# PROBLEMS ON PROFINITE KNOTS

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Let $\mathcal{K}$ be the set of isotopy classes of oriented (topological) knots, which forms a commutative monoid by the connected sum. Let $\widehat{\mathcal{K}}$ be the monoid of profinite knots constructed in [F]. The set $\widehat{\mathcal{K}}$ forms a topological commatative monoid by the connected sum and there is a natural monoid homomorphism

$$h : \mathcal{K} \to \widehat{\mathcal{K}}$$

whose image is dense in $\widehat{\mathcal{K}}$, as is shown in [F].

**Problem 1.** Is the map $h$ injective?

If it is non-injective, then the Kontsevich knot invariant fails to be perfect.

As for Artin braid group $B_n$ ($n \geqslant 2$), it is known that $B_n$ is residually finite, namely, the natural map

$$B_n \to \widehat{B_n}$$

is injective.

**Problem 2.** Is there any Alexander-Markov-like theorem for profinite links?

One can find several proofs of Alexander-Markov's theorem for topological links ([Bi, T, V, Y] etc). However they look heavily based on a certain finiteness property, which we may not expect the validity for profinite links.

Let Frac $\widehat{\mathcal{K}}$ be the fractional group of $\widehat{\mathcal{K}}$, which forms a topological commutative group. The action of the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of rationals $\mathbb{Q}$ on Frac $\widehat{\mathcal{K}}$ was constructed in [F].

**Problem 3.** Is the $G_{\mathbb{Q}}$-action on Frac $\widehat{\mathcal{K}}$ faithful?

As for the braid groups, the $G_{\mathbb{Q}}$-action on $\widehat{B_n}$ is known to be faithful for $n \geqslant 3$ by Belyĭ's theorem [Be].

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**Problem 4.** Does there exist any (co)homology theory $H_\star$ (or any fundamental group theory $\pi_1^\star$) and any (pro-)variety $X$ defined over $\mathbb{Q}$ such that $H_\star(X_{\overline{\mathbb{Q}}})$ (resp. $\pi_1^\star(X_{\overline{\mathbb{Q}}})$ ) carries a natural $G_{\mathbb{Q}}$-action and Frak $\widehat{\mathcal{K}}$ is identified with $H_\star(X_{\overline{\mathbb{Q}}})$ (resp. $\pi_1^\star(X_{\overline{\mathbb{Q}}})$ ) so that our $G_{\mathbb{Q}}$-action on Frak $\widehat{\mathcal{K}}$ can be derived from the $G_{\mathbb{Q}}$-action there?

## References


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