{k_1-1}B\cdots A^{k_m-1}B + $$

p-adic MZV (à la F) $\neq$ p-adic MZV (à la Deligne

- $$\begin{tikzcd} \phi : \mathbb{Q}_p\langle\langle A.B\rangle\rangle \arrow[r] & \mathbb{Q}_p\langle\langle A.B\rangle\rangle \\ A \arrow[r, mapsto] & \frac{A}{p} \\ B \arrow[r, mapsto] & \Phi^p_{De}(A.B)^{-1}\frac{B}{p}\Phi^p_{De}(A.B) \end{tikzcd}$$

<u>Th</u> $$\Phi^p_{De}(A.B) = \Phi^p_{KZ}(A.B)\cdot\Phi^p_{KZ}\left(\frac{A}{p}, \Phi^p_{De}(A.B)^{-1}\frac{B}{p}\Phi^p_{De}(A.B)\right)^{-1}$$

$$\rightsquigarrow$$

We can express pMZV (à la Deligne) in terms of pMZV (à la F) and vice versa!!

Example

- $\zeta_p^{\text{ш},\mathrm{De}}(k) = \left(1 - \frac{1}{p^k}\right) \cdot \zeta_p^{\text{ш}}(k)$

- $$\begin{aligned}\zeta_p^{\text{ш},\mathrm{De}}(a, b) = {} & \left(1 - \frac{1}{p^{a+b}}\right) \cdot \zeta_p^{\text{ш}}(a, b) \\ & - \left(\frac{1}{p^b} - \frac{1}{p^{a+b}}\right) \cdot \zeta_p^{\text{ш}}(a) \cdot \zeta_p^{\text{ш}}(b) \\ & - \sum_{r=0}^{a-1} (-1)^r \left(\frac{1}{p^{a-r}} - \frac{1}{p^{a+b}}\right) \cdot \binom{b-1+r}{b-1} \cdot \zeta_p^{\text{ш}}(a-r) \cdot \zeta_p^{\text{ш}}(b+r) \\ & - (-1)^{a+1} \sum_{s=0}^{b-1} \left(\frac{1}{p^{b-s}} - \frac{1}{p^{a+b}}\right) \cdot \binom{a-1+s}{a-1} \cdot \zeta_p^{\text{ш}}(a+s) \cdot \zeta_p^{\text{ш}}(b-s)\end{aligned}$$

Ch 4. GT

$\mathcal{M}_{0,n}$: the moduli space of curves with $(0,n)$-type

<u>e.g</u> $\mathcal{M}_{0,4} = X = \mathbb{P}^1 - \{0,1,\infty\}$

- <u>ℓ-adic étale</u>

$$T^{\ell et} = \left\{ \begin{tikzcd} {} \arrow[d, shift left] \\ \pi_1^{\ell et}(\mathcal{M}_{0,n}) \curvearrowleft \mathfrak{S}_n \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{\ell et}(\mathcal{M}_{0,n-1}) \curvearrowleft \mathfrak{S}_{n-1} \arrow[u, hook, shift left] \arrow[d, dotted, shift left] \\ \pi_1^{\ell et}(\mathcal{M}_{04}) \curvearrowleft \mathfrak{S}_4 \arrow[u, dotted, shift left] \arrow[d, shift left] \\ \pi_1^{\ell et}(\mathbb{G}_m) \arrow[u, hook, shift left] \end{tikzcd} \right\}$$

- <u>crystalline</u>

$$T^{p,rig} = \left\{ \begin{tikzcd} {} \arrow[d, shift left] \\ \pi_1^{p,rig}(\mathcal{M}_{06}) \curvearrowleft \mathfrak{S}_6 \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{p,rig}(\mathcal{M}_{05}) \curvearrowleft \mathfrak{S}_5 \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{p,rig}(\mathcal{M}_{04}) \curvearrowleft \mathfrak{S}_4 \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{p,rig}(\mathbb{G}_m) \arrow[u, hook, shift left] \end{tikzcd} \right\}$$

- <u>Betti</u>

$$T^{Be} = \left\{ \begin{tikzcd} {} \arrow[d, shift left] \\ \pi_1^{Be}(\mathcal{M}_{05}) \curvearrowleft \mathfrak{S}_5 \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{Be}(\mathcal{M}_{04}) \curvearrowleft \mathfrak{S}_4 \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{Be}(\mathbb{G}_m) \arrow[u, hook, shift left] \end{tikzcd} \right\}$$

- <u>de Rham</u>

$$T^{DR} = \left\{ \begin{tikzcd} {} \arrow[d, shift left] \\ \pi_1^{DR}(\mathcal{M}_{05}) \curvearrowleft \mathfrak{S}_5 \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{DR}(\mathcal{M}_{04}) \curvearrowleft \mathfrak{S}_4 \arrow[u, hook, shift left] \arrow[d, shift left] \\ \pi_1^{DR}(\mathbb{G}_m) \arrow[u, hook, shift left] \end{tikzcd} \right\}$$

- $\boxed{\text{bi torsor of tower}}$

$$\begin{array}{ccccc} \mathrm{Aut}\,T^{Be} & \circlearrowright & \mathrm{Isom}(T^{Be}, T^{DR}) & \circlearrowleft & \mathrm{Aut}\,T^{DR} \\ \cap & & \cap & & \cap \\ \prod_n \mathrm{Aut}\,\pi_1^{Be}(\mathcal{M}_{0n}) & & \prod_n \mathrm{Isom}(\pi_1^{Be}(\mathcal{M}_{0n}), \pi_1^{DR}(\mathcal{M}_{0n})) & & \prod_n \mathrm{Aut}\,\pi_1^{DR}(\mathcal{M}_{0n}) \end{array}$$

- bitorsor of Drinfel'd

$$\underline{GT}(k)=\left\{(\lambda,f)\in k^\times\times\underline{F_2}(k)\ \middle|\ \begin{array}{l} (0)\ f\in[\underline{F_2},\underline{F_2}](k)\\ (i)\ f(X,Y)f(Y,X)=1\\ (ii)\ f(Z,X)Z^m f(Y,Z)Y^m f(X,Y)X^m=1\\ \qquad\qquad\qquad \text{for } XYZ=1,\ m=\frac{\lambda-1}{2}\\ (iii)\ f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})\\ \qquad\cdot f(x_{23},x_{34})f(x_{45},x_{51})=1 \text{ in } \underline{P_5}(k) \end{array}\right\}$$

$$\underline{M}(k)=\left\{(\mu,\varphi)\in k^\times\times k\langle\langle A,B\rangle\rangle\ \middle|\ \begin{array}{l} (0)\ \varphi\in\exp[\hat{\mathbb{L}}_k,\hat{\mathbb{L}}_k]\\ (i)\ \varphi(A,B)\varphi(B,A)=1\\ (ii)\ e^{\frac{\mu}{2}A}\varphi(C,A)e^{\frac{\mu}{2}C}\varphi(B,C)e^{\frac{\mu}{2}B}\varphi(A,B)=1\\ \qquad\qquad\qquad \text{for } A+B+C=0\\ (iii)\ \varphi(X_{12},X_{23})\varphi(X_{34},X_{45})\varphi(X_{51},X_{12})\\ \qquad\cdot\varphi(X_{23},X_{34})\varphi(X_{45},X_{51})=1 \text{ in } \underline{UP_5}(k) \end{array}\right\}$$

$$\underline{GRT}(k)=\left\{(c,g)\in k^\times\times k\langle\langle A,B\rangle\rangle\ \middle|\ \begin{array}{l} (0)\ g\in\exp[\hat{\mathbb{L}}_k,\hat{\mathbb{L}}_k]\\ (i)\ g(A,B)g(B,A)=1\\ (ii)\ g(C,A)g(B,C)g(A,B)=1\\ \qquad\qquad\qquad \text{for } A+B+C=0\\ (iii)\ g(X_{12},X_{23})g(X_{34},X_{45})g(X_{51},X_{12})\\ \qquad\cdot g(X_{23},X_{34})g(X_{45},X_{51})=1 \text{ in } \underline{UP_5}(k) \end{array}\right\}$$

$$\begin{tikzcd}
\underline{GT} & \circlearrowright & \underline{M} & \circlearrowleft & \underline{GRT} \\
\text{pro-algebraic group}/\mathbb{Q} \arrow[u] & & \text{pro-algebraic bi-torsor}/\mathbb{Q} \arrow[u] & & \text{pro-algebraic group}/\mathbb{Q} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma \in G_{\mathbb{Q}} \arrow[r] & \underline{\mathrm{Aut}\,T}^{\ell\mathrm{et}}(\mathbb{Q}_\ell) \arrow[d, phantom, sloped, "\simeq"] & \\
& \underline{\mathrm{Aut}\,T}^{\mathrm{Be}\times\mathbb{Q}_\ell} \arrow[r, "\simeq"] \arrow[d, phantom, "\circlearrowright"] & \underline{GT}(\mathbb{Q}_\ell) \ni (\chi_\ell(\sigma), \Phi^\ell_\sigma) \arrow[d, phantom, "\circlearrowright"] \\
& p \in \underline{\mathrm{Isom}(T^{\mathrm{Be}}, T^{\mathrm{DR}})}(\mathbb{C}) \arrow[r, "\simeq"] & \underline{M}(\mathbb{C}) \ni (2\pi i, \Phi_{KZ}) \\
& \underline{\mathrm{Aut}\,T^{\mathrm{DR}}}_{\times\mathbb{Q}_p} \arrow[r, "\simeq"] \arrow[u, phantom, "\circlearrowleft"] \arrow[d, phantom, sloped, "\simeq"] & \underline{GRT}(\mathbb{Q}_p) \ni (\frac{1}{p}, \Phi^p_{KZ}) \arrow[u, phantom, "\circlearrowleft"] \\
& \phi \in \underline{\mathrm{Aut}\,T^{p.r.g}}(\mathbb{Q}_p) &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{étale} & \underline{GT} \arrow[r, leftrightarrow] \arrow[d, phantom, "\circlearrowright"] & \mathbb{G}_m \times \underline{Gal}^{\ell} \\
& {} & \\
\text{Hodge} & \underline{M} \arrow[r, leftrightarrow] \arrow[d, phantom, "\circlearrowleft"] & \operatorname{Spec} \mathcal{Z}_\bullet[\frac{1}{2\pi i}] \\
& {} & \\
\text{crys} & \underline{GRT} \arrow[r, leftrightarrow] \arrow[d, equal] & \mathbb{G}_m \times \operatorname{Spec} \mathcal{Z}^{p}_\bullet \\
& \mathbb{G}_m \times \underline{GRT}_1 &
\end{tikzcd}$$

$\underline{Gal}^{\ell}$: ℓ-adic Galois image pro-algebraic group

$$\mathcal{Z}_\bullet = \bigoplus_{w=0}^{\infty} \langle MZV : wt = w \rangle_{\mathbb{Q}}$$

$$\mathcal{Z}^{p}_\bullet = \bigoplus_{w=0}^{\infty} \langle pMZV : wt = w \rangle_{\mathbb{Q}}$$

$$\rightsquigarrow$$

"a graded quotient"

$$\begin{tikzcd}
\text{étale} & \underline{GrGT} \arrow[r, leftrightarrow, "\overset{?}{\cong}"] \arrow[d, equal] & \underline{Gr\,Gal}^{\ell} \\
\text{Hodge} & \underline{Gr\,M} \arrow[r, leftrightarrow, "\overset{??}{\cong}"] \arrow[d, equal] & \operatorname{Spec} \mathcal{Z}_\bullet/\pi^2 \\
\text{crys} & \underline{GRT}_1 \arrow[r, leftrightarrow, "\overset{???}{\cong}"] & \operatorname{Spec} \mathcal{Z}^{p}_\bullet
\end{tikzcd}$$

conjectured to be $\simeq$.
(by Ihara)

expected to be $\simeq$.

guess Is it $\simeq$?

Drinfel'd