ON PROFINITE KNOTS AND GALOIS ACTION THERE

HIDEKAZU FURUSHO
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY

ABSTRACT. This is a short survey of the paper [F1] where the notion of profinite knots is introduced and the action of the absolute Galois group of the rational number field there is constructed.

1. PROFINITE BRAIDS

We briefly review several known facts on Galois actions on profinite braid groups.

Let B_n ($n \geqslant 2$) be the *Artin braid group* B_n *with* n*-strings* is the group with generators σ_i ($1 \leqslant i \leqslant n-1$) subject to two relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$. The profinite braid group $\widehat{B}_n$ means its profinite completion. The following is one of the basic properties of braid groups (consult [KT], for example).

Lemma 1. *The braid group B_n ($n \geqslant 2$) is residually-finite, that is, the natural map $B_n \to \widehat{B}_n$ is injective.*

The *absolute Galois group* $G_{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$ means the profinite group

$$G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) := \varprojlim Gal(F/\mathbb{Q})$$

where the limit runs over all finite Galois extension F of $\mathbb{Q}$ and $Gal(F/\mathbb{Q})$ means its Galois group. A geometric continuous $G_{\mathbb{Q}}$-action

$$\rho_n : G_{\mathbb{Q}} \to \mathrm{Aut}\ \widehat{B}_n.$$

($n \geqslant 2$), associated with the arithmetic Galois action on the profinite fundamental group of the moduli space of curves with $(0,n)$-type, is discussed intensively by Drinfeld [D], Ihara [I], etc.

Proposition 2 ([D, IM]). *A pair $(\lambda, f) \in \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2$ can be associated with each $\sigma \in G_{\mathbb{Q}}$ so that the action $\rho_n(\sigma)$ ($n \geqslant 2$) is given by*

$$\sigma(\sigma_i) = f^{-1}_{1\cdots i-1,i,i+1}\sigma_i^{\lambda} f_{1\cdots i-1,i,i+1} \qquad (2 \leqslant i \leqslant n-1).$$

Here the symbol $\widehat{\mathbb{Z}}$ is the profinite completion of the ring $\mathbb{Z}$ of integers, which is nothing but the product $\prod_p \mathbb{Z}_p$ of the ring $\mathbb{Z}_p$ of p-adic integers (p: a prime). The symbol $\widehat{F}_2$ is the profinite completion of the free group F_2. For the symbol $f_{1\cdots i-1,i,i+1}$, see [F1]. The

Date: June 19, 2014.
This article is for the RIMS-kokyuroku of the conference 'Intelligence of Low-dimensional Topology' held during 21st-23rd May, 2014 in RIMS, Kyoto.

following is one of the basic properties of the Galois action ρ_n, which is a consequence of Belyĭ's theorem [Be].

Proposition 3. *The action ρ_n is faithful for $n \geqslant 3$.*

The action is extended into the one of the Grothendieck-Teichmüller group $\widehat{GT}$, a profinite group introduced by Drinfeld [D]. It contains $G_{\mathbb{Q}}$ and is closely related to the philosophy of Teichmüller-Lego posed by Grothendieck in his article 'Esquisse d'un programme' [G].

2. Profinite Knots

This section is a short survey of the paper [F1], where the notion of profinite knots is introduced and the Galois actions on profinite braids (explained in §1) are developed into the ones on profinite knots.

Let $k, l \geqslant 0$. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ and $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_l)$ be sequences (including the empty sequence $\emptyset$) of symbols $\uparrow$ and $\downarrow$.

Definition 4. The set of *fundamental profinite tangles* means the disjoint union of the following three sets A, $\widehat{B}$ and C [1] of symbols:

$$A := \left\{ a^{\epsilon}_{k,l} \;\middle|\; k, l = 0, 1, 2, \ldots,\ \epsilon = (\epsilon_i)_{i=1}^{k+l+1} \in \{\uparrow, \downarrow\}^k \times \{\curvearrowright, \curvearrowleft\} \times \{\uparrow, \downarrow\}^l \right\},$$

$$\widehat{B} := \left\{ b^{\epsilon}_n \;\middle|\; b^{\epsilon}_n = (b_n, \epsilon = (\epsilon_i)_{i=1}^{n}) \in \widehat{B}_n \times \{\uparrow, \downarrow\}^n, n = 1, 2, 3, 4, \ldots \right\},$$

$$C := \left\{ c^{\epsilon}_{k,l} \;\middle|\; k, l = 0, 1, 2, \ldots,\ \epsilon = (\epsilon_i)_{i=1}^{k+l+1} \in \{\uparrow, \downarrow\}^k \times \{\circlearrowleft, \circlearrowright\} \times \{\uparrow, \downarrow\}^l \right\}.$$

Here all arrows are merely regarded as symbols.

We occasionally depict these fundamental profinite tangles with ignorance of arrows (which represent orientation of each strings) as the pictures in Figure 1, which we call their topological pictures.

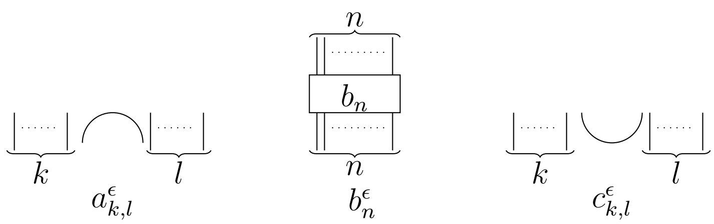

Figure 1. Topological picture of fundamental profinite tangles

For a fundamental profinite tangle γ, its *source* $s(\gamma)$ and its *target* $t(\gamma)$ are sequences of $\uparrow$ and $\downarrow$ defined below:

(1) When $\gamma = a^{\epsilon}_{k,l}$, $s(\gamma)$ is the sequence of $\uparrow$ and $\downarrow$ replacing $\curvearrowright$ (resp. $\curvearrowleft$) by $\uparrow\downarrow$ (resp.$\downarrow\uparrow$) in ϵ and $t(\gamma)$ is the sequence omitting $\curvearrowright$ and $\curvearrowleft$ in ϵ (cf. Figure2).

(2) When $\gamma = b^{\epsilon}_n$, $s(\gamma) = \epsilon$ and $t(\gamma)$ is the permutation of ϵ induced by the image of b^{ϵ}_n of the projection $\widehat{B}_n$ to the symmetric group $\mathfrak{S}_n$ (cf. Figure 3).

[1] A, B and C stand for Annihilations, Braids and Creations respectively.

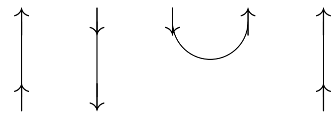

FIGURE 2. $a^{\epsilon}_{2,1}$ with $s(a^{\epsilon}_{2,1}) = \uparrow\downarrow\uparrow$ and $t(a^{\epsilon}_{2,1}) = \uparrow\downarrow\downarrow\uparrow\uparrow$

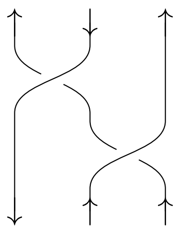

FIGURE 3. An example of b^{ϵ}_3 with $s(b^{\epsilon}_3) = \epsilon = \downarrow\uparrow\uparrow$ and $t(b^{\epsilon}_3) = \uparrow\downarrow\uparrow$

(3) When $\gamma = c^{\epsilon}_{k,l}$, $s(\gamma)$ is the set omitting $\circlearrowleft$ and $\circlearrowright$ in ϵ and $t(\gamma)$ is the set replacing $\circlearrowleft$ (resp. $\circlearrowright$) by $\downarrow\uparrow$ (resp.$\uparrow\downarrow$) in ϵ.

Definition 5. A *profinite (oriented) tangle diagram* means a finite *consistent* [2] sequence $T = \{\gamma_i\}_{i=1}^n$ of fundamental profinite tangles (which we denote by $\gamma_n \cdots \gamma_2 \cdot \gamma_1$). Its source and its target are defined by $s(T) := s(\gamma_1)$ and $t(T) := t(\gamma_n)$. A *profinite link diagram* means a profinite tangle T with $s(T) = t(T) = \emptyset$. A *profinite knot diagram* is a profinite link diagram with a single connected component. (The notion of connected components of profinite tangle diagrams are introduced in [F1].)

Definition 6. For profinite tangles diagram, the moves (T1)-(T6) are defined as follow.

(T1) *Trivial braids invariance*: for a profinite tangle T with $|s(T)| = m$ (resp. $|t(T)| = n$), [3]

$$e_n^{t(T)} \cdot T = T = T \cdot e_m^{s(T)}.$$

Here e^{ϵ}_k means the unit in $\widehat{B}_k$ whose source and targets are both ϵ. Figure 4 depicts the move.

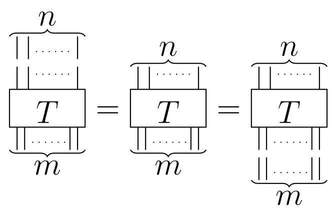

FIGURE 4. (T1): trivial braids invariance

(T2) *Braids composition*: for $b^{\epsilon_1}_n, b^{\epsilon_2}_n \in \widehat{B}$ with $t(b^{\epsilon_1}_n) = s(b^{\epsilon_2}_n)$,

$$b^{\epsilon_2}_n \cdot b^{\epsilon_1}_n = b^{\epsilon_3}_n.$$

[2]Here 'consistent' means successively composable, that is, $s(\gamma_{i+1}) = t(\gamma_i)$ holds for all $i = 1, 2, \dots, n-1$.
[3]For a set S, $|S|$ stands for its cardinality.

Here $b_n^{\epsilon_3}$ means the element in $\widehat{B}$ with $s(b_n^{\epsilon_3})=s(b_n^{\epsilon_1})$ and $t(b_n^{\epsilon_3})=t(b_n^{\epsilon_2})$ which represents the product $b_2\cdot b_1$ of two braids in $\widehat{B}_n$. Figure 5 depicts the move.

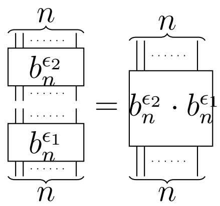

FIGURE 5. (T2): braids composition

(T3) *Independent tangles relation*: for profinite tangles T_1 and T_2 with $|s(T_1)|=m_1$, $|t(T_1)|=n_1$, $|s(T_2)|=m_2$ and $|t(T_2)|=n_2$,

$$(e_{n_1}^{t(T_1)}\otimes T_2)\cdot(T_1\otimes e_{m_2}^{s(T_2)})=(T_1\otimes e_{n_2}^{t(T_2)})\cdot(e_{m_1}^{s(T_1)}\otimes T_2).$$

For the symbol $\otimes$ see [F1]. We occasionally denote both hands side of the above equation simply by $T_1\otimes T_2$. Figure 6 depicts the move.

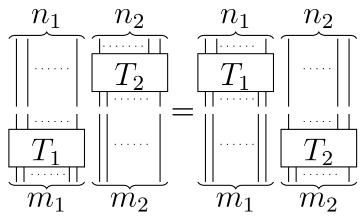

FIGURE 6. (T3): independent tangles relation

(T4) *Braid-tangle relations*: for $b_l^\epsilon\in\widehat{B}$, k with $1\leqslant k\leqslant l$ and a profinite tangle T with $|s(T)|=m$ and $|t(T)|=n$,

$$ev_{k,t(T)}(b_l^\epsilon)\cdot(e_{k-1}^{s_1}\otimes T\otimes e_{l-k}^{s_2})=(e_{k'-1}^{t_1}\otimes T\otimes e_{l-k'}^{t_2})\cdot ev^{k',s(T)}(b_l^\epsilon).$$

For ev, see [F1]. For $s(b_l^\epsilon)=\epsilon=(\epsilon_i)_{i=1}^l$ we put $s_1:=(\epsilon_i)_{i=1}^{k-1}$ and $s_2:=(\epsilon_i)_{i=k+1}^l$. Put $k'=b_l^\epsilon(k)$. Here $b_l^\epsilon(k)$ stands for the image of k by the permutation which corresponds to b_l^ϵ by the projection $B_l\to\mathfrak{S}_l$. For $t(b_l^\epsilon)=(\epsilon_i')_{i=1}^l$ we put $t_1:=(\epsilon_i')_{i=1}^{k'-1}$ and $t_2:=(\epsilon_i')_{i=k'+1}^l$. Figure 7 depicts the move.

(T5) *Creation-annihilation relation*: for $c_{k,l}^\epsilon\in C$ and $a_{k+1,l-1}^{\epsilon'}\in A$ with $t(c_{k,l}^\epsilon)=s(a_{k+1,l-1}^{\epsilon'})$

$$a_{k+1,l-1}^{\epsilon'}\cdot c_{k,l}^\epsilon=e_{k+l}^{s(c_{k,l}^\epsilon)}.$$

And for $c_{k,l}^\epsilon\in C$ and $a_{k-1,l+1}^{\epsilon'}\in A$ with $t(c_{k,l}^\epsilon)=s(a_{k-1,l+1}^{\epsilon'})$

$$a_{k-1,l+1}^{\epsilon'}\cdot c_{k,l}^\epsilon=e_{k+l}^{s(c_{k,l}^\epsilon)}.$$

Figure 8 depicts the move.

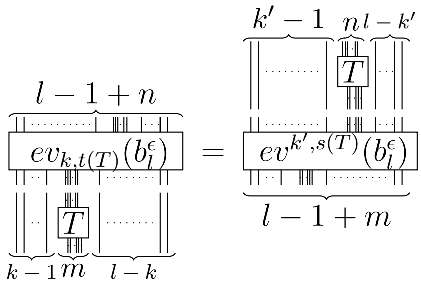

FIGURE 7. (T4): braid-tangle relation

FIGURE 8. (T5): creation-annihilation relations

(T6) *First Reidemeister move*: for $c \in \widehat{\mathbb{Z}}$ [4], $c_{k,l}^{\epsilon} \in C$ and $\sigma_{k+1}^{\epsilon'} \in \widehat{B}$ which represents $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $t(c_{k,l}^{\epsilon}) = s(\sigma_{k+1}^{\epsilon'})$

$$(\sigma_{k+1}^{\epsilon'})^c \cdot c_{k,l}^{\epsilon} = c_{k,l}^{\bar{\epsilon}}$$

where $\bar{\epsilon}$ is chosen to be $t(\bar{\epsilon}) = t((\sigma_{k+1}^{\epsilon'})^c)$.

For $c \in \widehat{\mathbb{Z}}$, $a_{k,l}^{\epsilon} \in A$ and $\sigma_{k+1}^{\epsilon'} \in \widehat{B}$ which represents $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $s(a_{k,l}^{\epsilon}) = t(\sigma_{k+1}^{\epsilon'})$

$$a_{k,l}^{\epsilon} \cdot (\sigma_{k+1}^{\epsilon'})^c = a_{k,l}^{\bar{\epsilon}}.$$

where $\bar{\epsilon}$ is chosen to be $s(\bar{\epsilon}) = s((\sigma_{k+1}^{\epsilon'})^c)$. Figure 9 depicts the moves.

FIGURE 9. (T6): first Reidemeister move

We note that in the first (resp. second) equation $c_{k,l}^{\epsilon} = c_{k,l}^{\bar{\epsilon}}$ (resp. $a_{k,l}^{\epsilon} = a_{k,l}^{\bar{\epsilon}}$) if and only if $c \equiv 0 \pmod 2$.

These moves (T1)-(T6) are profinite analogues of the so-called *Turaev moves* [Tu] for oriented tangles (consult also [CDM, K, O]). Our above formulation is stimulated by the moves (R1)-(R11) presented in [Ba].

Definition 7. Two profinite (oriented) tangle diagrams T_1 and T_2 are *isotopic*, denoted $T_1 = T_2$ by abuse of notation, if they are related by a *finite* number of the moves (T1)-(T6). An *(oriented) profinite tangle* stands for each isotopy class. We denote the set of

[4] It should be worthy to emphasize that c is assumed to be not in $\mathbb{Z}$ but in $\widehat{\mathbb{Z}}$.

profinite tangles by $\widehat{\mathcal{T}}$. Similarly a *profinite knot* stands for each isotopy class of profinite knot diagrams. The set $\widehat{\mathcal{K}}$ of profinite knots is the subset of $\widehat{\mathcal{T}}$ consisting of profinite knots.

We notice that the number of connected components is an isotopic invariant of profinite tangles. The profinite topology on $\widehat{B}_n$ $(n = 1, 2, \dots)$ and the discrete topology on A and on C yield a topology on the space of profinite tangles. Hence $\widehat{\mathcal{T}}$ carries a structure of topological space (cf. [F1]).

Theorem 8 ([F1])**.** *(1). Let $\mathcal{T}$ be the set of isotopy classes of (topological) oriented tangles. There is a natural map*

$$h : \mathcal{T} \to \widehat{\mathcal{T}},$$

which we call the profinite realization map.

(2). The above profinite realization map induces the map

$$h : \mathcal{K} \to \widehat{\mathcal{K}}.$$

Here $\mathcal{K}$ stands for the set of isotopy classes of topological oriented knots.

By abuse of notation, we occasionally denote the image $h(K)$ of $K \in \mathcal{K}$ by the same symbol K. As a knot analogue of the residually-finiteness (Lemma 1) of the braid group B_n, we raise the conjecture below.

Conjecture 9. The map $h : \mathcal{K} \to \widehat{\mathcal{K}}$ is injective.

Remark 10. If the above conjecture, i.e. the injectivity of h, failed, then the Kontsevich invariant [Ko] would fail to be a perfect invariant (cf. [F1]). We remind that the Kontsevich invariant is an invariant of oriented knots which is conjectured to be a perfect invariant, i.e. an invariant detecting all oriented knots.

The notion of connected sum for knots can be extended into profinite knots.

Theorem 11 ([F1])**.** *For any two profinite knot diagrams $K_1 = \alpha_m \cdots \alpha_1$ and $K_2 = \beta_n \cdots \beta_1$ with $(\alpha_m, \alpha_1) = (\curvearrowleft, \curvearrowright)$ and $(\beta_n, \beta_1) = (\curvearrowleft, \curvearrowright)$, their* connected sum *means the profinite tangle diagram defined by*

$$K_1 \sharp K_2 := \alpha_m \cdots \alpha_2 \cdot \beta_{n-1} \cdots \beta_1. \tag{2.1}$$

Then

(1). the above connected sum induces a well-defined product

$$\sharp : \widehat{\mathcal{K}} \times \widehat{\mathcal{K}} \to \widehat{\mathcal{K}}.$$

(2). By the product $\sharp$, the set $\widehat{\mathcal{K}}$ forms a topological (that is, the map $\sharp$ is continuous with respect to the topology given above) commutative associative monoid, whose unit is given by the oriented circle $\circlearrowleft := \curvearrowleft \cdot \curvearrowright$.

(3). The profinite realization map $h : \mathcal{K} \to \widehat{\mathcal{K}}$ forms a monoid homomorphism whose image is dense in $\widehat{\mathcal{K}}$.

For the proofs of Theorem 8 and 11, see [F1].

In knot theory, the so-called Alexander-Markov's theorem is fundamental on constructions of knot invariants. The theorem is to translate knots and links into purely algebraic objects:

Theorem 12 (Alexander-Markov's theorem)**.** *There is a one-to-one correspondence*

$$\mathcal{L} \longleftrightarrow \sqcup_n B_n / \sim_M$$

between the set $\mathcal{L}$ of isotopy classes of oriented links and the (disjoint) union $\sqcup_n B_n$ of braids groups modulo the equivalence $\sim_M$ given by the following Markov moves

(M1). $b_1 \cdot b_2 \sim_M b_2 \cdot b_1 \quad (b_1, b_2 \in B_n)$, *(M2).* $b \in B_n \sim_M b\sigma_n^{\pm 1} \in B_{n+1} \quad (b \in B_n)$

For more on the theorem, consult [CDM, O] for example. The question below is to ask a validity of profinite analogue of Alexander-Markov's theorem.

Problem 13. Is there a 'profinite analogue' of the Alexander-Markov's theorem which holds for the set $\widehat{\mathcal{L}}$ of isotopy classes of profinite links ?

There are several proofs of Alexander-Markov's theorem for topological links ([Bi, Tr, V, Y] etc). But they look heavily based on a certain finiteness property, which we (at least the author) may not expect the validity for profinite links.

Definition 14. The *fractional group of profinite knots* $\mathrm{Frac}\widehat{\mathcal{K}}$ is defined to be the group of fraction of the monoid $\widehat{\mathcal{K}}$, i.e., the quotient space of $\widehat{\mathcal{K}}^2$ by the equivalent relations $(r, s) \approx (r', s')$ if $r\sharp s'\sharp t \sim r'\sharp s\sharp t$ for some profinite knot t, i.e. $r\sharp s'\sharp t = r'\sharp s\sharp t$ holds in $\widehat{\mathcal{K}}$. Occasionally we denote the equivalent class $[(r, s)]$ by $\frac{r}{s}$.

We encode $\mathrm{Frac}\widehat{\mathcal{K}}$ with the quotient topology of $\widehat{\mathcal{K}}^2$. In [F1] it is shown that the product $\sharp$ yields a topological commutative non-trivial group structure on $\mathrm{Frac}\widehat{\mathcal{K}}$.

Problem 15. Is $\mathrm{Frac}\widehat{\mathcal{K}}$ a profinite group?

By [RZ], to show that $\mathrm{Frac}\widehat{\mathcal{K}}$ is a profinite group, we must show that it is compact, Hausdorff and totally-disconnected. The author is not aware of any one of their validities.

Definition 16. Let (r, s) be a pair of profinite knot diagrams with $r = \gamma_{1,m} \cdots \gamma_{1,2} \cdot \gamma_{1,1}$ and $s = \gamma_{2,n} \cdots \gamma_{2,2} \cdot \gamma_{2,1}$ ($\gamma_{i,j}$: profinite fundamental tangle). Let $\sigma \in G_{\mathbb{Q}}$ and (λ, f) be its associated pair (cf. Proposition 2). Define its action by

$$\sigma\left(\frac{r}{s}\right) := \frac{\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}\sharp(\Lambda_f)^{\sharp\alpha(s)}}{\{\sigma(\gamma_{2,n}) \cdots \sigma(\gamma_{2,2}) \cdot \sigma(\gamma_{2,1})\}\sharp(\Lambda_f)^{\sharp\alpha(r)}} \in \mathrm{Frac}\widehat{\mathcal{K}}. \tag{2.2}$$

Here $\sigma(\gamma_{i,j})$ and Λ_f are defined in $\mathrm{Frac}\widehat{\mathcal{K}}$ as follows:

(1) When $\gamma_{i,j} = a^{\epsilon}_{k,l}$, we define

$$\sigma(\gamma_{i,j}) := \gamma_{i,j} \cdot f^{s(\gamma_{i,j})}_{1\cdots k,k+1,k+2}.$$

Figure 10 depicts the action. Here the thickened black band stands for the trivial braid e_k with k-strings. Consult [F1] in more precise.

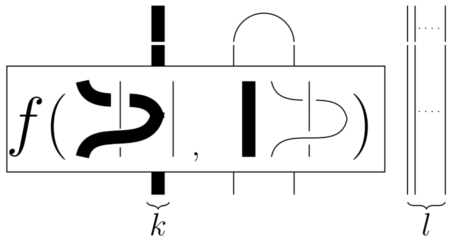

FIGURE 10. $\sigma(a^{\epsilon}_{k,l})$

(2) When $\gamma_{i,j} = b^{\epsilon}_n = (b_n, \epsilon) \in \widehat{B}$, we define

$$\sigma(\gamma_{i,j}) := (\sigma(b_n), \epsilon)$$

which is nothing but the image of $b_n \in \widehat{B}_n$ by the $G_{\mathbb{Q}}$-action on $\widehat{B}_n$ explained in Proposition 2.

(3) When $\gamma_{i,j} = c^{\epsilon}_{k,l}$, we define

$$\sigma(\gamma_{i,j}) := f^{-1,t(\gamma_{i,j})}_{1\cdots k,k+1,k+2} \cdot \gamma_{i,j}.$$

Figure 11 depicts the action.

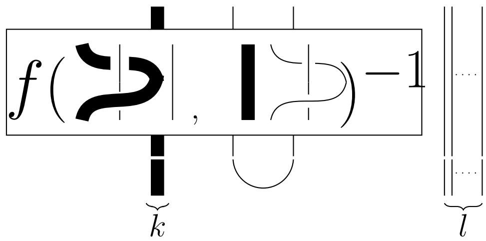

FIGURE 11. $\sigma(c^{\epsilon}_{k,l})$

The symbol Λ_f represents the profinite tangle

$$a^{\curvearrowleft}_{0,0} \cdot a^{\curvearrowleft\downarrow\uparrow}_{0,2} \cdot (e^{\downarrow}_1 \otimes f) \cdot c^{\downarrow\smile\uparrow}_{1,1} \cdot c^{\smile}_{0,0}$$

(cf. Figure 12).

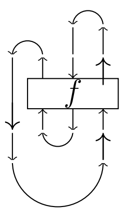

FIGURE 12. Λ_f

The symbol $\alpha(r)$ (resp. $\alpha(s)$) means the number of annihilations; the cardinality of the set $\{j|\gamma_{i,j} \in A\}$ for $i = 1$ (resp. $i = 2$) and $(\Lambda_f)^{\sharp\alpha(r)}$ (resp. $(\Lambda_f)^{\sharp\alpha(s)}$) means the $\alpha(r)$-th (resp. the $\alpha(s)$-th) power of Λ_f with respect to $\sharp$. Particularly we have

$$\sigma(\circlearrowleft)\sharp\Lambda_f = \circlearrowleft \ \in \mathrm{Frac}\widehat{\mathcal{K}}$$

Our main theorem is that the equation (2.2) yields a well-defined $G_{\mathbb{Q}}$-action on $\mathrm{Frac}\widehat{\mathcal{K}}$

$$\rho_0 : G_{\mathbb{Q}} \to \mathrm{Aut}\ \mathrm{Frac}\widehat{\mathcal{K}}.$$

Theorem 17 ([F1]). *(1). $\sigma(\frac{r_1}{s_1}) = \sigma(\frac{r_2}{s_2}) \in \mathrm{Frac}\widehat{\mathcal{K}}$ if $r_1 \sim r_2$ and $s_1 \sim s_2$, i.e. if $r_1 = r_2$ and $s_1 = s_2$ in $\widehat{\mathcal{K}}$.*

(2). $\sigma(\frac{r_1}{s_1}) = \sigma(\frac{r_2}{s_2}) \in \mathrm{Frac}\widehat{\mathcal{K}}$ if $(r_1, s_1) \approx (r_2, s_2)$, i.e. if $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ in $\mathrm{Frac}\widehat{\mathcal{K}}$.

(3). $\sigma_1(\sigma_2(x)) = (\sigma_1 \circ \sigma_2)(x)$ for any $\sigma_1, \sigma_2 \in G_{\mathbb{Q}}$ and $x \in \mathrm{Frac}\widehat{\mathcal{K}}$.

Furthermore $G\widehat{\mathcal{K}}$ forms a topological $G_{\mathbb{Q}}$-module. Namely,

(4). the action is compatible with the group structure, i.e.

$$\sigma(e) = e, \qquad \sigma(x\sharp y) = \sigma(x)\sharp\sigma(y), \qquad \sigma(x^{-1}) = \sigma(x)^{-1}$$

for any $\sigma \in G_{\mathbb{Q}}$ and $x, y \in \mathrm{Frac}\widehat{\mathcal{K}}$.

(5). the action is continuous.

For a proof of the theorem, consult [F1]. Particularly when $\sigma \in G_{\mathbb{Q}}$ is equal to the complex conjugation ς_0, it corresponds to $(\lambda, f) = (-1, 1)$ and its action on $\widehat{B}_n$ is given by $\sigma_i \mapsto \sigma_i^{-1}$ $(1 \leqslant i \leqslant n-1)$. Therefore

Example 18. The action of the complex conjugation ς_0 on $\mathrm{Frac}\widehat{\mathcal{K}}$ is given by

$$\rho_0(\varsigma_0)\,(K) = \overline{K}$$

for each $K \in \mathcal{K}$. Here $\overline{K}$ means the mirror image of K.

As a analogue of Belyĭ's theorem (Proposition 3), the following problem is posed.

Problem 19. Is our action ρ_0 also faithful?

If it turns that it is not faithful, then in that case detecting the corresponding kernel field, which is a Galois extension of $\mathbb{Q}$, might be also an interesting problem.

Remark 20. (1). Actually our $G_{\mathbb{Q}}$-action on profinite knots are extended to a $\widehat{GT}$-action not only on profinite knots but also on profinite *framed* knots (cf. [F1]).

(2). In the subsequent paper [F2], the notion of proalgebraic knots, which is nothing but knots completed by a filtration à la Vassiliev, is introduced and the action of the motivic Galois group there is discussed.

References

[Ba] Bar-Natan, D., *Non-associative tangles*, Geometric topology (Athens, GA, 1993), 139-183, AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.

[Be] Belyĭ, G. V., *Galois extensions of a maximal cyclotomic field*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 267-276, 479.

[Bi] Birman, J. S., *Braids, links, and mapping class groups*, Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974.

[CDM] Chmutov, S., Duzhin, S. and Mostovoy, J., *Introduction to Vassiliev Knot Invariants*, Cambridge University Press, 2012.

[D] Drinfel'd, V. G., *On quasitriangular quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991), no. 4, 829-860.

[F1] Furusho, H., *Galois action on knots I: Action of the absolute Galois group*, preprint, arXiv:1211.5469.

[F2] ———, *Galois action on knots II: Proalgebraic string links and knots*, preprint, arXiv:1405.4575.

[G] Grothendieck, A., *Esquisse d'un programme*, 1983, available on pp. 5-48. London Math. Soc. LNS **242**, Geometric Galois actions, 1, Cambridge Univ.

[I] Ihara, Y., *Braids, Galois groups, and some arithmetic functions*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 99-120, Math. Soc. Japan, Tokyo, 1991.

[IM] ——— and Matsumoto, M., *On Galois actions on profinite completions of braid groups*, Recent developments in the inverse Galois problem (Seattle, WA, 1993), 173-200, Contemp. Math., **186**, Amer. Math. Soc., Providence, RI, 1995.

[K] Kassel, C., *Quantum groups*, Graduate Texts in Mathematics, **155**, Springer-Verlag, New York, 1995.

[KT] ——— and Turaev, V., *Braid groups*, Graduate Texts in Mathematics, **247**, Springer, New York, 2008.

[Ko] Kontsevich, M., *Vassiliev's knot invariants*, I. M. Gelfand Seminar, 137-150, Adv. Soviet Math., **16**, Part 2, Amer. Math. Soc., Providence, RI, 1993.

[O] Ohtsuki, T., *Quantum invariants. A study of knots, 3-manifolds, and their sets*, Series on Knots and Everything, **29**, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[RZ] Ribes, L. and Zalesskii, P., *Profinite groups*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **40**, Springer-Verlag, Berlin, 2010.

[Tr] Traczyk, P., *A new proof of Markov's braid theorem*, Knot theory (Warsaw, 1995), 409–419, Banach Center Publ., **42**, Polish Acad. Sci., Warsaw, 1998.

[Tu] Turaev, V. G., *Operator invariants of tangles, and R-matrices*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 5, 1073-1107, 1135; translation in Math. USSR-Izv. **35** (1990), no. 2, 411-444

[V] Vogel, P., *Representation of links by braids: a new algorithm*, Comment. Math. Helv. **65** (1990), no. 1, 104-113.

[Y] Yamada, S., *The minimal number of Seifert circles equals the braid index of a link*, Invent. Math. **89** (1987), no. 2, 347-356.

Graduate School of Mathematics,
Nagoya University,
Furo-cho, Chikusa-ku, Nagoya, 464-8602,
JAPAN
E-mail address: furusho@math.nagoya-u.ac.jp

名古屋大学大学院多元数理科学研究科　古庄英和