ASSOCIATORS AND KONTSEVICH’S EYE

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Abstract. This is a report on Drinfeld’s associators and Kontsevich’s eye which is based on my talk at Algebraic Lie Theory and Representation Theory (ALTReT) 2018.

Contents

1. Drinfeld’s associator 1
2. Kontsevich’s eye 4
3. Alekseev-Torossian’s associator 8
References 11

1. DRINFELD’S ASSOCIATOR

The notion of associators was introduced by Drinfeld in [Dr]. They describe monodromies of the KZ-equations $^1$. They are essential for the construction of quasi-triangular quasi-Hopf quantized universal enveloping algebras ([Dr]), for the quantization of Lie-bialgebras (Etingof-Kazhdan quantization [EK]), for the proof of formality chain operad of little discs by Tamarkin [Ta] (see also Ševera and Willwacher [SW]), the formal solution of Kashiwara-Vergne conjecture by Alekseev and Torossian [AT12] and also for the combinatorial reconstruction of the universal Vassiliev knot invariant (the Kontsevich invariant [K93, B95]) by Bar-Natan [B97], Cartier [Ca], Kassel and Turaev [KaT], Le and Murakami [LM96a] and Piunikhin [P] (for some of these related topics, consult [Fu14, Fu16]).

Let us fix notations: Let $k$ be a field of characteristic 0 and $\bar{k}$ be its algebraic closure. Denote by $\bar{U}f_2 = k\langle\langle A, B\rangle\rangle$ the non-commutative formal power series ring defined as the completion (with respect to degree) of the universal enveloping algebra of the free Lie algebra $f_2$.

$^1$KZ stands for Knizhnik and Zamolodchikov.
over \( k \) with two variables \( A \) and \( B \). An element \( \varphi = \varphi(A, B) \) of \( \widehat{U_f}^2 \) is called group-like \(^2\) if it satisfies

\[
\Delta(\varphi) = \varphi \otimes \varphi \quad \text{and} \quad \varphi(0, 0) = 1
\]

where \( \Delta : \widehat{U_f}^2 \to \widehat{U_f}^2 \otimes \widehat{U_f}^2 \) is given by \( \Delta(A) = A \otimes 1 + 1 \otimes A \) and \( \Delta(B) = B \otimes 1 + 1 \otimes B \). For any \( k \)-algebra homomorphism \( \iota : \widehat{U_f}^2 \to S \), the image \( \iota(\varphi) \in S \) is denoted by \( \varphi(\iota(A), \iota(B)) \).

Denote by \( \widehat{U_a}^3 \) (resp. \( \widehat{U_a}^4 \)) the completion of the universal enveloping algebra of the pure braid Lie algebra \( a_3 \) (resp. \( a_4 \)) over \( k \) with 3 (resp. 4) strings, which is generated by \( t_{ij} \) \((1 \leq i, j \leq 3 \) (resp. 4)\) with defining relations

\[
t_{ii} = 0, \quad t_{ij} = t_{ji}, \quad [t_{ij}, t_{ik} + t_{jk}] = 0 \quad (i, j, k: \text{all distinct})
\]

and

\[
[t_{ij}, t_{kl}] = 0 \quad (i, j, k, l: \text{all distinct}).
\]

**Definition 1.1** ([Dr]). A pair \((\mu, \varphi)\) with a non-zero element \( \mu \) in \( k \) and a group-like series \( \varphi = \varphi(A, B) \in \widehat{U_f}^2 \) is called an associator if it satisfies one pentagon equation

\[
\varphi(t_{12}, t_{23} + t_{24}) \varphi(t_{13} + t_{23}, t_{34}) = \varphi(t_{23}, t_{34}) \varphi(t_{12} + t_{13}, t_{24} + t_{34}) \varphi(t_{12}, t_{23})
\]

in \( \widehat{U_a}^4 \) and two hexagon equations

\[
\exp\{\frac{\mu(t_{13} + t_{23})}{2}\} = \varphi(t_{13}, t_{12}) \exp\{\frac{\mu t_{13}}{2}\} \varphi(t_{13}, t_{23})^{-1} \exp\{\frac{\mu t_{23}}{2}\} \varphi(t_{12}, t_{23}),
\]

(1.4)

\[
\exp\{\frac{\mu(t_{12} + t_{13})}{2}\} = \varphi(t_{23}, t_{13})^{-1} \exp\{\frac{\mu t_{13}}{2}\} \varphi(t_{12}, t_{13}) \exp\{\frac{\mu t_{12}}{2}\} \varphi(t_{12}, t_{23})^{-1}
\]

in \( \widehat{U_a}^3 \).

Drinfeld [Dr] proved that such a pair always exists for any field \( k \) of characteristic 0. The equations (1.2)–(1.4) reflect the three axioms of braided monoidal categories introduced by Joyal and Street [JS].

Actually, the two hexagon equations are a consequence of the one pentagon equation:

**Theorem 1.2** ([Fu10]). Let \( \varphi = \varphi(A, B) \) be a group-like element of \( \widehat{U_f}^2 \). Suppose that \( \varphi \) satisfies the pentagon equation (1.2). Then there always exists \( \mu \in k \) (unique up to signature) such that the pair \((\mu, \varphi)\) satisfies two hexagon equations (1.3) and (1.4).

\(^2\)It is equivalent to \( \varphi \in \exp \widehat{f}_2 \).
We note that several different proofs of the above theorem were obtained (see [AT12, BD, W]).

A well-known example of associators is the KZ-associator:

**Example 1.3.** The *KZ-associator* (aka. Drinfeld associator) \( \Phi_{\text{KZ}} = \Phi_{\text{KZ}}(A, B) \in \mathbb{C}(\langle A, B \rangle) \) is defined to be the quotient \( \Phi_{\text{KZ}} = G_1(z)^{-1}G_0(z) \) where \( G_0 \) and \( G_1 \) are the solutions of the *formal KZ-equation*, which is the following differential equation for multi-valued functions \( G(z) \) of \( \mathbb{C}\backslash\{0, 1\} \) valued on \( \mathbb{C}(\langle A, B \rangle) \)

\[
dG(z) = \omega_{\text{KZ}} \cdot G(z)
\]

with

\[
\omega_{\text{KZ}} := \frac{dz}{z}A + \frac{dz}{z-1}B
\]

such that \( G_0(z) \approx z^A \) when \( z \to 0 \) and \( G_1(z) \approx (1-z)^B \) when \( z \to 1 \) (cf.[Dr]). It is shown in [Dr] that the pair \( (2\pi \sqrt{-1}, \Phi_{\text{KZ}}) \) forms an associator for \( k = \mathbb{C} \). Namely \( \Phi_{\text{KZ}} \) satisfies (1.1)–(1.4) with \( \mu = 2\pi \sqrt{-1} \).

**Remark 1.4.** (i). The KZ-associator is expressed as follows:

\[
\Phi_{\text{KZ}}(X_0, X_1) = 1 + \sum_{m, k_1, \ldots, k_m \in \mathbb{N}, k_m > 1} (-1)^m \zeta(k_1, \ldots, k_m) A^{k_m-1}B \cdots A^{k_1-1}B + \text{(regularized terms)}.
\]

Here \( \zeta(k_1, \ldots, k_m) \) is the *multiple zeta value* (MZV in short), the real number defined by the following power series

\[
\zeta(k_1, \ldots, k_m) := \sum_{0 < \ell_1 < \cdots < \ell_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}
\]

for \( m, k_1, \ldots, k_m \in \mathbb{N}(= \mathbb{Z}_{>0}) \) with \( k_m > 1 \) (its convergent condition). All of the coefficients of \( \Phi_{\text{KZ}} \) (including its regularized terms) are explicitly calculated in terms of MZV’s in [Fu03] Proposition 3.2.3 by Le-Murakami’s method in [LM96b] Theorem A.8.

(ii). Since all of the coefficients of \( \Phi_{\text{KZ}} \) are described by MZV’s, the equations (1.1)–(1.4) for \( (\mu, \varphi) = (2\pi \sqrt{-1}, \Phi_{\text{KZ}}) \) yield algebraic relations among them, which are called *associator relations*. It is expected that the associator relations might produce all algebraic relations among MZV’s.

In Definition 3.3, we will see another example of associators, the AT-associator \( \Phi_{\text{AT}} \), which was constructed by Alekseev and Torossian [AT10] by using a parallel transport of an analog (3.2) of the KZ-equation.
2. Kontsevich’s eye

We will recall the compactified configuration spaces and weights of Lie graphs [K03].

Let \( n \geq 1 \). For a topological space \( X \), we define \( \text{Conf}_n(X) := \{(x_1, \ldots, x_n) \mid x_i \neq x_j (i \neq j)\} \). The group \( \text{Aff}_+ := \{x \mapsto ax + b \mid a \in \mathbb{R}^+ \times, b \in \mathbb{C}\} \) acts on \( \text{Conf}_n(\mathbb{C}) \) diagonally by rescalling and parallel translations. We denote the quotient by

\[
C_n := \text{Conf}_n(\mathbb{C})/\text{Aff}_+
\]

for \( n \geq 2 \), which is a connected oriented smooth manifold with dimension \( 2n - 3 \). E.g.

- \( C_2 \simeq S^1 \)
- \( C_3 \simeq S^1 \times (\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) \).

For a finite set \( I \) with \( |I| = n \), we put \( C_I = C_n \). For \( I' \subset I \) with \( |I'| > 1 \), we have the pull-back map \( C_I \to C_{I'} \).

Put \( \text{Conf}_{n,m}(\mathbb{H}, \mathbb{R}) := \text{Conf}_n(\mathbb{H}) \times \text{Conf}_m(\mathbb{R}) \) with the coordinate \((z_1, \ldots, z_n, x_1, \ldots, x_m)\), where \( \mathbb{H} \) is the upper half plane. The group \( \text{Aff}_+ := \{x \mapsto ax + b \mid a \in \mathbb{R}_{+}^\times, b \in \mathbb{R}\} \) acts there diagonally and we denote the quotient by

\[
C_{n,m} := \text{Conf}_{n,m}(\mathbb{H}, \mathbb{R})/\text{Aff}_{+}^R
\]

for \( n, m \geq 0 \) with \( 2n + m \geq 2 \). It is an oriented smooth manifold with dimension \( 2n + m - 2 \) and with \( m! \) connected components. E.g.

- \( C_{0,2} \simeq \{\pm 1\} \), \( C_{0,2}^+ = \{+1\} \), \( C_{0,2}^- := \{-1\} \).
- \( C_{1,1} \simeq \{e^{\sqrt{-1} \theta} \mid 0 < \theta < 1\} \).
- \( C_{2,0} \simeq \mathbb{H} - \{\sqrt{-1}\} \).

For a finite set \( I \) and \( J \) with \( |I| = n \) and \( |J| = m \), we put \( C_{I,J} = C_{n,m} \). Then for \( I' \subset I \) and \( J' \subset J \), we have the pull-back map \( C_{I,J} \to C_{I',J'} \).

Below we recall 3 Kontsevich’s [K03] compactifications \( \overline{C}_n \) and \( \overline{C}_{n,m} \) of \( C_n \) and \( C_{n,m} \) à la Fulton-MacPherson (in more detail, consult [Si]):

For a finite set \( I \) with \( |I| = n \), we put \( \tilde{C}_I := \tilde{C}_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n z_i = 0\} \cap S^{2n-1} \). By identifying it with \( \mathbb{C}^n - \text{diag}/\text{Aff}_+ \) \((\text{diag} = \{(z, \ldots, z) \mid z \in \mathbb{C}\})\), we obtain an embedding \( C_I \hookrightarrow \tilde{C}_I \). The compactification \( \overline{C}_I = \overline{C}_n \) is a compact topological manifold with corners which is defined to be the closure of the image of the associated embedding \( \Phi : C_I \hookrightarrow \prod_{J \subset I, |J| < |I|} \tilde{C}_J \). While by the embedding \( \text{Conf}_{n,m}(\mathbb{H}, \mathbb{R}) \hookrightarrow \text{Conf}_{2n+m}(\mathbb{C}) \) sending \((z_1, \ldots, z_n, x_1, \ldots, x_m) \mapsto (z_1, \ldots, z_n, \tilde{z}_1, \ldots, \tilde{z}_n, x_1, \ldots, x_m)\), we have an embedding \( C_{n,m} \hookrightarrow C_{2n+m} \). By combining it with \( \Phi \), we obtain an embedding \( C_{n,m} \hookrightarrow \overline{C}_{2n+m} \).

\[\text{3Here we follow the conventions of Bruguières ([CKTB]).}\]
compactification $C_{n,m}$ is a compact topological manifold with corners which is defined to be the closure of the embedding.

They are functorial with respect to the inclusions of two finite sets, i.e. $I_1 \subset I_2$ and $J_1 \subset J_2$ with $\sharp(I_k) = n_k$ and $\sharp(J_k) = m_k$ ($k = 1, 2$) yield a natural map $C_{n_2,m_2} \rightarrow C_{n_1,m_1}$. The stratification of his compactification has a very nice description in terms of trees in [K03] (also refer [CKTB]). E.g.

- $\overline{C}_{0,2} = C_{0,2} \simeq \{\pm 1\}$,
- $\overline{C}_{1,1} = C_{1,1} \sqcup C_{0,2} = \{e^{\pm \frac{\pi \theta}{2}} \mid 0 \leq \theta \leq 1\}$,
- $\overline{C}_{2,0} = C_{2,0} \sqcup C_{1,1} \sqcup C_{1,1} \sqcup C_2 \sqcup C_{0,2}$.

The $\overline{C}_{2,0}$ is called Kontsevich’s eye and its each component bears a special name as is indicated in Figure 2.1. The upper (resp. lower) eyelid

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{kontsevich_eye.png}
\caption{Kontsevich’s eye $\overline{C}_{2,0}$}
\end{figure}

corresponds to $z_1$ (resp. $z_2$) on the the real line. The iris magnifies collisions of $z_1$ and $z_2$ on $\mathbb{H}$. LC (resp. RC) which stands for the left (resp. right) corner is the configuration of $z_1 > z_2$ (resp. $z_1 < z_2$) on the real line.

The angle map $\phi : \overline{C}_{2,0} \rightarrow \mathbb{R}/\mathbb{Z}$ is the map induced from the map $\text{Conf}_2(\mathbb{H}) \rightarrow \mathbb{R}/\mathbb{Z}$ sending

$$\phi : (z_1, z_2) \mapsto \frac{1}{2\pi} \arg\left(\frac{z_2 - z_1}{z_2 - \bar{z}_1}\right).$$

We note that $\phi$ is identically zero on the upper eyelid but is not on the lower eyelid.
Next we will recall the notion of Lie graphs and their weight functions and 1-forms.

**Definition 2.1.** Let \( n \geq 1 \). A Lie graph \( \Gamma \) of type \((n,2)\) is a graph consisting of two finite sets, the set of vertices \( V(\Gamma) := \{1, 2, 3, \ldots, n\} \) and the set of edges \( E(\Gamma) \subset V(\Gamma) \times V(\Gamma) \). The points \( 1 \) and \( 2 \) are called as the ground points, while the points \( 3, 4, \ldots, n \) are called as the air points. We equip \( V(\Gamma) \) with the total order \( 1 < 2 < 3 < \cdots < n \).

For each \( e \in E(\Gamma) \), under the inclusion \( E(\Gamma) \subset V(\Gamma) \times V(\Gamma) \), we call the corresponding first (resp. second) component \( s(e) \) (resp. \( t(e) \)) as the source (resp. the target) of \( e \) and denote as \( e = (s(e), t(e)) \). We equip \( E(\Gamma) \) with the lexicographic order induced from that of \( V(\Gamma) \).

Both \( V(\Gamma) \) and \( E(\Gamma) \) are subject to the following conditions:

1. An air point fires two edges: That means there always exist two edges with the source \( \hat{i} \) for each \( i = 1, \ldots, n \).
2. An air point is shot by one edge at most: That means there exists at most one edge with its target \( \hat{v} \) for each \( i = 1, \ldots, n \).
3. A ground point never fire edges: That means there is no edge with its source on ground points.
4. The graph \( \Gamma \) becomes a rooted trivalent tree after we cut off small neighborhoods of ground points: That means that the graph of \( \Gamma \) admits a unique vertex (called the root) shoot by no edges and it gives a rooted trivalent trees if we regard the vertex as a root and distinguish all targets of edges firing ground points.

Let \( \Gamma \) be a Lie graph of type \((n,2)\). We define a Lie monomial \( \Gamma(A, B) \in \tilde{f}_2 \) of degree \( n + 1 \) to be the associated element with the root by the following procedure: With \( 1 \) and \( 2 \), we assign \( A \) and \( B \in \tilde{f}_2 \) respectively. With each internal vertex \( v \) firing two edges \( e_1 = (v, w_1) \) and \( e_2 = (v, w_2) \) such that \( e_1 < e_2 \), we assign \( [\Gamma_1, \Gamma_2] \in \tilde{f}_2 \) where \( \Gamma_1 \) and \( \Gamma_2 \in \tilde{f}_2 \) are the corresponding Lie monomials with the vertices \( w_1 \) and \( w_2 \) respectively. Recursively we may assign Lie elements with all vertices of \( \Gamma \).

E.g. Figure 2.2 is an example of Lie graph of type \((3,2)\). Its root is \( 3 \). The associated Lie elements of the vertices \( 1, 2, 3 \) are \( A, B, [A, B], [B, [A, B]], [B, [B, [A, B]]] \) respectively.

Each \( e \in E(\Gamma) \) determines a subset \( \{s(e), t(e)\} \subset V(\Gamma) \) with \( |V(\Gamma)| = n + 2 \) which yields a pull-back \( p_e : C_{n+2, 0} \to C_{2, 0} \). By composing it with the angle map \((2.1)\), we get a map \( \phi_e : C_{n+2, 0} \to \mathbb{R}/\mathbb{Z} \). The
PA\(^4\) 2n-forms \(\Omega_\Gamma\) on \(\overline{C}_{n+2,0}\) (which is \(2n\)-dimensional compact space) associated with \(\Gamma\) is given by the ordered exterior product
\[
\Omega_\Gamma := \wedge_{e \in E(\Gamma)} d\phi_e \in \Omega_{\text{PA}}^{2n}(\overline{C}_{n+2,0}).
\]
Here \(\Omega_{\text{PA}}^{2n}(\overline{C}_{n+2,0})\) means the space of PA \(2n\)-forms of \(\overline{C}_{n+2,0}\).

**Definition 2.2.** (i). Put \(\pi : \overline{C}_{n+2,0} \to \overline{C}_{2,0}\) to be the above projection induced from the inclusion \(\{[1,2]\} \subset \{[1,2],[1,2,\ldots,n]\}\). The *weight function* (see [To]) of \(\Gamma\) is the smooth function \(w_\Gamma : \overline{C}_{2,0} \to \mathbb{C}\) defined by \(w_\Gamma := \pi_*(\Omega_\Gamma)\) where \(\pi_*\) is the push-forward (the integration along the fiber of the projection \(\pi\), cf. [HLTV]), that is, the function which assigns \(\xi \in \overline{C}_{2,0}\) with
\[
w_\Gamma(\xi) = \int_{\pi^{-1}(\xi)} \Omega_\Gamma \in \mathbb{C}.
\]

(ii). We denote \(L\Gamma\) (resp. \(R\Gamma\)) to be a graph obtained from \(\Gamma\) by adding one more edge \(e_L\) from \([1]\) (resp. \(e_R\) from \([2]\)) to the root of \(\Gamma\). The regular \((2n+1)\)-form \(\Omega_{L\Gamma}\) (resp. \(\Omega_{R\Gamma}\)) on \(\overline{C}_{n+2,0}\) is defined to be
\[
\Omega_{L\Gamma} := d\phi_{e_L} \land \Omega_\Gamma \quad \text{(resp.)} \quad \Omega_{R\Gamma} := d\phi_{e_R} \land \Omega_\Gamma
\]
in \(\Omega_{\text{PA}}^{2n}(\overline{C}_{n+2,0})\). The one-forms \(\omega_{L\Gamma}\) and \(\omega_{R\Gamma}\), which we call the *weight forms* of \(\Gamma\) here, are the PA one-forms of \(\overline{C}_{2,0}\) respectively defined by
\[
\omega_{L\Gamma} := \pi_*(\Omega_{L\Gamma}) \quad \text{and} \quad \omega_{R\Gamma} := \pi_*(\Omega_{R\Gamma})
\]
in \(\Omega_{\text{PA}}^1(\overline{C}_{2,0})\), i.e. they are one-forms respectively defined by
\[
\omega_{L\Gamma}(\xi) = \int_{\pi^{-1}(\xi)} \Omega_{L\Gamma}, \quad \text{and} \quad \omega_{R\Gamma}(\xi) = \int_{\pi^{-1}(\xi)} \Omega_{R\Gamma}
\]
where \(\xi\) runs over \(\overline{C}_{2,0}\).

\(^4\)‘PA’ stands for piecewise-algebraic (cf. [KS, HLTV, LV]).
Remark 2.3. (i). Particularly the special value \( w_\Gamma(RC) \) of the function \( w_\Gamma(\xi) \) at \( \xi = RC \) is called the Kontsevich weight of \( \Gamma \). It appears as a coefficient of Kontsevich’s formula on deformation quantization in [K03].

(ii). While its restriction \( w_\Gamma|_{C_2} \) to the iris \( C_2 \) is identically 0 because \( \Omega_\Gamma|_{C_2} = 0 \) due to the occurrence of double edges.

3. Alekseev-Torossian’s associator

We will recall the deformation \( Z(\xi) \) (\( \xi \in \overline{C}_{2,0} \)) of the Campbell-Baker-Hausdorff series and its associated differential equation. Then we will see how the AT-associator \( \Phi_{AT} \) will be constructed from the differential equation and give a new presentation of \( \Phi_{AT} \).

Consider the smooth function \( Z : \overline{C}_{2,0} \to \hat{f}_2 \) defined by

\[
\xi \in \overline{C}_{2,0} \mapsto Z(\xi) := A + B + \sum_{n \geq 1} \sum_{\Gamma \in \text{LieGra}_{n,2}^{\text{geom}}} \omega_\Gamma(\xi) \Gamma(A, B) \in \hat{f}_2.
\]

Here \( \text{LieGra}_{n,2}^{\text{geom}} \) is the set of geometric (it means non-labeled) Lie graphs of type \((n, 2)\). We note that both \( \Omega_\Gamma \) and \( \Gamma(A, B) \) require the order of \( E(\Gamma) \) however their product \( \Omega_\Gamma \cdot \Gamma(A, B) \) does not (cf. [CKTB]).

Remark 3.1. (i). It is obtained by Kathotia [Kat] that \( Z(RC) \) is equal to the Campbell-Baker-Hausdorff series \( \text{CBH}(A, B) = \log(e^A e^B) \).

(ii). While its its restriction \( Z|_{C_2} \) to the iris \( C_2 \) is simply equal to the addition \( A + B \) because we have \( \Omega_\Gamma|_{C_2} = 0 \) (cf. Remark 2.3).

We may say that \( Z \) is a series which deforms \( \text{CBH}(A, B) \).

Theorem 3.2 ([To]). The series \( Z(\xi) \) satisfies the differential equation

\[
dZ(\xi) = \omega_{AT} \cdot Z(\xi)
\]

where

\[
\omega_{AT} := \text{der} (\omega_L, \omega_R) \in \text{tder}_2 \Omega_{PA}^1(\overline{C}_{2,0}).
\]

Here \( \text{tder}_2 \) be the Lie algebra consisting of tangential derivations \( \text{der}(\alpha, \beta) : \hat{f}_2 \to \hat{f}_2 \) \((\alpha, \beta \in \hat{f}_2)\) such that \( A \mapsto [A, \alpha] \) and \( B \mapsto [B, \beta] \), and

\[
\omega_L := B : d\phi + \sum_{n \geq 1} \sum_{\Gamma \in \text{LieGra}_{n,2}^{\text{geom}}} \Gamma(A, B) \cdot \omega_{LF},
\]

\[
\omega_R := A : \sigma^*(d\phi) + \sum_{n \geq 1} \sum_{\Gamma \in \text{LieGra}_{n,2}^{\text{geom}}} \Gamma(A, B) \cdot \omega_{R\Gamma}.
\]

The symbol \( \sigma \) stands for the involution of \( \overline{C}_{2,0} \) caused by the switch of \( z_1 \) and \( z_2 \).
Related to (3.1), in [AT10] they considered the following differential equation on $C_{2,0}$:

\[(3.2) \quad dg(\xi) = -g(\xi) \cdot \omega_{AT}\]

with $g(\xi) \in {\text{exp\,tder}_2}$, the pro-algebraic subgroup of Aut$_2$ consisting of tangential automorphisms $\text{Int}(\alpha, \beta) : \hat{f}_2 \rightarrow \hat{f}_2 (\alpha, \beta \in \text{exp}\hat{f}_2)$ such that $A \mapsto \alpha^{-1}A\alpha$ and $B \mapsto \beta^{-1}B\beta$. They denote its parallel transport (its holonomy) of (3.2) for the straight path from $\alpha_0$ (the position 0 at the iris, see Figure 3.1) to RC by $F_{AT} \in \text{TAut}_2$.

![Figure 3.1. Parallel transport](image)

**Figure 3.1. Parallel transport**

**Definition 3.3 ([AT10]).** The AT-associator $\Phi_{AT}$ is defined to be

\[(3.3) \quad \Phi_{AT} := F^{1,23}_{AT} \circ F^{2,3}_{AT} \circ (F^{1,2}_{AT})^{-1} \circ (F^{12,3}_{AT})^{-1} \in \text{TAut}_3.\]

Here for any $T = \text{Int}(\alpha, \beta) \in \text{TAut}_2$, we denote

$T^{1,2} := \text{Int} (\alpha(A, B), \beta(A, B), 1)$, \quad $T^{2,3} := \text{Int} (1, \alpha(B, C), \beta(B, C))$, \quad $T^{1,23} := \text{Int} (\alpha(A, B + C), \beta(A, B + C), \beta(A, B + C))$, \quad $T^{12,3} := \text{Int} (\alpha(A + B, C), \alpha(A + B, C), \beta(A + B, C))$

in $\text{TAut}_3 := \text{exp\,tder}_3$ which is similarly defined to be the group of tangential automorphisms of the completed free Lie algebra $\hat{f}_3$ with variables $A$, $B$ and $C$.

We note that there is a Lie algebra inclusion $\hat{f}_2 \hookrightarrow \text{tder}_3$ sending

\[(3.4) \quad A \mapsto t_{12} := \text{der}(B, A, 0) \quad \text{and} \quad B \mapsto t_{23} := \text{der}(0, C, B)
\]

which induces an inclusion $\exp\hat{f}_2 \hookrightarrow \text{TAut}_3$.

**Theorem 3.4 ([AT12, SW]).** The AT-associator $\Phi_{AT}$ forms an associator. Namely it belongs to $\exp\hat{f}_2 (\subset \mathbb{C}\langle\langle A, B \rangle\rangle)$ and satisfies the
equations [Dr] (2.12), (2.13) and (5.3). Furthermore it is real (i.e. it belongs to the real structure $\mathbb{R}\langle\langle A, B\rangle\rangle$) and even.\footnote{It means $\Phi_{AT}(-A, -B) = \Phi_{AT}(A, B)$, from which it follows that $\Phi_{KZ} \neq \Phi_{AT}$ because $\Phi_{KZ}$ is not even.}

The following gives a more direct presentation of $\Phi_{AT}$.

**Theorem 3.5 ([Fu18]).** We have

\[ \Phi_{AT} = \left( \mathcal{P} \exp \int_{RC}^{a_0} (l_\tilde{\omega} + D_\tilde{\omega}) \right) \in \mathbb{C}\langle\langle A, B\rangle\rangle. \]

Here $l_\tilde{\omega}$ is the left multiplication by $\tilde{\omega}$ and $D_\tilde{\omega}$ is given by

\[ D_\tilde{\omega} := \text{der}(0, \tilde{\omega}) \in \text{tder}_2 \mathbb{R}\langle\langle A, B\rangle\rangle \]

with

\[ \tilde{\omega} := \sum_{\gamma \geq 1} \sum_{\Gamma \in \text{LieGraph}_{n,2}} \tilde{\omega}_\Gamma \cdot \Gamma(A, B) \quad \text{and} \quad \tilde{\omega}_\Gamma := \omega_{R\Gamma} - \omega_{L\Gamma}. \]

and for any one-form $\Omega \in \Omega^2_{PA}(C_{2,0})$ we define

\[
\mathcal{P} \exp \int_{RC}^{a_0} \Omega := \text{id} + \int_{RC}^{a_0} \Omega + \int_{RC}^{a_0} \Omega \cdot \Omega + \cdots \\
:= \text{id} + \int_{0<s_1<1} \ell^s \Omega(s_1) + \int_{0<s_1<s_2<1} \ell^s \Omega(s_2) \wedge \ell^s \Omega(s_1) + \cdots .
\]

with the straight path $\ell$ from RC to $a_0$ in Figure 3.1.

This theorem enables us to calculate explicitly all the coefficients of the AT-associator $\Phi_{AT}$ as rational linear combinations of iterated integrals of weight forms of Lie graphs (see [Fu18] for explicit computations in depth 1 and 2).

Explicit formulae to describe all the coefficients of the KZ-associator $\Phi_{KZ}$ in terms of multiple zeta values are given in [Fu03, LM96b]. Whereas, as far as the author knows, any explicit formulae to present all the coefficients of $\Phi_{AT}$ as linear combinations of multiple zeta values have not been presented so far, other than the computation

\[ (\Phi_{AT}|A^2BA^4B) = \frac{2048\zeta(3, 5) - 6293\zeta(3)\zeta(5)}{524288\pi^8} \]

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REFERENCES


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