p-adic multiple L-functions and cyclotomic multiple harmonic values

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We show that the special values at tuples of positive integers of the p-adic multiple L-function introduced by Furusho et al. can be expressed in terms of the cyclotomic multiple harmonic values introduced by Jarossay.

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0. Introduction

We start with the multiple zeta function in r variables (r ≥ 1), the following generalization of the Riemann zeta function which is defined by ζ_r((s)_r) := ∑_0<m_1<⋯<m_r, m_1^{-s_1}⋯m_r^{-s_r} for (s)_r := (s_1, ⋯, s_r) ∈ C^r such that Re(s_r−r′+1 + ⋯ + s_r) > r′ for all 1 ≤ r′ ≤ r. Its meromorphic continuation to the whole space C^r has been discussed in several papers among which [11, 14]. In this paper, for a prime p, we consider the p-adic multiple L-function L_{p,r}((s)_r; (ω^{k_i})_r; (1)_r; c), a p-adic function which is defined for ((s)_r; (k_i)_r) ∈ C_p^r × Z^r with |s_i|_p ≤ p^{−1} for all 1 ≤ i ≤ r, where c ≥ 2 is a positive integer prime to p, and ω is the Teichmüller character (cf. Definition 2). It is introduced by Furusho et al. [5], and serves as a p-adic analogue of the above ζ_r and also a generalization of the Kubota–Leopoldt p-adic L-function.
Multiple zeta values are the values of multiple zeta functions at tuples of positive integers, namely \( \zeta((n_1)_r) \) with \( n_i \) positive integers and \( n_r \geq 2 \). They have been intensively studied over the last decade, as an example of periods, and with relations to knot theory, mathematical physics and other branches of mathematics. Cyclotomic multiple zeta values are a generalization of multiple zeta values which are also periods [5, Theorem 3.41]. In this paper, we give a different presentation.

Cyclotomic multiple harmonic values (cf. Definition 3) are a tool for studying \( p \)-adic cyclotomic multiple zeta values, recently introduced by Jarossay [9], as sequences of weighted cyclotomic version of multiple harmonic sums of Hoffman [6] and lifts of cyclotomic version of finite multiple zeta values of Kaneko–Zagier (unpublished). Given a positive integer \( c \) prime to \( p \), cyclotomic multiple harmonic values are certain explicit elements \( \mathcal{H}((n_i)_r; (\epsilon_i)_r) \in \prod_{p \in \mathcal{P}_c} \mathbb{Q}_p(\mu_c) \), where \( \mathcal{P}_c \) is the set of prime numbers which do not divide \( c \), the \( n_i \)'s are positive integers and the \( \epsilon_i \)'s are in the group \( \mu_c \) of \( c \)-th roots of unity; one says that \( ((n_i)_r; (\epsilon_i)_r) \) have weight \( \sum_{i=1}^{r} n_i \) and depth \( r \). For any \( p \in \mathcal{P}_c \), the term indexed by \( p \) in \( \mathcal{H}((n_i)_r; (\epsilon_i)_r) \) has \( p \)-adic valuation \( \geq \sum_{i=1}^{r} n_i \).

The values \( L_{p,r}((n_i)_r; (\omega^{-n_i})_r; (1)_r; c) \) with \( n_1, \ldots, n_r \) positive integers can be expressed as finite \( \mathbb{Q}_c \)-linear combinations of the elements in \( \mathbb{Q}_p(\mu_{pc}) \) which are the values of the \( p \)-adic twisted multiple star polylogarithm at tuples of \( pc \)-th roots of unity [5, Theorem 3.41]. In this paper, we give a different presentation.

**Theorem 1.** For any \( r \)-tuple \( (n_i)_r \) of positive integers, the family

\[
\left( p^{\sum_{i=1}^{r} n_i} L_{p,r}((n_i)_r; (\omega^{-n_i})_r; (1)_r; c) \right)_{p \in \mathcal{P}_c}
\]

is expressed by Eq. (1.1) as an infinite sum of series whose terms are \( \mathbb{Q}(\mu_c) \)-linear combinations of cyclotomic multiple harmonic values with depth \( \leq r \) and weight tending to infinity, and whose convergence holds for the topology on \( \prod_{p \in \mathcal{P}_c} \mathbb{Q}_p(\mu_c) \) of the uniform convergence with respect to \( p \in \mathcal{P}_c \).

This theorem combined with the expression of cyclotomic multiple harmonic values in terms of \( p \)-adic cyclotomic multiple zeta values [5] gives an expression of \( (p^{\sum_{i=1}^{r} n_i} \sum_{c \in \mathcal{P}_c} L_{p,r}((n_i)_r; (\omega^{-n_i})_r; (1)_r; c)) \mathbb{Q}_p(\mu_{pc}) \) in terms of \( p \)-adic cyclotomic multiple zeta values, which are values of \( p \)-adic twisted multiple polylogarithms at tuples of \( c \)-th roots of unity. The sums of series of the above type are interpreted in terms of the Galois theory of \( p \)-adic cyclotomic multiple zeta values in [10].

Our plan of this paper is as follows: we recall the definitions of multiple \( L \)-functions \( L_{p,r} \) and cyclotomic multiple harmonic values in Sec. 1 and then we calculate the decomposition of the domain of the integration of \( L_{p,r} \) in Sec. 2 and a variant of cyclotomic multiple harmonic sums in Sec. 3 which are required to prove our Theorem 1 in Sec. 4.
1. Definitions

We review the definitions of the main objects involved in this paper: the \( p \)-adic multiple \( L \)-functions \( L_{p,r} \), cyclotomic multiple harmonic sums, and cyclotomic multiple harmonic values.

We denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( p \) be a prime number. Put \( \mathcal{O}_p^\times \) to be the ring of integers of \( \mathbb{C}_p \). Let \( \omega : \mathcal{O}_p^\times \to \mathcal{O}_p^\times \) be the Teichmüller character, and let \( \langle x \rangle = \frac{x}{\omega(x)} \) for \( x \in \mathcal{O}_p^\times \). Set \( \int_{\mathcal{O}_p^\times} (x) \, dx \, m(x) = \lim_{N \to -\infty} \sum_{a=0}^{p^{N-1}-1} f(a \mu_1(a + p^N \mathbb{Z}_p)). \) For \( z \in \mathbb{P}^1(\mathbb{C}_p) \) with \( |z - 1|_p \geq 1 \), let \( \mu_2 \) be the measure defined by \( m_2(j + p^N \mathbb{Z}_p) = \frac{z^j}{1 - z^p} \) \( (0 \leq j \leq p^N - 1) \). For \( c \in \mathbb{N} \) prime to \( p \), put \( \tilde{m}_c := \sum_{e=1}^{\xi} m_c \). Set \( r \in \mathbb{N} \) and \( \mathcal{X}_r(d) := \{(s_1, \ldots, s_r) \in C_p^r \mid \sum_{i=1}^{r} s_i \leq dp^{r-1} \} \).

**Definition 2** ([5, Definition 1.16]). Let \( (s_i)_r := (s_1, \ldots, s_r) \in \mathcal{X}_r(q^{-1}), (k_i)_r := (k_1, \ldots, k_r) \in \mathbb{Z}_r^r \), \( c \in \mathbb{N}_{>1} \) which is prime to \( p \), and

\[
(\mathbb{Z}_p^r)' := \{(x_1, \ldots, x_r) \in \mathbb{Z}_p^r \mid p \nmid x_1, p \nmid (x_1 + x_2), \ldots, p \nmid (x_1 + \cdots + x_r)\}. \tag{1.1}
\]

The \( p \)-adic multiple \( L \)-function is defined by

\[
L_{p,r}((s_i)_r; (w^{k_i})_r; (1)_r; c) := \int_{(\mathbb{Z}_p^r)'} \langle x_1 \rangle^{-s_1} \langle x_1 + x_2 \rangle^{-s_2} \cdots \langle x_1 + \cdots + x_r \rangle^{-s_r} \times \omega^{k_1}(x_1) \cdots \omega^{k_r}(x_1 + \cdots + x_r) \prod_{i=1}^{r} d\tilde{m}_c(x_j). \tag{1.2}
\]

In [5] Example 1.19, it is explained that when \( r = 1 \), the Kubota–Leopoldt \( p \)-adic \( L \)-function is recovered:

\[
L_{p,1}(s; w^{k-1}; \gamma; c) = \langle \gamma_1 \rangle^{-s} \omega^{k-1}(\gamma_1)(c^{1-s} \omega^{k}(c) - 1) \cdot L_{p}(s; w^{k}).
\]

**Definition 3.** Let \( (n_i)_r := (n_1, \ldots, n_r) \in \mathbb{N}_r^r \), and \( (\epsilon_i)_r := (\epsilon_1, \ldots, \epsilon_r) \in \mu_c^r \), and \( m \in \mathbb{N} \).

(i) The **cyclotomic multiple harmonic sum** is the element in \( \mathbb{Q}(\mu_c) \) defined by

\[
H_m((n_i)_r; (\epsilon_i)_r) := \sum_{0 < m_1 < \cdots < m_r < m} \left( \frac{\epsilon_2}{\epsilon_1} \right)^{n_1} \cdots \left( \frac{1}{\epsilon_r} \right)^{n_r} m_1^{m_1} \cdots m_r^{m_r} \in \mathbb{Q}(\mu_c).
\]

(ii) ([2, Definition 1.3.1]) Let \( \mathcal{P}_c \) be the set of prime numbers which do not divide \( c \). For \( p \in \mathcal{P}_c \), we also denote by \( \epsilon_i \) \( (1 \leq i \leq r) \) the image of \( \epsilon_i \) by the embedding \( \mathbb{Q}(\mu_c) \hookrightarrow \mathbb{Q}_p(\mu_c) \). The **cyclotomic multiple harmonic value** is the family of multiple harmonic sums defined by

\[
\tilde{f}((n_i)_r; (\epsilon_i)_r) := \langle p^{n_1 + \cdots + n_r} H_p((n_i)_r; (\epsilon_i)_r) \rangle_{p \in \mathcal{P}_c} \in \prod_{p \in \mathcal{P}_c} \mathbb{Q}_p(\mu_c).
\]
For differential forms $\eta_1, \ldots, \eta_n$ in the set \( \{ z_\alpha | z_0 \in \{ 0 \} \cup \mu_c \} \) with $\eta_1 \neq \frac{dz}{z}$, the formal iterated integral $I(\eta_1, \ldots, \eta_n) \in K[[z]]$ is defined, by induction on $n$, by $I(\eta_1) = \int_0^z \eta_1$ and $I(\eta_n, \ldots, \eta_1) = \int_0^z I(\eta_{n-1}, \ldots, \eta_1) \eta_n$. The cyclotomic multiple harmonic sums can be characterized in terms of iterated integrals as follows:

$$I(\omega_0^{j_i-1} \omega_1^{n_{ij}^{(r)}-1} \omega_{t_1-1}^{n_{ij}^{(p)}-1} \omega_{t_2-1}^{n_{ij}^{(p)}}) = (-1)^{r+1} \sum_{0 < m} H_m((n_{ij}^{(r)}); (\epsilon_i^{(r)})) \frac{z^m}{m!}. \quad (1.3)$$

**2. Decomposition of the Domain of the Integration of $L_{p,r}$**

We calculate the decomposition of the domain of the integration of $L_{p,r}(n_1;r; (\omega^{-n_1});(1);c)$ which is required to prove our Theorem 4 in Sec. 4.

For $i, j$ ($i < j$) in $\mathbb{N}_0$, we put $[i,j] := \{i,i+1, \ldots, j\}$. By Definition 2 any value of $L_{p,r}(n_1;r; (\omega^k));(1);c)$ at $(s_1, \ldots, s_r) = (s_1, \ldots, s_r) \in \mathcal{X}_r(q^{-1})$ is given by the limit of finite sums indexed by the finite set

$$D_{r,p,m} = \{ (x_i)_r \in [0,p^M-1]^r | p^i x_1 + \cdots + x_i (1 \leq i \leq r) \}$$

with $M \in \mathbb{N}^*$ tending to $\infty$. The goal of this section is to express $D_{r,p,m}$ in terms of domains of summation underlying multiple harmonic sums and variants. Let

$$\text{div} : (x_1, \ldots, x_r) \in \mathbb{N}_0^r \mapsto ((u_1, \ldots, u_r); (t_1, \ldots, t_r)) \in \mathbb{N}_0^r \times [0,p-1]^r$$

be the map defined by the Euclidean division $x_1 + \cdots + x_i = u_i p + t_i$ for all $i$ ($1 \leq i \leq r$).

**Lemma 4.** For natural numbers $r$ and $M$, the above map $\text{div}$ restricts to a bijection

$$D_{r,p,m} \to \bigcup_{J \in E_r} U_{r,p,m,J} \times T_{r,J},$$

where $E_r$ is the set of triples $J = (P_1, P_2, P_3)$ of subsets of $[1,r]$ such that $[1,r] = P_1 \uplus P_2 \uplus P_3$, and for $J = (P_1, P_2, P_3) \in E_r$,

$$U_{r,p,m,J} := \left\{ (u_1, \ldots, u_r) \in [0,p^M-1]^r \left| \begin{array}{l} i \in P_1 \Rightarrow u_{i-1} = u_i \\ i \in P_2 \Rightarrow u_{i-1} < u_i < u_{i-1} + p^{M-1} \\ i \in P_3 \Rightarrow u_i = u_{i-1} + p^{M-1} \end{array} \right. \right\},$$

$$T_{r,J} := \left\{ (t_1, \ldots, t_r) \in [1,p-1]^r \left| \begin{array}{l} i \in P_1 \Rightarrow t_{i-1} \leq t_i \\ i \in P_3 \Rightarrow t_{i-1} > t_i \end{array} \right. \right\}.$$  

Here we put $u_0 = 0$ and $t_0 = 0$.

**Proof.** The map $(x_1, \ldots, x_r) \mapsto (y_1, \ldots, y_r) = (x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_r)$ sends the set $D_{r,p,m}$ bijectively to $\{(y_1, \ldots, y_r) \in \mathbb{N}^r | 0 < y_1 \leq y_2 \leq \cdots \leq y_r$ and $\forall i, p \upharpoonright y_i \text{ and } y_i - y_{i-1} < p^M}$. If the Euclidean division of $y_i$ by $p$ is given by $y_i = pa_i + t_i,$
we have the equivalences as follows:
(a) \( y_i - 1 \leq y_i \Leftrightarrow (u_{i-1} < u_i) \) or \( (u_{i-1} = u_i \text{ and } t_{i-1} \leq t_i) \),
(b) \( y_i < y_{i-1} + p^M \Leftrightarrow (u_i < u_{i-1} + p^M) \) or \( (t_i < t_{i-1} \text{ and } u_i = u_{i-1} + p^M) \)
from which our result follows.

Let \( \Delta = \Delta_r \) be the set of quasi-simplices, where a quasi-simplex is a couple \((S, \Gamma)\), where \( S = \{a_{1,1}, \ldots, a_{1,t_1}, a_{2,1}, \ldots, a_{2,t_2}, \ldots, a_{k,1}\} \) is a (possibly empty) subset of \( \{1, \ldots, r\} \), called the support of the quasi-simplex, and \( \Gamma = \emptyset \) or
\[
\Gamma = \{(t_{a_{1,1}}, \ldots, t_{a_{k,1}}) \in [1, p - 1]^S \mid t_{a_{1,1}} = \cdots = t_{a_{1,i_1}} < t_{a_{2,1}} = \cdots = t_{a_{2,i_2}} < \cdots < t_{a_{k,1}} = \cdots = t_{a_{k,i_k}} \}.
\tag{2.1}
\]

We will frequently omit \((t_{a_{1,1}}, \ldots, t_{a_{k,1}}) \in [1, p - 1]^S\) in the notation. Let \( \Pi \Delta \) be the set of couples \((S, \Gamma)\) where for all \( i \), \((S, \Gamma_i)\) is in \( \Delta \). The quasi-simplices of support \( S \) form a partition of \([1, p - 1]^S\), thus, for an element \((S, \Gamma)\) of \( \Pi \Delta \), the expression \( \Gamma = \Pi_{i \in r} \Gamma_i \) with \((S, \Gamma_i) \in \Delta\) is unique.

**Proposition 5.** Let \( J = (P_1, P_2, P_3) \) and \( T_{r,j} \) be as in Lemma \[4\] Let \( P'_1 = P_1 \cup \{j \geq 1 \mid j + 1 \in P_1\}, i = 1, 3 \). Then \( (P'_1 \cup P'_2, T_{r,j}) \in \Pi \Delta \).

**Proof.** Let the quasi-shuffle product \(* : \Delta \times \Delta \rightarrow \Pi \Delta \) (a.k.a, harmonic, stuffle product, [4]) be \((S_1, \Gamma_1) * (S_2, \Gamma_2) = (S_1 \cup S_2, (\Gamma_1 \times [1, p - 1]^{S_2 \backslash S_1}) \cap (\Gamma_2 \times [1, p - 1]^{S_1 \backslash S_2}))\). For instance,
\[
\{(1),\{t_1\}\} * (\{2\},\{t_2\}) = \{(1,2),\{(t_1 < t_2) \mid t_2 < t_1\} \cup \{t_1 = t_2\}\}.
\]

Let the following variant \( \tilde{*} \) of the quasi-shuffle product: denoting by \((S, \Gamma) = (S_1, \Gamma_1) * (S_2, \Gamma_2), \) we let \((S_1, \Gamma_1) \tilde{*} (S_2, \Gamma_2) := (S, \Gamma \backslash (\Gamma_1 < \Gamma_2))\) where \( \Gamma_1 < \Gamma_2 \) is the quasi-simplex obtained by combining the “rightmost end” of \( \Gamma_1 \) with the “leftmost end” of \( \Gamma_2 \) in the presentation of \( \Delta_{21} \) by the edge labeled by \( < \). Namely,
\[
\{t_{a_{1,1}} = \cdots < \cdots < = t_{a_{k,1}}\}
\]
\[
< \{t_{b_{1,1}} = \cdots < \cdots < = t_{b'_{(i',j')}}\}
\]
\[
\{t_{a_{1,1}} = \cdots < \cdots < = t_{a_{k,1}} < t_{b_{1,1}} = \cdots < \cdots < = t_{b'_{(i',j')}}\}.
\]

More generally, if \((S, \Gamma) = (S_1, \Gamma_1) \cdots \cdots (S_r, \Gamma_r), \) we put \((S_1, \Gamma_1) \tilde{*} \cdots \tilde{*} (S_r, \Gamma_r) = (S, \Gamma \backslash U)\) where \( U \) is the union of the quasi-shuffle elements which contain a factor of the form \( \Gamma_{i-1} < \Gamma_i \). The products \( * \) and \( \tilde{*} \) extend to elements of \( \Pi \Delta \).

Let \( \Delta = \Delta_r \) be the set defined like \( \Delta \) except that the inequalities \( t_i < t_j \) are replaced by the inequalities \( t_i \leq t_j \). By following the rule \( \{t_i \leq t_j\} = \{t_i = t_j\} \cup \{t_i < t_j, \text{ we have a natural inclusion } \phi : \Delta \hookrightarrow \Pi \Delta \). For instance, \( \phi(((1,2,3) \times \{t_2 \leq t_1 = t_3\}) = \{(1,2,3), \{t_2 < t_1 = t_3\} \cup \{t_2 = t_1 = t_3\}\}.

The canonical increasing connected partition of a subset \( P \subset \{1, \ldots, r\} \) means the unique expression of \( P \) as the disjoint union of the maximal (for the inclusion) segments: \( P = [i_1, j_1] \Pi [i_2, j_2] \cdots [i_r, j_r] \) such that \( i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_r \leq j_r \).
and $i_s - j_{s-1} \geq 2$ for all $s$ ($2 \leq s \leq r$). For each subset $[i, j] = \{i, i+1, \ldots, j\} \subset \{1, \ldots, r\}$, we define $[i, j]_1 := ([i, j], \{t_{i-1} \leq t_i \leq \cdots \leq t_j\}) \in \Delta_r$ and $[i, j]_3 := ([i, j], \{t_j < \cdots < t_s \leq t_{s-1}\}) \in \Delta_r$. Let $C_1(P'_1 \cup P'_3) \Pi \cdots \Pi C_3(P'_1 \cup P'_3)$ be the canonical increasing connected partition of $P'_1 \cup P'_3$. For each $l$ ($1 \leq l \leq u$), let

$$[i'_l, j'_l] \Pi \cdots \Pi [i''_l, j''_l]$$

be the increasing connected partitions of $P'_1 \cap C_l(P'_1 \cup P'_3)$ and $P'_3 \cap C_l(P'_1 \cup P'_3)$, respectively. Then we have

$$P'_1 \cup P'_3, T_{r,j} = \bigstar_{1 \leq l \leq u} \phi([i'_l, j'_l]) \bigstar_{1 \leq l \leq u} \phi([i''_l, j''_l]).$$

(2.2)

3. Computation of a Variant of Cyclotomic Multiple Harmonic Sums

We introduce and investigate a variant of cyclotomic multiple harmonic sums which is necessary to prove Theorem 1.

Let $c \in \mathbb{Z}^r_+ \cap \mathbb{N}$, $c \geq 2$. Put $(l_i)_r = (l_1, \ldots, l_r) \in \mathbb{N}^r_0$ and $(\epsilon_i)_r = (\epsilon_1, \ldots, \epsilon_r) \in \mu^r_c$. Take $h \in \mathbb{N}$, and $(\kappa_i)_{r-1} = (\kappa_1, \ldots, \kappa_{r-1}) \in \mathbb{N}^{r-1}_0$. Let

$$S_{(\kappa_i)_{r-1}, h}((l_i)_r; (\epsilon_i)_r) := \sum_{(u_{1}, \ldots, u_r) \in \mathbb{N}^r_0} \prod_{1 \leq u \leq h} \left( \frac{\epsilon_2}{\epsilon_1} \right)^{u_1} \cdots \left( \frac{1}{\epsilon_r} \right)^{u_r} u_1^{l_1} \cdots u_r^{l_r} \in \mathbb{Q}(\mu_c).$$

(3.1)

The next lemma characterizes the dependence on $g$ of such functions.

**Lemma 6.** For any $(l_i)_r \in \mathbb{N}^r_0$, $(\epsilon_i)_r \in \mu^r_c$, and $(\kappa_i)_{r-1} \in \mathbb{N}^{r-1}_0$, there exists an element $B_{l, (\epsilon_i), (\kappa_i)_{r-1}} \in \mathbb{Q}(\mu_c)$ for each $l \in [0, l_1 + \cdots + l_r + r]$ and $\xi \in \mu_c$, such that, for all $h \in \mathbb{N}$, we have

$$S_{(\kappa_i)_{r-1}, h}((l_i)_r; (\epsilon_i)_r) = \sum_{0 \leq l_1 + \cdots + l_r + r} B_{l, (\epsilon_i), (\kappa_i)_{r-1}} \xi^h.$$ 

(3.2)

Moreover, we have $v_p(B_{l, (\epsilon_i), (\kappa_i)_{r-1}}) \geq -r \left( 1 + \frac{\log((l_1 + \cdots + l_r + r))}{\log(p)} \right)$.  

**Proof.** We prove the existence of the numbers $B_{l, \xi}$ by induction on $r$.

Let us prove the claim for $r = 1$. We have

$$S_{0, h}((l_1; \epsilon_1)) = \sum_{u_1=0}^{h-1} \epsilon_1^{-u_1} u_1^{l_1}.$$ 

- If $\epsilon_1 = 1$, we write $\sum_{u_1=0}^{h-1} u_1^{l_1} = \sum_{i=1}^{h} \frac{1}{i+1} (l_1+1)^{i} B_{l_1+1-i} h^i$, where $B$ denotes the Bernoulli numbers. This defines the coefficients $B_{l, \xi} = \frac{1}{l_1!} (l_1+1)^{i} B_{l_1+1-i}$ if $\xi = 1$ and $1 \leq l \leq l_1 + 1$ and $B_{l, \xi} = 0$ otherwise. By von Staudt–Clausen’s theorem, for all $l_1$, we have $v_p(B_{l_1+1-i}) \geq -1$ and $p^{v_p(l_1+1)} \leq l_1 + 1$ thus $v_p(\frac{1}{l_1!}) \geq -\frac{\log(l_1+1)}{\log(p)}$. Whence the desired bounds on $v_p(B_{l, \xi})$ follows.
If \( \epsilon_1 \neq 1 \), let \( E \) be a formal variable. We have \( \sum_{u=0}^{h-1} u_i^h E^{u_1} = \left(E \frac{d}{dE}\right)^h \left(\sum_{u=0}^{h-1} E^{u_1}\right) = \left(E \frac{d}{dE}\right)^h \left(\frac{E^{h+1}}{1-E}\right) \). Consider a two-variable polynomial \( R_t(x, y) \) with coefficients in \( \mathbb{Z}[x, y] \) inductively constructed by \( R_t(x, y) = (1 - E)^{2t}x^t y^t R_t(x, y) + 2^t E(1 - E)^{2^{t-1}} R_t(x, y) \) and \( R_0(x, y) = 1 - y \). This fulfills \( \left(E \frac{d}{dE}\right)^h \left(\frac{E^{h+1}}{1-E}\right) = \frac{R_t(b, \frac{E}{1-E})}{(1-\epsilon)^2} \) for all \( h \in \mathbb{N} \). Since our \( S_{0,h}(l_1; \epsilon_1) \) is obtained by substituting \( \frac{1}{\epsilon} \) to \( E \) there, it gets that \( S_{0,h}(l_1; \epsilon_1) \) is given by the finite linear combination of \( h^l(\frac{1}{\epsilon_1}) \) \((l, k \geq 0)\) with coefficients in \( \mathbb{Z}[\frac{1}{\epsilon_1}, \frac{1}{\epsilon_{l+1}}] \). By making an adjustment on \( \frac{1}{\epsilon_1} \), we get the presentation (3.2). It is easy to see the coefficients \( B^{(l_1), (\epsilon_1)}_l \) are all in \( \mathbb{Z}[1, \frac{1}{\epsilon_1}, \frac{1}{\epsilon_1-1}] \), which implies that their \( p \)-adic valuation is \( \geq 0 \) because of \( |\epsilon_1|_p = |\epsilon_1 - 1|_p \equiv 1 \).

Assume that our claim holds for \( r \), and let us prove it for \( r+1 \). For any \( (l_i)_{i=1}^{r+1} \in \mathbb{N}_0^{r+1} \), \( (\epsilon_i)_{i=1}^{r+1} \in \mu^{r+1} \), and \( (\kappa_i)_{i=1}^{r+1} \in \mathbb{N}_0^{r+1} \), we write

\[
S_{(\kappa_i)_{i=1}^{r+1}, h}((l_i)_{i=1}^{r+1}; (\epsilon_i)_{i=1}^{r+1})
\]

\[
= \sum_{u_r+\kappa_r<h u_{r+1}+\kappa_{r+1}} \prod_{i=1}^{r} \left( \frac{\epsilon_{i+1}}{\epsilon_i} \right)^{u_i} u_i^{l_i} \times \sum_{u_r+\kappa_r<u_{r+1}+\kappa_{r+1}} \left( \frac{1}{\epsilon_{r+1}} \right)^{u_{r+1}} u_{r+1}^{l_{r+1}}, \tag{3.3}
\]

We have \( [u_r + \kappa_r + 1, u_{r+1} + (\kappa_r + 1)h - 1] = [0, u_r + (\kappa_r + 1)h - 1] \) and \( [u_r + \kappa_r, h - 1] = [0, u_r + \kappa_r, h - 1] \). Thus, by the result for \( r = 1 \),

\[
\sum_{u_r+\kappa_r<h u_{r+1}+\kappa_{r+1}} \left( \frac{1}{\epsilon_{r+1}} \right)^{u_{r+1}} u_{r+1}^{l_{r+1}}
\]

\[
= \sum_{0 \leq l \leq l_{r+1}} \sum_{\xi \in \mu^{r+1}} B^{(l_{r+1}), (\epsilon_{r+1})}_{l, \xi} \left( (u_r + (\kappa_r + 1)h) \xi((u_r + (\kappa_r + 1)h) \right) - (u_r + \kappa_r) \xi((u_r + \kappa_r, h) \right) \left( \frac{1}{\epsilon_{r+1}} \right)^{u_{r+1}} u_{r+1}^{l_{r+1}}. \tag{3.4}
\]

By expanding \( (u_r + (\kappa_r + 1)h) \)

\[
= \sum_{l=0}^{l_{r+1}} \left( \frac{l}{l_{r+1}} \right) u_r^{l_r} \kappa_r^{l_r} \xi((\kappa_r) l_{r+1}^{l_{r+1}}, \kappa_r, h) l_{r+1}^{l_{r+1}}
\]

and \( (u_r + \kappa_r, h) l_{r+1}^{l_{r+1}} = \sum_{l=0}^{l_{r+1}} \left( \frac{l}{l_{r+1}} \right) u_r^{l_r} \kappa_r^{l_r} \xi((\kappa_r) l_{r+1}^{l_{r+1}} - h \xi((\kappa_r + 1)h) \right) \) in (3.4), we transform (3.3) to the following induction formula:

\[
S_{(\kappa_i)_{i=1}^{r+1}, h}((l_i)_{i=1}^{r+1}; (\epsilon_i)_{i=1}^{r+1})
\]

\[
= \sum_{0 \leq l \leq l_{r+1}} \sum_{\xi \in \mu^{r+1}} B^{(l_{r+1}), (\epsilon_{r+1})}_{l, \xi} \left( \frac{l}{l_{r+1}} \right) h^{l_{r+1}} \left( (\kappa_r + 1)h \xi((\kappa_r + 1)h) \right) \right) .
\]


The definition of the coefficients \( v_{(\kappa_i),h}((l_i)_{r-1}, l_r + \tilde{r}; (\epsilon_r^{-1} \epsilon_i)_r) \) are immediate by our induction hypotheses:

\[
S_{(\kappa_i),h}((l_i)_{r-1}, l_r + \tilde{r}; (\epsilon_i)_r) = \sum_{i=0}^{l_r+1} \binom{l_r+1}{i} (\kappa_i h)^{l_r+1-i} \epsilon_r^{-1} \epsilon_i \cdot S_{(\kappa_i),h}((l_i)_{r-1}, l_r + \tilde{r}; (\epsilon_i)_r).
\]

(3.5)

The definition of the coefficients \( B_{(\xi)}^{((l_i)_{r-1}, (\epsilon_i)_{r+1}, (\kappa_i)_r)} \) are deduced from (3.5) by applying the induction hypothesis to \( S_{(\kappa_i),h}((l_i)_{r-1}, l_r + \tilde{r}, \epsilon_1, \ldots, \epsilon_{r-1}, \epsilon_r \xi) \) and \( S_{(\kappa_i),h}((l_i)_{r-1}, l_r + \tilde{r}, \epsilon_1, \ldots, \epsilon_{r-1}, \epsilon_r \epsilon_{r+1}) \). The bounds on their valuations are immediate by our induction hypotheses:

\[
v_p(B_{(\xi)}^{((l_i)_{r-1}, (\epsilon_i)_{r+1}, (\kappa_i)_r)}) \geq - \left( 1 + \frac{\log(l_r+1+1)}{\log p} \right) - r \left( 1 + \frac{\log(l_1+\ldots+l_r+r)}{\log p} \right) \geq -(r+1) \left( 1 + \frac{\log(l_1+\ldots+l_r+1+r)}{\log p} \right).
\]

\[\Box\]

The next lemma expresses certain sums over the set \( U_{r,p^M,J} \) appearing in Lemma 4 in terms of the functions of (3.1).

**Lemma 7.** Let \( J = (P_1, P_2, P_3) \) be as in Lemma 4 (we take the convention that \( 1 \in P_2 \)). For any \((l_i)_r \in \mathbb{N}_0^r, (\epsilon_i)_r \in \mathbb{M}_r^r\), we have

\[
\sum_{(u_i) \in U_{r,p^M,J}} \left( \frac{\epsilon_2}{\epsilon_1} u_1 \cdots \frac{1}{\epsilon_r} u_1 \cdots u_r \right) = \sum_{0 \leq l_i \leq (P_{2,3})} \left( \prod_{j \in P_3} \left( \frac{1}{\tilde{l}_j} \left( \frac{\epsilon_j (P_{2,3})}{\tilde{l}_j} \right)^{\kappa_j p^{M-1}} \right) \right)
\]

\[
S_{(\kappa_i) \in P_2 \setminus (1), p^{M-1}} \left( \left( \frac{1}{l_i (P_{2,3})} + \sum_{j \in P_3} \tilde{l}_j \right) ; (\epsilon_i)_i \in P_2 \right).
\]

(3.6)

Here, for \( j \in P_2 \cup P_3 \), we put

\[
\begin{align*}
l_j (P_{2,3}) &= \min((P_2 \cup P_3) \cap [j+1, r]) - 1, \\
\epsilon_j (P_{2,3}) &= \epsilon_{\min((P_2 \cup P_3) \cap [j+1, r])}.
\end{align*}
\]

For \( j \in P_3 \), we put

\[
\begin{align*}
l_j (P_2) &= \max[1, j - 1] \cap P_2, \\
k_j &= \#(P_3 \cap [j, l_j]) - 1.
\end{align*}
\]
And, for $i \in P_2$, we put
\[
\begin{align*}
\{ i(P_2) &= \min \{ i + 1, r \} \cap P_3, \\
\kappa_i &= \min \{ i, i(P_2) \}. 
\end{align*}
\]

**Proof.** Put $\epsilon_{r+1} = 1$. We have, for any $(u_i)_r \in U_{r, p^M, J}$,
\[
\prod_{i \in [1, r]} \left( \frac{\epsilon_{i+1}}{\epsilon_i} \right)^{u_i} u_i^{l_i} = \prod_{i \in P_2 \cup P_3} \left( \frac{\epsilon_i^{(P_2, 3)}}{\epsilon_i} \right)^{u_i} u_i^{l_i^{(P_2, 3)}} \\
= \prod_{i \in P_2} \left( \frac{\epsilon_i^{(P_2, 3)}}{\epsilon_i} \right)^{u_i} u_i^{l_i^{(P_2, 3)}} \prod_{j \in P_3} \left( \frac{\epsilon_j^{(P_2, 3)}}{\epsilon_j} \right)^{u_j^{(P_2, 3) + \kappa_j p^{M-1}}} \\
\times (u_j^{(P_2, 3) + \kappa_j p^{M-1}})^{l_j^{(P_2, 3)}} \\
= \prod_{i \in P_2} \left( \frac{\epsilon_i^{(P_2, 3)}}{\epsilon_i} \right)^{u_i} u_i^{l_i^{(P_2, 3)}} \prod_{j \in P_3} \left( \frac{\epsilon_j^{(P_2, 3)}}{\epsilon_j} \right)^{u_j^{(P_2, 3) + \kappa_j p^{M-1}}} \\
\times \left\{ \sum_{l_j = 0}^{l_j^{(P_2, 3)}} \left( l_j^{(P_2, 3)} \right)^{u_j^{(P_2, 3) + \kappa_j p^{M-1}}} \right\}.
\]

For $i \in P_2$, let $i^{(P_2)} = \min (P_2 \cap [i + 1, r])$ and $\epsilon_i^{(P_2)} = \epsilon_i^{(P_2)}$. We have
\[
\prod_{i \in P_2} \left( \frac{\epsilon_i^{(P_2, 3)}}{\epsilon_i} \right)^{u_i} \prod_{j \in P_3} \left( \frac{\epsilon_j^{(P_2, 3)}}{\epsilon_j} \right)^{u_j^{(P_2, 3)}} = \prod_{i \in P_2} \left( \frac{\epsilon_i^{(P_2, 3)}}{\epsilon_i} \right)^{u_i} \left( \prod_{j \in P_3 : j^{(P_2)} = i} \frac{\epsilon_j^{(P_2, 3)}}{\epsilon_j} \right)^{u_i} \\
= \prod_{i \in P_2} \left( \frac{\epsilon_i^{(P_2, 3)}}{\epsilon_i} \right)^{u_i}.
\]

We obtain the result by summing over all $(u_i)_r \in U_{r, p^M, J}$.

\[\square\]

4. **End of the Proof of Theorem**

We finish the proof of Theorem

**Proof.** Let $(n_i)_r \in \mathbb{N}^r$. By the definition of $L_{p, r}$ and by the definition of $D_{r, p^M}$ (Sec. 2.1), we have
\[
L_{p, r}((n_i)_r; (\omega^{-n_i})_r; (1)_r; c) = \lim_{n \to \infty} \sum_{(x_i)_r \in D_{r, p^M}} \prod_{i=1}^{r} \frac{1}{(x_1 + \cdots + x_i)^{n_i}} \sum_{\xi_i \in \mu \setminus \{1\}} \frac{\xi_i^{2}}{1 - \xi_i^{p^M}}.
\]
Put \( M \in \mathbb{N} \) and \((x_i)_i \in D_{r,p,M} \). Let \((u_i)_i; (t_i)_i\) be its image by the map div. For all \( r \in [1,r] \), we write \((x_1 + \cdots + x_r)^{-n_i} = (pu_i + t_i)^{-n_i} = \sum_{i \in \mathbb{N}_0} (-n_i) r_i^{-n_i} \left( \frac{M}{\xi_i} \right)^{l_i} \).

Then, by Lemma 4, \( \Delta \left( \frac{M}{\xi_i} \right) \) be its image by the map div. For each quasi-simplex \( \delta \in \Delta(T_{r,J}) \) with the presentation \( \mu \), the index \( n \) with \( n \in \mathbb{N}_0 \), and the pair \( w := ((n_i)_i; (\xi_i)_i) \in \mathbb{N}^r \times \mu^r \), we set \( w(\delta) := \mu(n) \delta; (\xi(1) \delta) \) with \( \delta \in (\mu_\mathcal{C} \backslash \{1\})^r \), then Eq. (2.2) implies

\[
\sum_{(t_i)_i \in T_{r,J}} \prod_{i=1}^r (pu_i)^{\xi_i, p=1} \prod_{(t_i)_i \in T_{r,J}} \delta \left( \frac{M}{\xi_i} \right)^{l_i} \sum_{\delta \in \Delta(T_{r,J})} H_{\mu(w(\delta))}.
\] (4.2)

By applying Lemma 7 and then Lemma 6 we have, for all \( (l_i)_i \in \mathbb{N}^r_0 \) and \( (\xi_i)_i \in (\mu_\mathcal{C} \backslash \{1\})^r \),

\[
\sum_{(u_i)_i \in U_{r,p,M,J}} \prod_{i=1}^r (u_i)^{\xi_i, p=1} \left\{ \prod_{j \in P_3} (K_j p^{M-1})^{-1} \left( \frac{\xi_j (P_{2,3})}{\xi_j} \right) \left( \frac{(P_{2,3})}{L_j} \right) \right\}
\]

\[
\cdot \mathcal{S}_{(\xi(1)_i)_{i \in P_3}, p=1} \left( \left( \frac{(P_{2,3})}{L_j} \right) + \sum_{j \in P_3} \hat{l}_j \right) ; (\xi_i^{-p})_{i \in P_2}
\]

\[
= \sum_{0 \leq \hat{l}_j \leq (P_{2,3})} \left\{ \prod_{j \in P_3} (K_j p^{M-1})^{-1} \left( \frac{\xi_j (P_{2,3})}{\xi_j} \right) \left( \frac{(P_{2,3})}{L_j} \right) \right\}
\]

\[
= \sum_{0 \leq \hat{l}_j \leq (P_{2,3})} \left\{ \prod_{j \in P_3} (K_j p^{M-1})^{-1} \left( \frac{\xi_j (P_{2,3})}{\xi_j} \right) \left( \frac{(P_{2,3})}{L_j} \right) \right\}
\]


\[ \sum_{0 \leq l \leq l_1 + \ldots + l_r + r} B_{l, \xi} \]

\[ \times (p^{M-1})^{l_1 \xi^{p^{M-1}}} \]  

(4.3)

Substituting Eqs. 4.2 and 4.3 in Eq. 4.1, and multiplying by \( p^{n_1 + \ldots + n_r} \), we obtain

\[ p^{n_1 + \ldots + n_r} L_{p,r}((n_i)_r; (\omega^{-n_i})_r; (1)_r; c) \]

\[ = \lim_{M \to \infty} \sum_{l=1}^{(l, r) \in \mathcal{N}_0} \prod_{1=1}^{r} \left( -n_1 \right) \left( \frac{\xi_j}{\xi_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j^{(P_2, 3)}} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \]  

\[ \sum_{0 \leq l \leq l_1 + \ldots + l_r + r} B_{l, \xi} \]

\[ \sum_{\delta \in \Delta(T_{r, l})} p^{\text{weight}(\omega(l)_\delta)} H_p(\omega(l)_\delta). \]  

(4.4)

For any \( b \in \mathbb{N} \), \( c \in \mu_\xi \), \( (\xi)_r \in (\mu_\xi \setminus \{1\})^r \), we define \( A_{b, r, (\xi)_r}^{(n_1)_r} \) as follows, which does not depend on \( M \):

\[ A_{b, r, (\xi)_r}^{(n_1)_r} = \sum_{J=(P_1, P_2, P_3) \in E_r} \sum_{l=1}^{(l, r) \in \mathcal{N}_0} \prod_{1=1}^{r} \left( -n_1 \right) \left( \frac{\xi_j}{\xi_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j^{(P_2, 3)}} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \]

\[ \times \prod_{i=1}^{r} \left( -n_1 \right) \left( \frac{\xi_j}{\xi_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j^{(P_2, 3)}} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \]

\[ \times \prod_{i=1}^{r} \left( -n_1 \right) \left( \frac{\xi_j}{\xi_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j^{(P_2, 3)}} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \left( \frac{\xi_j^{(P_2, 3)}}{l_j} \right) \]  

\[ \times B_{l, \xi} \]

\[ \sum_{\delta \in \Delta(T_{r, l})} p^{\text{weight}(\omega(l)_\delta)} H_p(\omega(l)_\delta). \]  

(4.5)
The convergence of the right-hand side of (4.5) follows from that it is a sum of series over \((l_1)_r \in \mathbb{N}_0^r\) whose general term has valuation \(\geq n_1 + \cdots + n_r + l_1 + \cdots + l_r - r(1 + \log(l_1 + \cdots + l_r + r))\), which tends to \(\infty\) when \(\sum_{i=1}^r l_i \to \infty\). Indeed, for all \((m_i)_r \in \mathbb{N}^r\) and \((\epsilon_i)_r \in \mu^r_c\), we have clearly
\[
v_p(p^{n_1 + \cdots + n_r}H_p((m_i)_r, (\epsilon_i)_r)) \geq m_1 + \cdots + m_r
\]
and for all \(J \in E_r\), \((l_i)_r \in \mathbb{N}_0^r\) and \((\tilde{l}_i)_i \in \mathbb{P}_2 \cup \{0\}, \tilde{l}_i(\mathbb{P}_2, 3)\), the following first inequality follows from Lemma 6 and the second one follows from \(\sharp P_2 \leq r\) and Lemma 7.

Reordering Eq. (4.4), we obtain
\[
p^{n_1 + \cdots + n_r}L_{p, r}((n_i)_r; (\omega^{-n_i})_r; (1)_{r}; c) = \lim_{M \to \infty} \sum_{0 \leq k \leq M} \sum_{\xi \in \mu^r_c} (p^{-M})^k c^{-M} A_{b, c, (\xi)_r} \prod_{i=1}^r (1 - \xi_i p^{-M}) \]

The convergence of (4.6) is proved as follows:
\[
v_p(A_{b, c, (\xi)_r}^{(n_i)_r}) \geq n_1 + \cdots + n_r + \inf \left\{ l_1 + \cdots + l_r - r \left( 1 + \frac{\log(l_1 + \cdots + l_r + r)}{\log(p)} \right) \right\} \geq n_1 + \cdots + n_r + \min \left\{ l_1 + \cdots + l_r - r \left( 1 + \frac{\log(l_1 + \cdots + l_r + r)}{\log(p)} \right) \right\} \times \sum_{i=1}^r l_i \in [0, 3r] \]
\[
\geq n_1 + \cdots + n_r - r \left( 1 + \frac{\log(4r)}{\log(p)} \right).
\]
Here the first inequality follows from the above discussion; the second inequality follows from the fact that the function \( f : t \in (-r, \infty) \mapsto t - r \{ 1 + \frac{\log(1+t)}{\log(p)} \} \in \mathbb{R} \) is increasing on \( \left[ \frac{r}{\log(p)} - r, \infty \right) \) which contains the interval \([3r, \infty)\); the third inequality follows from the fact that, for \((l_i)_r \in \mathbb{N}_0^r\) such that \(\sum_{i=1}^r l_i \in [0, 3r]\), we have \(\sum_{i=1}^r l_i \geq 0\) and \( -\log(\sum_{i=1}^r l_i + r) \geq -\log(4r) \).

We can compute the limit in (4.6) by restricting \( M \) to the case with \( p^{M-1} \equiv 1 \mod c \). For such \( M \)'s, we have \( \xi^{p^{-M+1}} = \xi \) for all \( \xi \in \mu_c \), and, since \( p^{M-1} \to 0 \) when \( M \to \infty \), the limit in (4.6) is equal to

\[
\sum_{\epsilon \in \mu_c \setminus \{1\}} \sum_{\substack{(\xi_i)_r \in (\mu_r \setminus \{1\})^r \setminus \{0\}}} \prod_{i=1}^r (1 - \xi_i^p) A^{(n_i)_r}_{\epsilon, (\xi_i)_r}.
\]

This is true for any \( p \in \mathcal{P}_c \). Thus, we deduce

\[
(p^{n_1 + \cdots + n_r} \mathcal{L}_{p,r}((n_i)_r; (\omega^{-n_i})_r; (1)_r; c))_{p \in \mathcal{P}_c} = \sum_{l=(l_i)_r \in \mathbb{N}_0^r} \sum_{\substack{\epsilon \in \mu_c \setminus \{1\} \setminus \{0\}}} \prod_{i=1}^r (1 - \xi_i^p) \]

\[
\times \sum_{J=(P_1, P_2, P_3) \in \mathcal{E}_r} \sum_{\xi \in \mu_c \setminus \{1\}} \prod_{i=1}^r (-n_i) \]

\[
\Pi_{j \in P_3} \xi_j^{\xi_j''} \xi_j = \epsilon \]

\[
\Delta((l_i^{(P_2, 3)} + \sum_{j \in P_3} i_j^{(P_2, 3)})_{P_2, (\xi''_r)_r} \in \mathcal{P}_2, (\kappa_i)_r \in \mathcal{P}_2 \setminus \{1\}) B_{0, \xi} \sum_{\delta \in \Delta(T_{r, r})} \delta(w(1)_\delta) \]

By adjusting \( p \)-th powers, thus we reformulate it to be

\[
(p^{n_1 + \cdots + n_r} \mathcal{L}_{p,r}((n_i)_r; (\omega^{-n_i})_r; (1)_r; c))_{p \in \mathcal{P}_c} = \sum_{l=(l_i)_r \in \mathbb{N}_0^r} \sum_{\substack{\epsilon \in \mu_c \setminus \{1\} \setminus \{0\}}} \prod_{i=1}^r (1 - \xi_i^p) \]

\[
\times \sum_{J=(P_1, P_2, P_3) \in \mathcal{E}_r} \sum_{\xi \in \mu_c \setminus \{1\}} \prod_{i=1}^r (-n_i) \]

\[
\Pi_{j \in P_3} \xi_j^{\xi_j''} \xi_j = \epsilon \]

\[
\Delta((l_i^{(P_2, 3)} + \sum_{j \in P_3} i_j^{(P_2, 3)})_{P_2, (\xi''_r)_r} \in \mathcal{P}_2, (\kappa_i)_r \in \mathcal{P}_2 \setminus \{1\}) B_{0, \xi} \sum_{\delta \in \Delta(T_{r, r})} \delta(w(1)_\delta)^{\text{prob}^{-1}}. \tag{4.7}
\]
Here for $(x_p)_p \in \prod_{p \in \mathbb{P}} \mathbb{Q}_p(\mu_c)$, we define $(x_p)^{\text{Frob}^{-1}}$ to be $(\text{Frob}^{-1}_p(x_p))_p$ with the Frobenius automorphism $\text{Frob}_p : \mathbb{Q}_p(\mu_c) \to \mathbb{Q}_p(\mu_c)$ sending $\xi \mapsto \xi^p$ for $\xi \in \mu_c$.

Finally, in the right-hand side of (4.7), the term indexed by any $l_i \in \mathbb{N}_0^d$, $r \in \mathbb{N}_0^d$, has valuation $\geq n_1 + \cdots + n_r + l_1 + \cdots + l_r - r(1 + \frac{\log(l_1 + \cdots + l_r + 2)}{\log(p)})$, which tends to $\infty$ when $\sum_{i=1}^r l_i \to \infty$, uniformly with respect to $p$.

**Example 8.** For $r = 1$, Eq. (1.1) is
\[(p^\alpha L_p(n, \omega^{-n}, 1, c))_{p \in \mathbb{P}} = \sum_{l \geq 0} \binom{-n}{l} \sum_{\mu \in \mu_c \setminus \{1\}} \sum_{\epsilon \in N} B_{l, \epsilon, c}^{(n, \mu, \epsilon)} \frac{\epsilon}{1 - \xi \epsilon} \delta(n + l, \xi)^{\text{Frob}^{-1}}.
\]
This formula is a variant of the following formula which is a particular case of Theorem 5.11:
\[(p^\alpha L_p(m; \omega^{-1-m}))_p = \sum_{s \geq m-1} \frac{(-1)^{s+m+1}}{m-1} \frac{s-1}{m-2} B_{s+1-m} \delta(s).
\]

**Remark 9.** A generalization $L_{p,r,\alpha}$ of $L_{p,r}$ with $\alpha \in \mathbb{N}$ can be defined by replacing the condition $p \nmid x_1 \gamma_1 + \cdots + x_i \gamma_i$ by the condition $p^\alpha \nmid x_1 \gamma_1 + \cdots + x_i \gamma_i$ ($1 \leq i \leq r$) in Eq. (1.1). The results of [5] and Theorem 1 can be generalized to $L_{p,r,\alpha}$ by similar proofs, provided $p^{\alpha(n_1 + \cdots + n_r)} H_p((n_i)_r; (\epsilon_i)_r)$ is replaced by $p^{\alpha(n_1 + \cdots + n_r)} H_p((n_i)_r; (\epsilon_i)_r)$ in Definition 3, which defines a more general type of cyclotomic multiple harmonic values satisfying the same properties [7][10].

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