**References**

Four Groups \((\infty\text{-dim unip alg.gps}/\mathbb{Q})\)

- **GRT**: Grothendieck-Teichmüller group by Drinfeld (’91)
- **DMR**: Double shuffle group by Racinet (’02)
- **KRV**: Kashiwara-Vergne group by Alekseev-Torossian (’08)
- **Gal^M(\mathbb{Z})**: Motivic Galois group by Deligne-Goncharov (’05)

\[ \iff \] **Associators**

Conjecture: \( GRT = DMR = KRV = Gal^M(\mathbb{Z})? \)
Grothendieck-Teichmüller group $GRT$: Drinfeld’s work (’91) on KZ (Knizhnik-Zamolodchikov) equations, Teichmüller-Lego game by Grothendieck (’84)

Double shuffle group $DMR$: double shuffle + regularization relations among multiple zeta values by Zagier, Ecalle,.. (in early ’90’s).

Kashiwara-Vergne group $KRV$: Kashiwara-Vergne conjecture (’78) proved by Alekseev-Torossian (’08)

Motivic Galois group $Gal^M(\mathbb{Z})$: unramified mixed Tate motives theory by Deligne-Goncharov (’05)
Associators
\[\downarrow\]
Grothendieck-Teichmüller group
Double shuffle group
Kashiwara-Vergne group
Motivic Galois group
\[\downarrow\]
Comparison
The pair \((\mu, \varphi)\) with \(\mu \in \mathbb{K}^\times\) (\(\mathbb{K}\): a field of \(\text{ch} = 0\)) and group-like series \(\varphi = \varphi(A, B) \in \mathbb{K}\langle\langle A, B \rangle\rangle\) (i.e. \(\log \varphi\) is a Lie series) forms an \textbf{associator} if and only if it satisfies Drinfeld’s \textbf{one pentagon equation}

\[
\varphi(t_{12}, t_{23}+t_{24})\varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\varphi(t_{12}+t_{13}, t_{24}+t_{34})\varphi(t_{12}, t_{23})
\]

and his \textbf{two hexagon equations}

\[
\exp\left\{ \frac{\mu(t_{13} + t_{23})}{2} \right\} = \varphi(t_{13}, t_{12}) \exp\left\{ \frac{\mu t_{13}}{2} \right\} \varphi(t_{13}, t_{23})^{-1} \exp\left\{ \frac{\mu t_{23}}{2} \right\} \varphi(t_{12}, t_{23}),
\]

\[
\exp\left\{ \frac{\mu(t_{12} + t_{13})}{2} \right\} = \varphi(t_{23}, t_{13})^{-1} \exp\left\{ \frac{\mu t_{13}}{2} \right\} \varphi(t_{12}, t_{13}) \exp\left\{ \frac{\mu t_{12}}{2} \right\} \varphi(t_{12}, t_{23})^{-1}
\]

\([\{t_{ij}\}: \text{the braid parameters}].\)
Associators: A Categorical Background

Definition: A symmetric braided monoidal category (tensor category) (by Joyal-Street ’93) is a category $C$ with

- left unit $l_U : 1 \otimes U \to U$,
- right unit $r_U : U \otimes 1 \to U$,
- associativity $a : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$
- commutativity $c : U \otimes V \to V \otimes U$

satisfying

- triangle axiom $(id \otimes l) \circ a = r \otimes id$,
- pentagon axiom $(id \otimes a) \circ a \circ (a \otimes id) = a \circ a$,
- hexagon axiom $a \circ (c \otimes id) \circ a^{-1} = c \circ a^{-1} \circ (id \otimes c^{-1})$,
- involution axiom $c \circ c = id$. hexagon axiom (II)

$a \circ (c^{-1} \otimes id) \circ a^{-1} = c^{-1} \circ a^{-1} \circ (id \otimes c)$. 
Drinfeld-Cartier construction: For a $\mathbb{K}$-linear infinitesimal tensor category $C$, each associator gives $C[[h]]$ a braided monoidal structure.

- Quantization of certain Hopf algebras (quasi-triangular quasi-Hopf quantum groups) related with the monodromy of KZ-systems (Kohno-Drinfeld theorem) by Drinfeld (’91).
- Combinatorial reconstruction of universal Vassiliev knot invariant (Kontsevich invariant) by Le-Murakami (’96) and Bar-Natan (’97).
Kontsevich invariant ('93): \( Z(K) := Z'(K) \cdot Z'(U)^{-c/2} \) with

\[
Z'(K) := \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_1 < \cdots < t_m} \sum_P (-1)^{\#P} \wedge_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'} D_P
\]
Combinatorial reconstruction by Le-Murakami (’96) and Bar-Natan (’97): Any (framed) knot can be decomposed into elementary quasi-tangle diagrams. Kontsevich inv’t can be reconstructed by using an associator $\varphi(A, B)$:

\[
\begin{align*}
\includegraphics[width=\textwidth]{knot-diagram.png}
\end{align*}
\]
Theorem (by F., Ann of Math,’10)

Let \( \varphi(A, B) \) be a group-like series. If it satisfies the one pentagon equation, then there always exists \( \mu \) which satisfies the two hexagon equations.

Proof. Based on a direct Lie algebra calculation. □

One Pentagon implies Two Hexagons!!
KZ (Knizhnik-Zamolodchikov) equation over $\mathbb{C}\setminus\{0, 1\}$

$$\frac{dG}{dz} = \left(\frac{A}{z} + \frac{B}{z - 1}\right)G(z) \quad \text{on} \quad \mathbb{C}\langle\langle A, B\rangle\rangle.$$

The Drinfeld associator $\Phi_{KZ} \in \mathbb{C}\langle\langle A, B\rangle\rangle$ is the connection matrix from 0 to 1:

$$\Phi_{KZ} = 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m) A^{k_m-1} B \cdots A^{k_1-1} B + \text{(regularized terms)}.$$

Explicit formula was given by Le-Murakami('96).
Multiple Zeta Value (MZV) : Associator relations

For \( k_1, \ldots , k_{m-1} \geq 1 \) and \( k_m \geq 2 \),

\[
\zeta(k_1, \ldots , k_m) := \sum_{0 < n_1 < \cdots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \in \mathbb{R}.
\]

Theorem (Drinfeld, '91): \((\Phi_{KZ}, 2\pi i)\) forms an associator, i.e. \(\Phi_{KZ}\) is a group-like series satisfying one pentagon and two hexagon equations with \(\mu = 2\pi i\).

This yields many relations, associator relations, among MZV’s.

Associator Conjecture: Associator relations might produce all algebraic relations among MZV’s.
MZV: Corollaries of Associator relations

- Euler’s formula

\[ \zeta(2n) = (-1)^{n+1} \frac{B_{2n} \pi^{2n}}{2(2n)!} \]

follows from Hexagons + Integral Shuffle relations (by Deligne ’89).

- Le-Murakami’s formula

\[
\sum_{w: \text{admissible}} (-1)^{dp(w)} \frac{\zeta(w)}{\pi^{2k}} = \frac{(-1)^k}{(2k+1)!} \sum_{r=0}^{k-s} \binom{2k+1}{2r} (2-2^{2r}) B_{2r}
\]

follows from Associator relations.
Double Shuffle relations for MZV’s
= Series Shuffle relation + Integral Shuffle relation
+ $\epsilon$ (regularization relation)

Example:
$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(a + b) + \zeta(b, a) = \sum_i \binom{b-1+i}{i} \zeta(a - i, b + i) + \sum_j \binom{a-1+j}{j} \zeta(b - j, a + j).$$
Series Shuffle relation

\[\zeta(a)\zeta(b) = \sum_{0<k} \frac{1}{k^a} \cdot \sum_{0<l} \frac{1}{l^b} = (\sum_{0<k<l} + \sum_{0<k=l} + \sum_{0<l<k}) \frac{1}{k^a l^b}\]

\[= \zeta(a, b) + \zeta(a + b) + \zeta(b, a).\]

Integral Shuffle relation

\[\zeta(a)\zeta(b) = \int_{0<s_1<\ldots<s_a<1} \frac{ds_1}{1 - s_1} \wedge \frac{ds_2}{s_2} \wedge \cdots \wedge \frac{ds_a}{s_a}\]

\[\times \int_{0<t_1<\ldots<t_b<1} \frac{dt_1}{1 - t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_b}{t_b}\]

\[= \sum \int_{0}^{1} \text{all shuffles}\]

\[= \sum_i \left(\frac{b - 1}{i} + i\right)\zeta(a - i, b + i) + \sum_j \left(\frac{a - 1}{j} + j\right)\zeta(b - j, a + j).\]
Multiple zeta values

**MZV: Conjectures**

Example: 
\[ \zeta(a) \cdot \zeta(b) = \zeta(a, b) + \zeta(a + b) + \zeta(b, a) = \sum_i \binom{b-1+i}{i} \zeta(a - i, b + i) + \sum_j \binom{a-1+j}{j} \zeta(b - j, a + j). \]

**Double Shuffle relations** = Series Shuffle relation + Integral Shuffle relation + \[ \varepsilon \text{(regularization relation)} \]

**Double Shuffle Conjecture:** Double Shuffle relations might produce all algebraic relations among MZV’s.

But we also had

**Associator Conjecture:** Associator relations might produce all algebraic relations among MZV’s.
Theorem (by F., Ann of Math, ’11)

Double Shuffle Conjecture ⇒ Associator Conjecture, i.e. Associator relations imply Double Shuffle relations.

Consequence: \( GRT \subset DMR \)

This attains the goal of the project of Deligne-Terasoma. Their approach was to use a convolution of perverse sheaves, whereas ours is to use Chen’s bar construction calculus.
Sketch of Proof: For $\mathcal{M} = \mathcal{M}_{0,4} = \{ z \in A^1 | z \neq 0, 1 \}$ or $\mathcal{M}_{0,5} = \{(x, y) \in A^2 | x, y \neq 0, 1, xy \neq 1 \}$,

$$H^0 \text{Ch}^\cdot(\Omega_* \mathcal{M}) \cong \text{ULie} \pi_1(\mathcal{M}, *)^\vee.$$  

$Li_a(z), Li_b(z), Li_{a+b}(z), Li_{a,b}(1, z), Li_{b,a}(1, z) \in H^0 \text{Ch}^\cdot(\Omega_* \mathcal{M}_{0,4})$ and $Li_a(x), Li_b(y), Li_{a+b}(xy), Li_{a,b}(x, y), Li_{b,a}(y, x) \in H^0 \text{Ch}^\cdot(\Omega_* \mathcal{M}_{0,5})$.

$$Li_a(x)Li_b(y) = Li_{a+b}(xy) + Li_{a,b}(x, y) + Li_{b,a}(y, x).$$

Evaluate at $\varphi^{2,3,4}\varphi^{1,23,4}\varphi^{1,2,3} \in \text{ULie} \pi_1(\mathcal{M}_{0,5}, *)$:

$Li_a(x)(\varphi^{2,3,4}\varphi^{1,23,4}\varphi^{1,2,3}) = Li_a(z)(\varphi),$

$Li_b(x)(\varphi^{2,3,4}\varphi^{1,23,4}\varphi^{1,2,3}) = Li_b(z)(\varphi),$

$Li_{a+b}(xy)(\varphi^{2,3,4}\varphi^{1,23,4}\varphi^{1,2,3}) = Li_{a+b}(z)(\varphi),$

$Li_{a,b}(x, y)(\varphi^{2,3,4}\varphi^{1,23,4}\varphi^{1,2,3}) = Li_{a,b}(1, z)(\varphi),$

$Li_{b,a}(y, x)(\varphi^{2,3,4}\varphi^{1,23,4}\varphi^{1,2,3}) = Li_{b,a}(y, x)(\varphi^{1,2,3}\varphi^{12,3,4}) = Li_{b,a}(1, z)(\varphi).$

$$Li_a(z)(\varphi) \cdot Li_b(z)(\varphi) = Li_{a,b}(1, z)(\varphi) + Li_{a+b}(z)(\varphi) + Li_{b,a}(1, z)(\varphi). \square$$
\[ GRT := \{ \text{associators } \varphi \text{ with } '\mu = 0' \} \]

- a kind of ‘degenerations’ of the set of associators.
- a deformation group of a ‘universal’ braided monoidal category.
- the Drinfeld associator \( \Phi_{KZ} \notin GRT(\mathbb{C}) \)
- the \( p \)-adic Drinfeld associator \( \Phi_{KZ}^p \in GRT(\mathbb{Q}_p) \)
  (by [F] Inv math, ’04, and [Unver] preprint).
Double shuffle group (by Racinet, ’02)

\[ DMR := \left\{ \text{series } \varphi(A, B) \text{ whose all coefficients satisfy the double shuffle relations and } c(2) = 0 \right\} \]

\[ \varphi(A, B) = 1 + \sum (-1)^m c(k_1, \ldots, k_m) A^{k_m-1} B \cdots A^{k_1-1} B + (\text{reg.terms}) \]

- \( DMR \) stands for ‘\textit{double mélange régulaisede}’.
- it forms a group (a nontrivial fact by Racinet).
- the Drinfeld associator \( \Phi_{KZ} \notin DMR(\mathbb{C}) \)
- the \( p \)-adic Drinfeld associator \( \Phi_{KZ}^p \in DMR(\mathbb{Q}_p) \) (by [Besser-F.]’06 and [F.-Jafari]’07).
Kashiwara-Vergne Conjecture (’78)

Kashiwara-Vergne Conjecture (a formal version)
There should exist Lie series $F(A, B)$ and $G(A, B)$ s.t.

$$A + B - \text{CBH}(B, A) = (1 - e^{-adA})F + (e^{adB} - 1)G,$$

$$\text{Tr}\left\{\partial_A(F)A + \partial_B(G)B\right\} = \frac{1}{2}\text{Tr}\left\{g(A) + g(B) - g(\text{CBH}(A, B))\right\}$$

in $\text{Cyc}_2$ for some $g = g(X) \in \mathbb{Q}[[X]]$.

- It generalises the Duflo isomorphism.
- It was solved: **1st proof** by Alekseev-Meinrenken (’06) and **2nd proof** by Alekseev-Torossian (’08).
Kashiwara-Vergne group (by Alekseev-Torossian,’08)

Proof (by AT): Let \((\mu, \varphi(A, B))\) be an associator. Then
\[(F, G) = \left( \varphi(A/\mu, C/\mu)A\varphi(A/\mu, C/\mu)^{-1}, e^{C/2}\varphi(B/\mu, C/\mu)B\varphi(B/\mu, C/\mu)^{-1}e^{-C/2} \right)\]
with \(C = -A - B\) is a solution of KV-conjecture.

\[\emptyset \neq \{\text{associators}\} \iff \text{SolKV} \tag{\square}\]

\[\text{KRV} := \{\text{solution } (F, G) \text{ of KV-prob. with } \mu = 0\}\]

- a kind of ‘degeneration’ of \(\text{SolKV}\).

Theorem (by AT): \(\text{GRT} \hookrightarrow \text{KRV}\).

We recall that

Theorem (by F): \(\text{GRT} \hookrightarrow \text{DMR}\).
Motivic Galois group

$\mathcal{DMM}(\mathbb{Q})_\mathbb{Q}$: the triangulated category of mixed motives by Voevodsky, Levine and Hanamura.

$\mathcal{MTM}(\mathbb{Z})_\mathbb{Q}$: the tannakian category of unramified mixed Tate motives by Deligne-Goncharov.

$\mathcal{Rep}_G$: the category of graded representations of $G$.

$G =: \Gal^M(\mathbb{Z})$: motivic Galois group
Motivic Galois representation

\[ MTM(\mathbb{Z})_{\mathbb{Q}} \cong \pi_1^{M}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) : 0 \] : the motivic fundamental group by Deligne-Goncharov (’05).

\[ RepGal^M(\mathbb{Z}) \cong F_2(\mathbb{Q}) \] : the free pro-unipotent group with two variables \( e^A \) and \( e^B \).

\[ \Phi : Gal^M(\mathbb{Z}) \to AutF_2 \] : motivic Galois rep.

Brown’s arguments and Zagier’s formula yield a breakthrough

Theorem: \( \Phi \) is injective!
\[ \text{Gal}^M(\mathbb{Z}) \subseteq \text{Aut} F_2 \] by Brown

\[ \bigcap \] by a geometric interpretation

\[ \text{GRT} \subseteq \text{KRV} \] by Alekseev-Torossian

\[ \varphi \mapsto \begin{cases} \varphi(e^A) = e^A, \\ \varphi(e^B) = \varphi^{-1}e^B \varphi \end{cases} \]

\[ \bigcap \] by F.

\[ \text{DMR} \subseteq \text{Aut} F_2 \]

\[ \bigcap \] by Schneps

\[ \text{KRV} \subseteq \text{Aut} F_2 \]

Conjecture \[ \text{Gal}^M(\mathbb{Z}) = \text{GRT} = \text{DMR} = \text{KRV}? \]
That is ALL!!
Thank you very much for your attentions!!