KONTSEVICH'S EYE, LIE GRAPHS AND THE ALEKSEEV-TOROSSIAN ASSOCIATOR

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Abstract. After we recall the definition of Kontsevich's eye $\overline{C}_{2,0}$ and the notion of Lie graphs, we explain how to construct the new associator Φ_{AT} of Alekseev and Torossian by using a holonomy of differential equation, made by Lie graphs, on $\overline{C}_{2,0}$, and also introduce the AT-analogues of multiple zeta values.

We start by recalling the compactified configuration spaces and weights of Lie graphs [K03].

Let $n \geqslant 1$. For a topological space X, we define

$$\mathrm{Conf}_n(X) := \{(x_1, \dots, x_n) \mid x_i \neq x_j \ (i \neq j)\}.$$

The group

$$\mathrm{Aff}_+ := \{x \mapsto ax + b \mid a \in \mathbb{R}_+^\times, b \in \mathbb{C}\}$$

acts on $\mathrm{Conf}_n(\mathbb{C})$ diagonally by rescallings and parallel translations. We denote the quotient by

$$C_n := \mathrm{Conf}_n(\mathbb{C})/\mathrm{Aff}_+$$

for $n \geqslant 2$, which is a connected oriented smooth manifold with dimension $2n - 3$.

Example 1.
- $C_2 \simeq S^1$.
- $C_3 \simeq S^1 \times (\mathrm{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})$.

For a finite set I with $|I| = n$, we put $C_I = C_n$. For $I' \subset I$ with $|I'| > 1$, we have the pull-back map $C_I \twoheadrightarrow C_{I'}$.

Put

$$\mathrm{Conf}_{n,m}(\mathbb{H}, \mathbb{R}) := \mathrm{Conf}_n(\mathbb{H}) \times \mathrm{Conf}_m(\mathbb{R})$$

with the coordinate $(z_1, \dots, z_n, x_1, \dots, x_m)$, where $\mathbb{H}$ is the upper half plane. The group

$$\mathrm{Aff}_+^{\mathbb{R}} := \{x \mapsto ax + b \mid a \in \mathbb{R}_+^\times, b \in \mathbb{R}\}$$

Date: November 16, 2018.

This article is for the proceedings of the 26th Number Theory Summer School on 2018.

acts there diagonally and we denote the quotient by

$$C_{n,m} := \mathrm{Conf}_{n,m}(\mathbb{H},\mathbb{R})/\mathrm{Aff}_+^{\mathbb{R}}$$

for $n,m \geqslant 0$ with $2n+m \geqslant 2$. It is an oriented smooth manifold with dimension $2n+m-2$ and with $m!$ connected components.

Example 2.
- $C_{0,2} \simeq \{\pm 1\}$, $C_{0,2}^+ = \{+1\}$, $C_{0,2}^- := \{-1\}$.
- $C_{1,1} \simeq \{e^{\sqrt{-1}\pi\theta} \bigm| 0<\theta<1\}$.
- $C_{2,0} \simeq \mathbb{H} - \{\sqrt{-1}\}$.

For a finite set I and J with $|I| = n$ and $|J| = m$, we put $C_{I,J} = C_{n,m}$. Then for $I' \subset I$ and $J' \subset J$, we have the pull-back map $C_{I,J} \twoheadrightarrow C_{I',J'}$.

Below we recall [1] Kontsevich's [K03] compactifications $\overline{C}_n$ and $\overline{C}_{n,m}$ of C_n and $C_{n,m}$ à la Fulton-MacPherson (in more detail, consult [Si]):

Definition 3. For a finite set I with $|I| = n$, we put

$$\tilde{C}_I := \tilde{C}_n := \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n z_i = 0\} \cap S^{2n-1}.$$

By identifying it with $\mathbb{C}^n$–diag/Aff_+ (diag $= \{(z,\dots,z) \mid z \in \mathbb{C}\}$), we obtain an embedding $C_I \hookrightarrow \tilde{C}_I$. The compactification

$$\overline{C}_I = \overline{C}_n$$

is a compact topological manifold *with corners* which is defined to be the closure of the image of the associated embedding

$$\mathit{\Phi} : C_I \hookrightarrow \prod_{J \subset I, 1<|J|} \tilde{C}_J.$$

While by the embedding $\mathrm{Conf}_{n,m}(\mathbb{H},\mathbb{R}) \hookrightarrow \mathrm{Conf}_{2n+m}(\mathbb{C})$ sending $(z_1,\dots,z_n,x_1,\dots,x_m) \mapsto (z_1,\dots,z_n,\bar{z}_1,\dots,\bar{z}_n,x_1,\dots,x_m)$, we have an embedding $C_{n,m} \hookrightarrow C_{2n+m}$. By combining it with $\mathit{\Phi}$, we obtain an embedding $C_{n,m} \hookrightarrow \overline{C}_{2n+m}$. The compactification

$$\overline{C}_{I,J} = \overline{C}_{n,m}$$

is a compact topological manifold with corners which is defined to be the closure of the embedding.

They are functorial with respect to the inclusions of two finite sets, i.e. $I_1 \subset I_2$ and $J_1 \subset J_2$ with $\sharp(I_k) = n_k$ and $\sharp(J_k) = m_k$ ($k = 1,2$) yield a natural map $\overline{C}_{n_2,m_2} \to \overline{C}_{n_1,m_1}$.

The stratification of his compactification has a very nice description in terms of trees in [K03] (also refer [CKTB]).

[1] Here we follow the conventions of Bruguières ([CKTB]).

Example 4.
- $\overline{C}_{0,2} = C_{0,2} \simeq \{\pm 1\}$,
- $\overline{C}_{1,1} = C_{1,1} \sqcup C_{0,2} = \{e^{\sqrt{-1}\pi\theta} \mid 0 \leqslant \theta \leqslant 1\}$,
- $\overline{C}_{2,0} = C_{2,0} \sqcup C_{1,1} \sqcup C_{1,1} \sqcup C_2 \sqcup C_{0,2}$.

The $\overline{C}_{2,0}$ is called *Kontsevich's eye* and its each component bears a special name as is indicated in Figure 1. The *upper* (resp. *lower*) *eyelid*

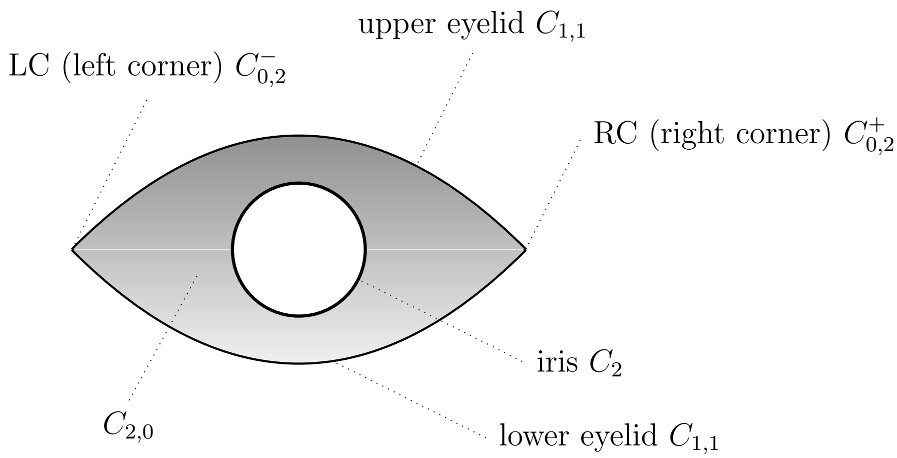

FIGURE 1. Kontsevich's eye $\overline{C}_{2,0}$

corresponds to z_1 (resp. z_2) on the the real line. The *iris* magnifies collisions of z_1 and z_2 on $\mathbb{H}$. LC (resp. RC) which stands for the *left* (resp. *right*) *corner* is the configuration of $z_1 > z_2$ (resp. $z_1 < z_2$) on the real line.

Definition 5. The *angle map* $\phi : \overline{C}_{2,0} \to \mathbb{R}/\mathbb{Z}$ is the map induced from the map $\mathrm{Conf}_2(\mathbb{H}) \to \mathbb{R}/\mathbb{Z}$ sending

$$\phi : (z_1, z_2) \mapsto \frac{1}{2\pi} \arg\left(\frac{z_2 - z_1}{z_2 - \bar{z}_1}\right). \tag{1}$$

We note that ϕ is identically zero on the upper eyelid but is not on the lower eyelid.

Next we will recall the notion of Lie graphs and their weight functions and 1-forms.

Definition 6. Let $n \geqslant 1$. A *Lie graph* Γ *of type* $(n, 2)$ is a graph consisting of two finite sets, the *set of vertices* $V(\Gamma) := \{\boxed{1}, \boxed{2}, (1), (2), \ldots, (n)\}$ and the *set of edges* $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. The points $\boxed{1}$ and $\boxed{2}$ are called as the *ground points*, while the points $(1), (2), \ldots, (n)$ are called as the *air points*. We equip $V(\Gamma)$ with the total order $\boxed{1} < \boxed{2} < ① < ② < \cdots < ⓝ$.

For each $e \in E(\Gamma)$, under the inclusion $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$, we call the corresponding first (resp. second) component $s(e)$ (resp. $t(e)$) as the *source* (resp. the *target*) of e and denote as $e = (s(e), t(e))$. We equip $E(\Gamma)$ with the lexicographic order induced from that of $V(\Gamma)$. Both $V(\Gamma)$ and $E(\Gamma)$ are subject to the following conditions:

(i) An air point fires two edges: That means there always exist two edges with the source ⓘ for each $i = 1, \ldots, n$.
(ii) An air point is shot by one edge at most: That means there exists at most one edge with its target ⓘ for each $i = 1, \ldots, n$.
(iii) A ground point never fire edges: That means there is no edge with its source on ground points.
(iv) The graph Γ becomes a rooted trivalent tree after we cut off small neighborhoods of ground points: That means that the graph of Γ admits a unique vertex (called *the root*) shoot by no edges and it gives a rooted trivalent trees if we regard the vertex as a root and distinguish all targets of edges firing ground points.

Let Γ be a Lie graph of type $(n, 2)$. We define a Lie monomial $\Gamma(A, B) \in \widehat{\mathfrak{f}}_2$ of degree $n+1$ to be the associated element with the root by the following procedure: With $\boxed{1}$ and $\boxed{2}$, we assign A and $B \in \widehat{\mathfrak{f}}_2$ respectively. With each internal vertex v firing two edges $e_1 = (v, w_1)$ and $e_2 = (v, w_2)$ such that $e_1 < e_2$, we assign $[\Gamma_1, \Gamma_2] \in \widehat{\mathfrak{f}}_2$ where Γ_1 and $\Gamma_2 \in \widehat{\mathfrak{f}}_2$ are the corresponding Lie monomials with the vertices w_1 and w_2 respectively. Recursively we may assign Lie elements with all vertices of Γ.

Example 7. Figure 2 is an example of Lie graph of type $(3, 2)$. Its root is ③. The associated Lie elements of the vertices $\boxed{1}$, $\boxed{2}$, ①, ②, ③ are A, B, $[A, B]$, $[B, [A, B]]$, $[B, [B, [A, B]]]$ respectively.

Each $e \in E(\Gamma)$ determines a subset $\{s(e), t(e)\} \subset V(\Gamma)$ with $|V(\Gamma)| = n+2$ which yields a pull-back $p_e : \overline{C}_{n+2,0} \to \overline{C}_{2,0}$. By composing it with the angle map (1), we get a map $\phi_e : \overline{C}_{n+2,0} \to \mathbb{R}/\mathbb{Z}$. The PA[2] $2n$-forms Ω_Γ on $\overline{C}_{n+2,0}$ (which is $2n$-dimensional compact space) associated with Γ is given by the ordered exterior product

$$\Omega_\Gamma := \wedge_{e \in E(\Gamma)} d\phi_e \ \in \Omega^{2n}_{\mathrm{PA}}(\overline{C}_{n+2,0}).$$

Here $\Omega^{2n}_{\mathrm{PA}}(\overline{C}_{n+2,0})$ means the space of PA $2n$-forms of $\overline{C}_{n+2,0}$

[2]'PA' stands for piecewise-algebraic (cf. [KS, HLTV, LV]).

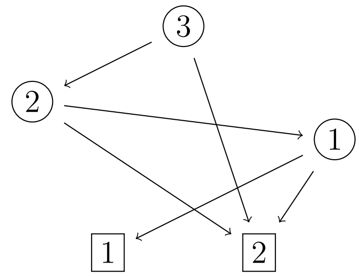

FIGURE 2. $\Gamma(A,B) = [B,[B,[A,B]]]$

Definition 8. (i). Put $\pi : \overline{C}_{n+2,0} \to \overline{C}_{2,0}$ to be the above projection induced from the inclusion $\{\boxed{1},\boxed{2}\} \subset \{\boxed{1},\boxed{2},1,2,\dots,n\}$. The *weight function* (see [To]) of Γ is the smooth function $w_\Gamma : \overline{C}_{2,0} \to \mathbb{C}$ defined by $w_\Gamma := \pi_*(\Omega_\Gamma)$ where π_* is the push-forward (the integration along the fiber of the projection π, cf. [HLTV]), that is, the function which assigns $\xi \in \overline{C}_{2,0}$ with

$$w_\Gamma(\xi) = \int_{\pi^{-1}(\xi)} \Omega_\Gamma \in \mathbb{C}.$$

(ii). We denote $L\Gamma$ (resp. $R\Gamma$) to be a graph obtained from Γ by adding one more edge e_L from $\boxed{1}$ (resp. e_R from $\boxed{2}$) to the root of Γ. The regular $(2n+1)$-form $\Omega_{L\Gamma}$ (resp. $\Omega_{R\Gamma}$) on $\overline{C}_{n+2,0}$ is defined to be

$$\Omega_{L\Gamma} := d\phi_{e_L} \wedge \Omega_\Gamma \qquad (\text{resp.} \quad \Omega_{R\Gamma} := d\phi_{e_R} \wedge \Omega_\Gamma)$$

in $\Omega^{2n}_{\mathrm{PA}}(\overline{C}_{n+2,0})$. The one-forms $\omega_{L\Gamma}$ and $\omega_{R\Gamma}$, which we call the *weight forms* of Γ here, are the PA one-forms of $\overline{C}_{2,0}$ respectively defined by

$$\omega_{L\Gamma} := \pi_*(\Omega_{L\Gamma}) \qquad \text{and} \qquad \omega_{R\Gamma} := \pi_*(\Omega_{R\Gamma})$$

in $\Omega^{1}_{\mathrm{PA}}(\overline{C}_{2,0})$, i.e. they are one-forms respectively defined by

$$\omega_{L\Gamma}(\xi) = \int_{\pi^{-1}(\xi)} \Omega_{L\Gamma}, \qquad \text{and} \quad \omega_{R\Gamma}(\xi) = \int_{\pi^{-1}(\xi)} \Omega_{R\Gamma}$$

where ξ runs over $\overline{C}_{2,0}$.

Remark 9. (i). Particularly the special value $w_\Gamma(\mathrm{RC})$ of the function $w_\Gamma(\xi)$ at $\xi = \mathrm{RC}$ is called the *Kontsevich weight* of Γ and denoted simply by w_Γ. It appears as a coefficient of Kontsevich's formula on deformation quantization in [K03].

(ii). While its restriction $w_\Gamma|_{C_2}$ to the iris C_2 is identically 0 because $\Omega_\Gamma|_{C_2} = 0$ due to the occurrence of double edges.

Example 10. (i). For Γ depicted in Figure 3, by calculations of Torossian [To] we have

- $\omega_\Gamma = (-1)^n \frac{B_n}{n!}$
- $\omega_\Gamma(\theta) = (-1)^n \frac{B_n(\frac{\theta}{\pi})}{n!}$ where θ is the local parameter of the upper eyelid $C_{1,1}$ and $B_n(x)$ is the Bernoulli polynomial defined by $\sum_{n\geqslant 0} \frac{B_n(x)t^n}{n!} = \frac{te^{xt}}{e^t-1}$.
- While the restriction of ω_Γ to lower eyelid is not well-understood.

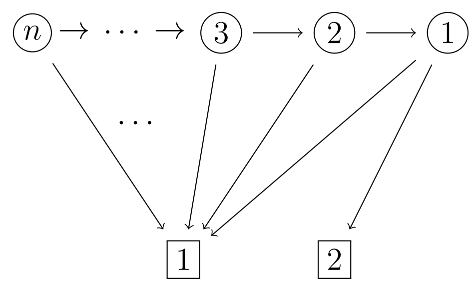

Figure 3. $\Gamma(A,B) = (\mathrm{ad}A)^n(B)$

(ii). G. Felder and Willwacher [FeW] showed that for Γ depicted in Figure 4 we have

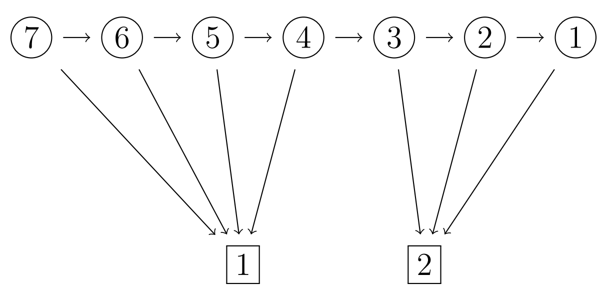

Figure 4. $\Gamma(A,B) = (\mathrm{ad}A)^4(\mathrm{ad}B)^2([A,B])$

$$\omega_\Gamma = a\frac{\zeta(3)^2}{\pi^6} + b$$

with some rational numbers a and b. Since it is conjectured that $\frac{\zeta(3)^2}{\pi^6} \notin \mathbb{Q}$, the Kontsevich weights might not be always rational.

Remark 11. It looks unknown if Kontsevich weights of Lie graphs can be expressed as algebraic combinations of multiple zeta values and $(2\pi\sqrt{-1})^{\pm 1}$ or not.

Let tder_2 be the Lie algebra consisting of tangential derivations $\mathrm{der}(\alpha,\beta):\widehat{\mathfrak{f}}_2\to\widehat{\mathfrak{f}}_2$ $(\alpha,\beta\in\widehat{\mathfrak{f}}_2)$ such that $A\mapsto[A,\alpha]$ and $B\mapsto[B,\beta]$. A connection valued there

$$\omega_{\mathrm{AT}}=\mathrm{der}\,(\omega_L,\omega_R)\in\mathrm{tder}_2\widehat{\otimes}\Omega^1_{\mathrm{PA}}(\overline{C}_{2,0})$$

is introduced in [AT10, To]. Here $\Omega^1_{\mathrm{PA}}(\overline{C}_{2,0})$ means the space of PA one-forms of $\overline{C}_{2,0}$ and

$$\omega_L := B\cdot \quad d\phi+\sum_{n\geqslant 1}\sum_{\Gamma\in\mathrm{LieGra}^{\mathrm{geom}}_{n,2}}\Gamma(A,B)\cdot\omega_{L\Gamma},$$
$$\omega_R := A\cdot\sigma^*(d\phi)+\sum_{n\geqslant 1}\sum_{\Gamma\in\mathrm{LieGra}^{\mathrm{geom}}_{n,2}}\Gamma(A,B)\cdot\omega_{R\Gamma}.$$

with the set $\mathrm{LieGra}^{\mathrm{geom}}_{n,2}$ of *geometric* (it means non-labeled) Lie graphs of type $(n,2)$ (cf. Definition 6). We note that both Ω_Γ and $\Gamma(A,B)$ require the order of $E(\Gamma)$ however their product $\Omega_\Gamma\cdot\Gamma(A,B)$ does not (cf. [CKTB]), whence both ω_L and ω_R do not require labels. The symbol σ stands for the involution of $\overline{C}_{2,0}$ caused by the switch of z_1 and z_2.

In [AT10] they considered the following differential equation on $\overline{C}_{2,0}$ which was shown to be flat:

$$dg(\xi)=-g(\xi)\cdot\omega_{\mathrm{AT}} \tag{2}$$

with $g(\xi)\in\mathrm{TAut}_2:=\exp\mathrm{tder}_2$, the pro-algebraic subgroup of Aut_2 consisting of tangential automorphisms $\mathrm{Int}(\alpha,\beta):\widehat{\mathfrak{f}}_2\to\widehat{\mathfrak{f}}_2$ $(\alpha,\beta\in\exp\widehat{\mathfrak{f}}_2)$ such that $A\mapsto\alpha^{-1}A\alpha$ and $B\mapsto\beta^{-1}B\beta$. They denote its parallel transport (its holonomy) of (2) for the straight path from α_0 (the position 0 at the iris, see Figure 5) to RC by $F_{\mathrm{AT}}\in\mathrm{TAut}_2$.

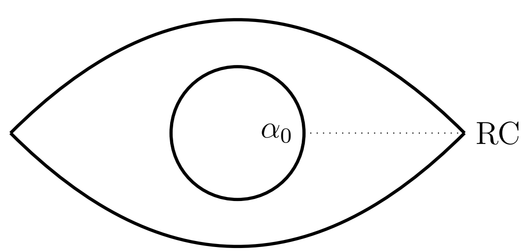

FIGURE 5. Parallel transport

Definition 12 ([AT10])**.** The *AT-associator* $\Phi_{\rm AT}$ is defined to be

$$\Phi_{\rm AT} := F_{\rm AT}^{1,23}\circ F_{\rm AT}^{2,3}\circ (F_{\rm AT}^{1,2})^{-1}\circ (F_{\rm AT}^{12,3})^{-1}\in {\rm TAut}_3. \tag{3}$$

Here for any $T={\rm Int}(\alpha,\beta)\in{\rm TAut}_2$, we denote

$$T^{1,2}:={\rm Int}\left(\alpha(A,B),\beta(A,B),1\right),\quad T^{2,3}:={\rm Int}\left(1,\alpha(B,C),\beta(B,C)\right),$$
$$T^{1,23}:={\rm Int}\left(\alpha(A,B+C),\beta(A,B+C),\beta(A,B+C)\right),$$
$$T^{12,3}:={\rm Int}\left(\alpha(A+B,C),\alpha(A+B,C),\beta(A+B,C)\right)$$

in ${\rm TAut}_3:=\exp{\rm tder}_3$ which is similarly defined to be the group of tangential automorphisms of the completed free Lie algebra $\widehat{\mathfrak f}_3$ with variables A, B and C.

We note that there is a Lie algebra inclusion $\widehat{\mathfrak f}_2\hookrightarrow{\rm tder}_3$ sending

$$A\mapsto t_{12}:={\rm der}(B,A,0)\quad\text{and}\quad B\mapsto t_{23}:={\rm der}(0,C,B) \tag{4}$$

which induces an inclusion $\exp\widehat{\mathfrak f}_2\hookrightarrow{\rm TAut}_3$.

Theorem 13 ([AT12, SW])**.** *The AT-assocciator $\Phi_{\rm AT}$ forms an associator. Namely it belongs to $\exp\widehat{\mathfrak f}_2$ ($\subset\mathbb{C}\langle\langle A,B\rangle\rangle$) and satisfies the equations* [Dr] *(2.12), (2.13) and (5.3). Furthermore it is real (i.e. it belongs to the real structure $\mathbb{R}\langle\langle A,B\rangle\rangle$) and even.* [3]

The following gives a more direct presentation of $\Phi_{\rm AT}$.

Theorem 14 ([Fu18])**.** *We have*

$$\Phi_{\rm AT}=\left(\mathcal{P}\exp\int_{\rm RC}^{\alpha_0}(l_{\widehat\omega}+D_{\widehat\omega})\right)(1)\ \in\mathbb{C}\langle\langle A,B\rangle\rangle. \tag{5}$$

Here $l_{\widehat\omega}$ is the left multiplication by $\widehat\omega$ and $D_{\widehat\omega}$ is given by

$$D_{\widehat\omega}:={\rm der}\left(0,\widehat\omega\right)\in{\rm tder}_2\widehat\otimes\Omega^1_{\rm PA}(\overline{C}_{2,0})$$

with

$$\widehat\omega:=\sum_{n\geqslant1}\sum_{\Gamma\in{\rm LieGra}_{n,2}^{\rm geom}}\widehat\Gamma(A,B)\cdot\omega_\Gamma\quad\text{and}\quad\widehat\omega_\Gamma:=\omega_{R\Gamma}-\omega_{L\Gamma}. \tag{6}$$

and for any one-form $\Omega\in\Omega^1_{\rm PA}(\overline{C}_{2,0})$ we define

$$\mathcal{P}\exp\int_{\rm RC}^{\alpha_0}\Omega:={\rm id}+\int_{\rm RC}^{\alpha_0}\Omega+\int_{\rm RC}^{\alpha_0}\Omega\cdot\Omega+\cdots$$
$$:={\rm id}+\int_{0<s_1<1}\ell^*\Omega(s_1)+\int_{0<s_1<s_2<1}\ell^*\Omega(s_2)\wedge\ell^*\Omega(s_1)+\cdots.$$

with the straight path ℓ from RC to α_0 in Figure 5.

[3] It means $\Phi_{\rm AT}(-A,-B)=\Phi_{\rm AT}(A,B)$, from which it follows that $\Phi_{\rm KZ}\neq\Phi_{\rm AT}$ because $\Phi_{\rm KZ}$ is not even.

This theorem enables us to calculate explicitly all the coefficients of the AT-associator Φ_{AT} as rational linear combinations of iterated integrals of weight forms of Lie graphs (see [Fu18] for explicit computations in depth 1 and 2).

As is explained in [Ha] that *multiple zeta values*, the real numbers defined by the following power series

$$\zeta(k_1,\dots,k_m) := \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}}$$

with $k_1,\dots,k_m \in \mathbb{N}$ and $k_m > 1$ (the condition to be convergent), appear as coefficients of the *KZ-associator* Φ_{KZ}. Particularly its coefficient $(\Phi_{\mathrm{KZ}}|A^{k_m-1}B\cdots A^{k_1-1}B)$ of the monominal $A^{k_m-1}B\cdots A^{k_1-1}B$ is given by

$$(\Phi_{\mathrm{KZ}}|A^{k_m-1}B\cdots A^{k_1-1}B) = (-1)^m\zeta(k_1,\dots,k_m)$$

(cf. [Fu03, LM96b]).

Alm introduced the following AT-analogue of multiple zeta values:

Definition 15 ([Alm])**.** For $k_1,\dots,k_m \in \mathbb{N}$, we define the AT-analogue of multiple zeta values by

$$\zeta_{\mathrm{AT}}(k_1,\dots,k_m) := (-1)^m(\Phi_{\mathrm{AT}}|A^{k_m-1}B\cdots A^{k_1-1}B) \in \mathbb{R}.$$

It was shown in [Alm] that

$$\zeta_{\mathrm{AT}}(n) = -\frac{B_n}{2(n!)},$$

whence in particular it is 0 for all odd n. M. Felder [Fe] calculated

$$\zeta_{\mathrm{AT}}(5,3) = \frac{2048\zeta(3,5) - 6293\zeta(3)\zeta(5)}{524288\pi^8}.$$

It is a challenging problem to present closed formulae describing all $\zeta_{\mathrm{AT}}(k_1,\dots,k_m)$ for general indices $(k_1,\dots,k_m)$ in terms of multiple zeta values.

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