ASSOCIATORS AND KONTSEVICH'S EYE

HIDEKAZU FURUSHO

Abstract. This is a report on Drinfeld's associators and Kontsevich's eye which is based on my talk at Algebraic Lie Theory and Representation Theory (ALTReT) 2018.

Contents

1. Drinfeld's associator

The notion of associators was introduced by Drinfeld in [Dr]. They describe monodromies of the KZ-equations [1]. They are essential for the construction of quasi-triangular quasi-Hopf quantized universal enveloping algebras ([Dr]), for the quantization of Lie-bialgebras (Etingof-Kazhdan quantization [EK]), for the proof of formality chain operad of little discs by Tamarkin [Ta] (see also Ševera and Willwacher [SW]), the formal solution of Kashiwara-Vergne conjecture by Alekseev and Torossian [AT12] and also for the combinatorial reconstruction of the universal Vassiliev knot invariant (the Kontsevich invariant [K93, B95]) by Bar-Natan [B97], Cartier [Ca], Kassel and Turaev [KaT], Le and Murakami [LM96a] and Piunikhin [P] (for some of these related topics, consult [Fu14, Fu16]).

Let us fix notations: Let k be a field of characteristic 0 and $\bar{k}$ be its algebraic closure. Denote by $\widehat{U\mathfrak{f}_2} = k\langle\langle A, B\rangle\rangle$ the non-commutative formal power series ring defined as the completion (with respect to degree) of the universal enveloping algebra of the free Lie algebra $\mathfrak{f}_2$

Date: September 22, 2018.

[1]KZ stands for Knizhnik and Zamolodchikov.

over k with two variables A and B. An element $\varphi=\varphi(A,B)$ of $\widehat{U\mathfrak{f}_2}$ is called *group-like* [2] if it satisfies

$$\Delta(\varphi)=\varphi\otimes\varphi \text{ and } \varphi(0,0)=1 \tag{1.1}$$

where $\Delta:\widehat{U\mathfrak{f}_2}\to\widehat{U\mathfrak{f}_2}\hat{\otimes}\widehat{U\mathfrak{f}_2}$ is given by $\Delta(A)=A\otimes 1+1\otimes A$ and $\Delta(B)=B\otimes 1+1\otimes B$. For any k-algebra homomorphism $\iota:\widehat{U\mathfrak{f}_2}\to S$, the image $\iota(\varphi)\in S$ is denoted by $\varphi(\iota(A),\iota(B))$.

Denote by $\widehat{U\mathfrak{a}_3}$ (resp. $\widehat{U\mathfrak{a}_4}$) the completion of the universal enveloping algebra of the *pure braid Lie algebra* $\mathfrak{a}_3$ (resp. $\mathfrak{a}_4$) over k with 3 (resp. 4) strings, which is generated by t_{ij} ($1\leqslant i,j\leqslant 3$ (resp. 4)) with defining relations

$$t_{ii}=0,\ t_{ij}=t_{ji},\ [t_{ij},t_{ik}+t_{jk}]=0 \ (i,j,k\text{: all distinct})$$
$$\text{and}\ \ [t_{ij},t_{kl}]=0\ (i,j,k,l\text{: all distinct}).$$

Definition 1.1 ([Dr]). A pair (μ,φ) with a *non-zero* element μ in k and a group-like series $\varphi=\varphi(A,B)\in\widehat{U\mathfrak{f}_2}$ is called an *associator* if it satisfies *one pentagon equation*

(1.2)
$$\varphi(t_{12},t_{23}+t_{24})\varphi(t_{13}+t_{23},t_{34})=\varphi(t_{23},t_{34})\varphi(t_{12}+t_{13},t_{24}+t_{34})\varphi(t_{12},t_{23})$$

in $\widehat{U\mathfrak{a}_4}$ and *two hexagon equations*

(1.3)
$$\exp\{\frac{\mu(t_{13}+t_{23})}{2}\}=\varphi(t_{13},t_{12})\exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{13},t_{23})^{-1}\exp\{\frac{\mu t_{23}}{2}\}\varphi(t_{12},t_{23}),$$

(1.4)
$$\exp\{\frac{\mu(t_{12}+t_{13})}{2}\}=\varphi(t_{23},t_{13})^{-1}\exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{12},t_{13})\exp\{\frac{\mu t_{12}}{2}\}\varphi(t_{12},t_{23})^{-1}$$

in $\widehat{U\mathfrak{a}_3}$.

Drinfeld [Dr] proved that such a pair always exists for any field k of characteristic 0. The equations (1.2)–(1.4) reflect the three axioms of braided monoidal categories introduced by Joyal and Street [JS].

Actually, the two hexagon equations are a consequence of the one pentagon equation:

Theorem 1.2 ([Fu10]). *Let $\varphi=\varphi(A,B)$ be a group-like element of $\widehat{U\mathfrak{f}_2}$. Suppose that φ satisfies the pentagon equation* (1.2). *Then there always exists $\mu\in\bar{k}$ (unique up to signature) such that the pair (μ,φ) satisfies two hexagon equations* (1.3) *and* (1.4).

[2] It is equivalent to $\varphi\in\exp\widehat{\mathfrak{f}}_2$.

We note that several different proofs of the above theorem were obtained (see [AT12, BD, W]).

A well-known example of associators is the KZ-associator:

Example 1.3. The *KZ-associator* (aka. Drinfeld associator) $\varPhi_{\mathrm{KZ}} = \varPhi_{\mathrm{KZ}}(A,B) \in \mathbb{C}\langle\langle A,B\rangle\rangle$ is defined to be the quotient $\varPhi_{\mathrm{KZ}} = G_1(z)^{-1}G_0(z)$ where G_0 and G_1 are the solutions of the *formal KZ-equation*, which is the following differential equation for multi-valued functions $G(z)$ of $\mathbb{C}\backslash\{0,1\}$ valued on $\mathbb{C}\langle\langle A,B\rangle\rangle$

$$dG(z) = \omega_{\mathrm{KZ}} \cdot G(z) \tag{1.5}$$

with

$$\omega_{\mathrm{KZ}} := \frac{dz}{z}A + \frac{dz}{z-1}B$$

such that $G_0(z) \approx z^A$ when $z \to 0$ and $G_1(z) \approx (1-z)^B$ when $z \to 1$ (cf.[Dr]). It is shown in [Dr] that the pair $(2\pi\sqrt{-1}, \varPhi_{\mathrm{KZ}})$ forms an associator for $k = \mathbb{C}$. Namely $\varPhi_{\mathrm{KZ}}$ satisfies (1.1)$\sim$(1.4) with $\mu = 2\pi\sqrt{-1}$.

Remark 1.4. (i). The KZ-associator is expressed as follows:

$$\varPhi_{\mathrm{KZ}}(X_0,X_1) = 1 + \sum_{\substack{m,k_1,\dots,k_m\in\mathbb{N}\\ k_m>1}} (-1)^m\zeta(k_1,\cdots,k_m)A^{k_m-1}B\cdots A^{k_1-1}B + (\text{regularized terms}).$$

Here $\zeta(k_1,\cdots,k_m)$ is the *multiple zeta value* (MZV in short), the real number defined by the following power series

$$\zeta(k_1,\cdots,k_m) := \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}} \tag{1.6}$$

for m, $k_1,\dots,$ $k_m \in \mathbb{N}(=\mathbb{Z}_{>0})$ with $k_m > 1$ (its convergent condition). All of the coefficients of $\varPhi_{\mathrm{KZ}}$ (including its regularized terms) are explicitly calculated in terms of MZV's in [Fu03] Proposition 3.2.3 by Le-Murakami's method in [LM96b] Theorem A.8.

(ii). Since all of the coefficients of $\varPhi_{\mathrm{KZ}}$ are described by MZV's, the equations (1.1)$\sim$(1.4) for $(\mu,\varphi) = (2\pi\sqrt{-1}, \varPhi_{\mathrm{KZ}})$ yield algebraic relations among them, which are called *associator relations*. It is expected that the associator relations might produce all algebraic relations among MZV's.

In Definition 3.3, we will see another example of associators, the AT-associator $\varPhi_{\mathrm{AT}}$, which was constructed by Alekseev and Torossian [AT10] by using a parallel transport of an analog (3.2) of the KZ-equation.

2. Kontsevich's eye

We will recall the compactified configuration spaces and weights of Lie graphs [K03].

Let $n \geqslant 1$. For a topological space X, we define $\mathrm{Conf}_n(X) := \{(x_1,\dots,x_n) \bigm| x_i \neq x_j \ (i \neq j)\}$. The group $\mathrm{Aff}_+ := \{x \mapsto ax+b \bigm| a \in \mathbb{R}_+^\times, b \in \mathbb{C}\}$ acts on $\mathrm{Conf}_n(\mathbb{C})$ diagonally by rescallings and parallel translations. We denote the quotient by

$$C_n := \mathrm{Conf}_n(\mathbb{C})/\mathrm{Aff}_+$$

for $n \geqslant 2$, which is a connected oriented smooth manifold with dimension $2n-3$. E.g.

- $C_2 \simeq S^1$.
- $C_3 \simeq S^1 \times (\mathrm{P}^1(\mathbb{C}) \setminus \{0,1,\infty\})$.

For a finite set I with $|I| = n$, we put $C_I = C_n$. For $I' \subset I$ with $|I'| > 1$, we have the pull-back map $C_I \twoheadrightarrow C_{I'}$.

Put $\mathrm{Conf}_{n,m}(\mathbb{H},\mathbb{R}) := \mathrm{Conf}_n(\mathbb{H}) \times \mathrm{Conf}_m(\mathbb{R})$ with the coordinate $(z_1,\dots,z_n,x_1,\dots,x_m)$, where $\mathbb{H}$ is the upper half plane. The group $\mathrm{Aff}_+^{\mathbb{R}} := \{x \mapsto ax+b \bigm| a \in \mathbb{R}_+^\times, b \in \mathbb{R}\}$ acts there diagonally and we denote the quotient by

$$C_{n,m} := \mathrm{Conf}_{n,m}(\mathbb{H},\mathbb{R})/\mathrm{Aff}_+^{\mathbb{R}}$$

for $n,m \geqslant 0$ with $2n+m \geqslant 2$. It is an oriented smooth manifold with dimension $2n+m-2$ and with $m!$ connected components. E.g.

- $C_{0,2} \simeq \{\pm 1\}$, $\quad C_{0,2}^+ = \{+1\}$, $C_{0,2}^- := \{-1\}$.
- $C_{1,1} \simeq \{e^{\sqrt{-1}\pi\theta} \bigm| 0 < \theta < 1\}$.
- $C_{2,0} \simeq \mathbb{H} - \{\sqrt{-1}\}$.

For a finite set I and J with $|I| = n$ and $|J| = m$, we put $C_{I,J} = C_{n,m}$. Then for $I' \subset I$ and $J' \subset J$, we have the pull-back map $C_{I,J} \twoheadrightarrow C_{I',J'}$.

Below we recall [3] Kontsevich's [K03] compactifications $\overline{C}_n$ and $\overline{C}_{n,m}$ of C_n and $C_{n,m}$ à la Fulton-MacPherson (in more detail, consult [Si]) : For a finite set I with $|I| = n$, we put $\tilde{C}_I := \tilde{C}_n := \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n z_i = 0\} \cap S^{2n-1}$. By identifying it with $\mathbb{C}^n$–diag/Aff_+ (diag $= \{(z,\dots,z) \mid z \in \mathbb{C}\}$), we obtain an embedding $C_I \hookrightarrow \tilde{C}_I$. The compactification $\overline{C}_I = \overline{C}_n$ is a compact topological manifold *with corners* which is defined to be the closure of the image of the associated embedding $\varPhi : C_I \hookrightarrow \prod_{J \subset I, 1<|J|} \tilde{C}_J$. While by the embedding $\mathrm{Conf}_{n,m}(\mathbb{H},\mathbb{R}) \hookrightarrow \mathrm{Conf}_{2n+m}(\mathbb{C})$ sending $(z_1,\dots,z_n,x_1,\dots,x_m) \mapsto (z_1,\dots,z_n,\bar{z}_1,\dots,\bar{z}_n,x_1,\dots,x_m)$, we have an embedding $C_{n,m} \hookrightarrow C_{2n+m}$. By combining it with $\varPhi$, we obtain an embedding $C_{n,m} \hookrightarrow \overline{C}_{2n+m}$. The

[3] Here we follow the conventions of Bruguières ([CKTB]).

compactification $\overline{C}_{n,m}$ is a compact topological manifold with corners which is defined to be the closure of the embedding.

They are functorial with respect to the inclusions of two finite sets, i.e. $I_1 \subset I_2$ and $J_1 \subset J_2$ with $\sharp(I_k) = n_k$ and $\sharp(J_k) = m_k$ $(k = 1, 2)$ yield a natural map $\overline{C}_{n_2,m_2} \to \overline{C}_{n_1,m_1}$. The stratification of his compactification has a very nice description in terms of trees in [K03] (also refer [CKTB]). E.g.

- $\overline{C}_{0,2} = C_{0,2} \simeq \{\pm 1\}$,
- $\overline{C}_{1,1} = C_{1,1} \sqcup C_{0,2} = \{e^{\sqrt{-1}\pi\theta} \mid 0 \leqslant \theta \leqslant 1\}$,
- $\overline{C}_{2,0} = C_{2,0} \sqcup C_{1,1} \sqcup C_{1,1} \sqcup C_2 \sqcup C_{0,2}$.

The $\overline{C}_{2,0}$ is called *Kontsevich's eye* and its each component bears a special name as is indicated in Figure 2.1. The *upper* (resp. *lower*) *eyelid*

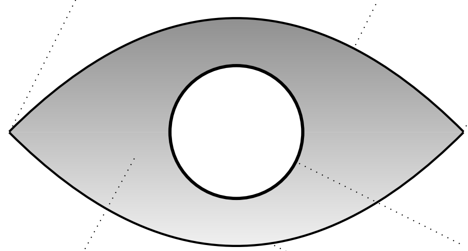

FIGURE 2.1. Kontsevich's eye $\overline{C}_{2,0}$

corresponds to z_1 (resp. z_2) on the the real line. The *iris* magnifies collisions of z_1 and z_2 on $\mathbb{H}$. LC (resp. RC) which stands for the *left* (resp. *right*) *corner* is the configuration of $z_1 > z_2$ (resp. $z_1 < z_2$) on the real line.

The *angle map* $\phi : \overline{C}_{2,0} \to \mathbb{R}/\mathbb{Z}$ is the map induced from the map $\mathrm{Conf}_2(\mathbb{H}) \to \mathbb{R}/\mathbb{Z}$ sending

$$\phi : (z_1, z_2) \mapsto \frac{1}{2\pi} \arg\left(\frac{z_2 - z_1}{z_2 - \bar{z}_1}\right). \tag{2.1}$$

We note that ϕ is identically zero on the upper eyelid but is not on the lower eyelid.

Next we will recall the notion of Lie graphs and their weight functions and 1-forms.

Definition 2.1. Let $n \geqslant 1$. A *Lie graph* Γ *of type* $(n, 2)$ is a graph consisting of two finite sets, the *set of vertices* $V(\Gamma) := \{\boxed{1}, \boxed{2}, ①, ②, \dots, Ⓝ\}$ and the *set of edges* $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. The points $\boxed{1}$ and $\boxed{2}$ are called as the *ground points*, while the points ①, ②, …, ⓝ are called as the *air points*. We equip $V(\Gamma)$ with the total order $\boxed{1} < \boxed{2} <$ ① $<$ ② $< \cdots <$ ⓝ.

For each $e \in E(\Gamma)$, under the inclusion $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$, we call the corresponding first (resp. second) component $s(e)$ (resp. $t(e)$) as the *source* (resp. the *target*) of e and denote as $e = (s(e), t(e))$. We equip $E(\Gamma)$ with the lexicographic order induced from that of $V(\Gamma)$. Both $V(\Gamma)$ and $E(\Gamma)$ are subject to the following conditions:

(1) An air point fires two edges: That means there always exist two edges with the source ⓘ for each $i = 1, \dots, n$.
(2) An air point is shot by one edge at most: That means there exists at most one edge with its target ⓘ for each $i = 1, \dots, n$.
(3) A ground point never fire edges: That means there is no edge with its source on ground points.
(4) The graph Γ becomes a rooted trivalent tree after we cut off small neighborhoods of ground points: That means that the graph of Γ admits a unique vertex (called *the root*) shoot by no edges and it gives a rooted trivalent trees if we regard the vertex as a root and distinguish all targets of edges firing ground points.

Let Γ be a Lie graph of type $(n, 2)$. We define a Lie monomial $\Gamma(A, B) \in \widehat{\mathfrak{f}}_2$ of degree $n+1$ to be the associated element with the root by the following procedure: With $\boxed{1}$ and $\boxed{2}$, we assign A and $B \in \widehat{\mathfrak{f}}_2$ respectively. With each internal vertex v firing two edges $e_1 = (v, w_1)$ and $e_2 = (v, w_2)$ such that $e_1 < e_2$, we assign $[\Gamma_1, \Gamma_2] \in \widehat{\mathfrak{f}}_2$ where Γ_1 and $\Gamma_2 \in \widehat{\mathfrak{f}}_2$ are the corresponding Lie monomials with the vertices w_1 and w_2 respectively. Recursively we may assign Lie elements with all vertices of Γ.

E.g. Figure 2.2 is an example of Lie graph of type $(3, 2)$. Its root is ③. The associated Lie elements of the vertices $\boxed{1}, \boxed{2}$, ①, ②, ③ are A, B, $[A, B]$, $[B, [A, B]]$, $[B, [B, [A, B]]]$ respectively.

Each $e \in E(\Gamma)$ determines a subset $\{s(e), t(e)\} \subset V(\Gamma)$ with $|V(\Gamma)| = n + 2$ which yields a pull-back $p_e : \overline{C}_{n+2,0} \to \overline{C}_{2,0}$. By composing it with the angle map (2.1), we get a map $\phi_e : \overline{C}_{n+2,0} \to \mathbb{R}/\mathbb{Z}$. The

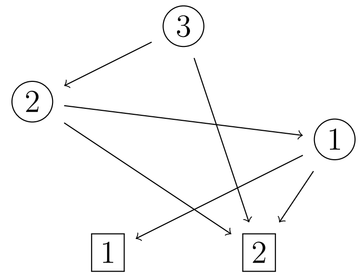

FIGURE 2.2. $\Gamma(A,B) = [B,[B,[A,B]]]$

PA[4] $2n$-forms Ω_Γ on $\overline{C}_{n+2,0}$ (which is $2n$-dimensional compact space) associated with Γ is given by the ordered exterior product

$$\Omega_\Gamma := \wedge_{e\in E(\Gamma)} d\phi_e \ \in \Omega^{2n}_{\mathrm{PA}}(\overline{C}_{n+2,0}).$$

Here $\Omega^{2n}_{\mathrm{PA}}(\overline{C}_{n+2,0})$ means the space of PA $2n$-forms of $\overline{C}_{n+2,0}$

Definition 2.2. (i). Put $\pi : \overline{C}_{n+2,0} \to \overline{C}_{2,0}$ to be the above projection induced from the inclusion $\{\boxed{1},\boxed{2}\} \subset \{\boxed{1},\boxed{2},①,②,\dots,ⓝ\}$. The *weight function* (see [To]) of Γ is the smooth function $w_\Gamma : \overline{C}_{2,0} \to \mathbb{C}$ defined by $w_\Gamma := \pi_*(\Omega_\Gamma)$ where π_* is the push-forward (the integration along the fiber of the projection π, cf. [HLTV]), that is, the function which assigns $\xi \in \overline{C}_{2,0}$ with

$$w_\Gamma(\xi) = \int_{\pi^{-1}(\xi)} \Omega_\Gamma \in \mathbb{C}.$$

(ii). We denote $L\Gamma$ (resp. $R\Gamma$) to be a graph obtained from Γ by adding one more edge e_L from $\boxed{1}$ (resp. e_R from $\boxed{2}$) to the root of Γ. The regular $(2n+1)$-form $\Omega_{L\Gamma}$ (resp. $\Omega_{R\Gamma}$) on $\overline{C}_{n+2,0}$ is defined to be

$$\Omega_{L\Gamma} := d\phi_{e_L} \wedge \Omega_\Gamma \qquad (\text{resp.} \quad \Omega_{R\Gamma} := d\phi_{e_R} \wedge \Omega_\Gamma)$$

in $\Omega^{2n}_{\mathrm{PA}}(\overline{C}_{n+2,0})$. The one-forms $\omega_{L\Gamma}$ and $\omega_{R\Gamma}$, which we call the *weight forms* of Γ here, are the PA one-forms of $\overline{C}_{2,0}$ respectively defined by

$$\omega_{L\Gamma} := \pi_*(\Omega_{L\Gamma}) \qquad \text{and} \qquad \omega_{R\Gamma} := \pi_*(\Omega_{R\Gamma})$$

in $\Omega^1_{\mathrm{PA}}(\overline{C}_{2,0})$, i.e. they are one-forms respectively defined by

$$\omega_{L\Gamma}(\xi) = \int_{\pi^{-1}(\xi)} \Omega_{L\Gamma}, \qquad \text{and} \quad \omega_{R\Gamma}(\xi) = \int_{\pi^{-1}(\xi)} \Omega_{R\Gamma}$$

where ξ runs over $\overline{C}_{2,0}$.

[4]'PA' stands for piecewise-algebraic (cf. [KS, HLTV, LV]).

Remark 2.3. (i). Particularly the special value $w_\Gamma(\mathrm{RC})$ of the function $w_\Gamma(\xi)$ at $\xi = \mathrm{RC}$ is called the *Kontsevich weight* of Γ. It appears as a coefficient of Kontsevich's formula on deformation quantization in [K03].

(ii). While its restriction $w_\Gamma|_{C_2}$ to the iris C_2 is identically 0 because $\Omega_\Gamma|_{C_2} = 0$ due to the occurrence of double edges.

3. Alekseev-Torossian's associator

We will recall the deformation $Z(\xi)$ ($\xi \in \overline{C}_{2,0}$) of the Campbell-Baker-Hausdorff series and its associated differential equation. Then we will see how the AT-associator Φ_{AT} will be constructed from the differential equation and give a new presentation of Φ_{AT}.

Consider the smooth function $Z : \overline{C}_{2,0} \to \widehat{\mathfrak{f}}_2$ defined by

$$\xi \in \overline{C}_{2,0} \mapsto Z(\xi) := A + B + \sum_{n \geqslant 1} \sum_{\Gamma \in \mathrm{LieGra}^{\mathrm{geom}}_{n,2}} \omega_\Gamma(\xi)\Gamma(A, B) \ \in \widehat{\mathfrak{f}}_2.$$

Here $\mathrm{LieGra}^{\mathrm{geom}}_{n,2}$ is the set of *geometric* (it means non-labeled) Lie graphs of type $(n, 2)$. We note that both Ω_Γ and $\Gamma(A, B)$ require the order of $E(\Gamma)$ however their product $\Omega_\Gamma \cdot \Gamma(A, B)$ does not (cf. [CKTB]).

Remark 3.1. (i). It is obtained by Kathotia [Kat] that $Z(\mathrm{RC})$ is equal to the Campbell-Baker-Hausdorff series $\mathrm{CBH}(A, B) = \log(e^A e^B)$.

(ii). While its its restriction $Z|_{C_2}$ to the iris C_2 is simply equal to the addition $A + B$ because we have $\Omega_\Gamma|_{C_2} = 0$ (cf. Remark 2.3).

We may say that Z is a series which deforms $\mathrm{CBH}(A, B)$.

Theorem 3.2 ([To])**.** *The series $Z(\xi)$ satisfies the differential equation*

$$dZ(\xi) = \omega_{\mathrm{AT}} \cdot Z(\xi) \tag{3.1}$$

with

$$\omega_{\mathrm{AT}} := \mathrm{der}\,(\omega_L, \omega_R) \in \mathrm{tder}_2 \widehat{\otimes} \Omega^1_{\mathrm{PA}}(\overline{C}_{2,0}).$$

Here tder_2 be the Lie algebra consisting of tangential derivations $\mathrm{der}(\alpha, \beta) : \widehat{\mathfrak{f}}_2 \to \widehat{\mathfrak{f}}_2$ ($\alpha, \beta \in \widehat{\mathfrak{f}}_2$) such that $A \mapsto [A, \alpha]$ and $B \mapsto [B, \beta]$, and

$$\omega_L := B \cdot \quad d\phi + \sum_{n \geqslant 1} \sum_{\Gamma \in \mathrm{LieGra}^{\mathrm{geom}}_{n,2}} \Gamma(A, B) \cdot \omega_{L\Gamma},$$

$$\omega_R := A \cdot \sigma^*(d\phi) + \sum_{n \geqslant 1} \sum_{\Gamma \in \mathrm{LieGra}^{\mathrm{geom}}_{n,2}} \Gamma(A, B) \cdot \omega_{R\Gamma}.$$

The symbol σ stands for the involution of $\overline{C}_{2,0}$ caused by the switch of z_1 and z_2.

Related to (3.1), in [AT10] they considered the following differential equation on $\overline{C}_{2,0}$:

$$dg(\xi) = -g(\xi) \cdot \omega_{\mathrm{AT}} \tag{3.2}$$

with $g(\xi) \in \mathrm{TAut}_2 := \exp \mathrm{tder}_2$, the pro-algebraic subgroup of Aut_2 consisting of tangential automorphisms $\mathrm{Int}(\alpha, \beta) : \widehat{\mathfrak{f}_2} \to \widehat{\mathfrak{f}_2}$ ($\alpha, \beta \in \exp \widehat{\mathfrak{f}_2}$) such that $A \mapsto \alpha^{-1} A \alpha$ and $B \mapsto \beta^{-1} B \beta$. They denote its parallel transport (its holonomy) of (3.2) for the straight path from α_0 (the position 0 at the iris, see Figure 3.1) to RC by $F_{\mathrm{AT}} \in \mathrm{TAut}_2$.

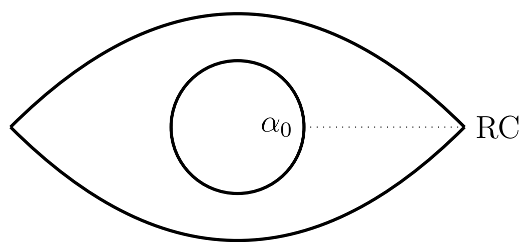

FIGURE 3.1. Parallel transport

Definition 3.3 ([AT10]). The *AT-associator* Φ_{AT} is defined to be

$$\Phi_{\mathrm{AT}} := F_{\mathrm{AT}}^{1,23} \circ F_{\mathrm{AT}}^{2,3} \circ (F_{\mathrm{AT}}^{1,2})^{-1} \circ (F_{\mathrm{AT}}^{12,3})^{-1} \in \mathrm{TAut}_3. \tag{3.3}$$

Here for any $T = \mathrm{Int}(\alpha, \beta) \in \mathrm{TAut}_2$, we denote

$$T^{1,2} := \mathrm{Int}\left(\alpha(A,B), \beta(A,B), 1\right), \quad T^{2,3} := \mathrm{Int}\left(1, \alpha(B,C), \beta(B,C)\right),$$
$$T^{1,23} := \mathrm{Int}\left(\alpha(A, B+C), \beta(A, B+C), \beta(A, B+C)\right),$$
$$T^{12,3} := \mathrm{Int}\left(\alpha(A+B, C), \alpha(A+B, C), \beta(A+B, C)\right)$$

in $\mathrm{TAut}_3 := \exp \mathrm{tder}_3$ which is similarly defined to be the group of tangential automorphisms of the completed free Lie algebra $\widehat{\mathfrak{f}_3}$ with variables A, B and C.

We note that there is a Lie algebra inclusion $\widehat{\mathfrak{f}_2} \hookrightarrow \mathrm{tder}_3$ sending

$$A \mapsto t_{12} := \mathrm{der}(B, A, 0) \quad \text{and} \quad B \mapsto t_{23} := \mathrm{der}(0, C, B) \tag{3.4}$$

which induces an inclusion $\exp \widehat{\mathfrak{f}_2} \hookrightarrow \mathrm{TAut}_3$.

Theorem 3.4 ([AT12, SW]). *The AT-assocciator Φ_{AT} forms an associator. Namely it belongs to $\exp \widehat{\mathfrak{f}_2}$ ($\subset \mathbb{C}\langle\langle A, B \rangle\rangle$) and satisfies the* *equations* [Dr] *(2.12), (2.13) and (5.3). Furthermore it is real (i.e. it belongs to the real structure* $\mathbb{R}\langle\langle A,B\rangle\rangle$*) and even.* [5]

The following gives a more direct presentation of Φ_{AT}.

Theorem 3.5 ([Fu18])**.** *We have*

$$\Phi_{\mathrm{AT}} = \left(\mathcal{P}\exp\int_{\mathrm{RC}}^{\alpha_0}(l_{\widehat{\omega}} + D_{\widehat{\omega}})\right)(1) \ \in \mathbb{C}\langle\langle A,B\rangle\rangle. \tag{3.5}$$

Here $l_{\widehat{\omega}}$ is the left multiplication by $\widehat{\omega}$ and $D_{\widehat{\omega}}$ is given by

$$D_{\widehat{\omega}} := \mathrm{der}\,(0,\widehat{\omega}) \in \mathrm{tder}_2\widehat{\otimes}\Omega^1_{\mathrm{PA}}(\overline{C}_{2,0})$$

with

$$\widehat{\omega} := \sum_{n\geqslant 1}\ \sum_{\Gamma\in\mathrm{LieGra}_{n,2}} \widehat{\omega}_\Gamma\cdot\Gamma(A,B) \quad \text{and} \quad \widehat{\omega}_\Gamma := \omega_{R\Gamma} - \omega_{L\Gamma}. \tag{3.6}$$

and for any one-form $\Omega \in \Omega^1_{\mathrm{PA}}(\overline{C}_{2,0})$ we define

$$\begin{aligned}\mathcal{P}\exp\int_{\mathrm{RC}}^{\alpha_0}\Omega &:= \mathrm{id} + \int_{\mathrm{RC}}^{\alpha_0}\Omega + \int_{\mathrm{RC}}^{\alpha_0}\Omega\cdot\Omega + \cdots \\ &:= \mathrm{id} + \int_{0<s_1<1}\ell^*\Omega(s_1) + \int_{0<s_1<s_2<1}\ell^*\Omega(s_2)\wedge\ell^*\Omega(s_1) + \cdots .\end{aligned}$$

with the straight path ℓ from RC to α_0 in Figure 3.1.

This theorem enables us to calculate explicitly all the coefficients of the AT-associator Φ_{AT} as rational linear combinations of iterated integrals of weight forms of Lie graphs (see [Fu18] for explicit computations in depth 1 and 2).

Explicit formulae to describe all the coefficients of the KZ-associator Φ_{KZ} in terms of multiple zeta values are given in [Fu03, LM96b]. Whereas, as far as the author knows, any explicit formulae to present all the coefficients of Φ_{AT} as linear combinations of multiple zeta values have not been presented so far, other than the computation

$$(\Phi_{\mathrm{AT}}|A^2BA^4B) = \frac{2048\zeta(3,5) - 6293\zeta(3)\zeta(5)}{524288\pi^8}$$

by M. Felder [Fe].

---

[5] It means $\Phi_{\mathrm{AT}}(-A,-B) = \Phi_{\mathrm{AT}}(A,B)$, from which it follows that $\Phi_{\mathrm{KZ}} \neq \Phi_{\mathrm{AT}}$ because Φ_{KZ} is not even.

References

[AT10] Alekseev, A. and Torossian, C., *Kontsevich deformation quantization and flat connections*, Comm. Math. Phys. **300** (2010), no. 1, 47–64.

[AT12] Alekseev, A. and Torossian, C., *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Ann. of Math. (2) **175** (2012), no. 2, 415–463.

[B95] Bar-Natan, D.; *On the Vassiliev knot invariants*, Topology 34 (1995), no. 2, 423–472.

[B97] Bar-Natan, D.; *Non-associative tangles*, Geometric topology (Athens, GA, 1993), 139-183, AMAPIP Stud. Adv. Math., **2.1**, Amer. Math. Soc., Providence, RI, 1997.

[BD] Bar-Natan, D. and Dancso, Z.; *Pentagon and hexagon equations following Furusho*, Proc. Amer. Math. Soc. **140** (2012), no. 4, 1243–1250.

[Ca] Cartier, P.; *Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds*, C. R. Acad. Sci. Paris Ser. I Math. **316** (1993), no. 11, 1205-1210.

[CKTB] Cattaneo, A., Bernhard, B., Torossian, C. and Bruguières, A., *Déformation, quantification, théorie de Lie*, Panoramas et Synthèses, **20**. Société Mathématique de France, Paris, 2005.

[Dr] Drinfeld, V. G., *On quasitriangular quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991), no. 4, 829–860.

[EK] Etingof, P. and Kazhdan, D.; *Quantization of Lie bialgebras. II*, Selecta Math. **4** (1998), no. 2, 213–231.

[Fe] Felder, M., *On the irrationality of certain coefficients of the Alekseev-Torossian associator*, J. Lie Theory 27 (2017), no. 2, 501–528.

[Fu03] Furusho, H., *The multiple zeta value algebra and the stable derivation algebra*, Publ. Res. Inst. Math. Sci. Vol **39**. no 4. (2003). 695–720.

[Fu10] Furusho, H., *Pentagon and hexagon equations*, Ann. of Math. (2) **171** (2010), no. 1, 545–556.

[Fu14] Furusho, H., *Around associators*, Automorphic forms and Galois representations, 2, London Math.Soc. Lecture Note Ser. **415** (2014), 105-117, Cambridge Univ. Press, Cambridge,
which is a revised version of '*Four groups related to associators*', the report on Mathematische Arbeitstagung. 2011.

[Fu16] Knots and Grothendieck-Teichmuller group (Japanese), Math for Industry Lecture note series, Vol 68, Kyushu. Univ.

[Fu18] Furusho, H., *On the coefficients of the Alekseev Torossian associator*, Journal of Algebra **506** (2018), 364–378.

[HLTV] Hardt, R., Lambrechts, P., Turchin, V. and Volić, I., *Real homotopy theory of semi-algebraic sets*, Algebr. Geom. Topol. **11** (2011), no. 5, 2477–2545.

[JS] Joyal, A. and Street, R.; *Braided tensor categories*, Adv. Math. 102 (1993), no. 1, 20–78.

[KaT] Kassel, C. and Turaev, V.; *Chord diagram invariants of tangles and graphs*, Duke Math. J. **92** (1998), no. 3, 497–552.

[Kat] Kathotia, V.; *Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula*, Internat. J. Math. **11** (2000), no. 4, 523–551.

[K93] Kontsevich, M.; *Vassiliev's knot invariants*, I. M. Gel'fand Seminar, 137–150, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.

[K03] Kontsevich, M., *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216.

[KS] Kontsevich, M. and Soibelman, Y., *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), 255–307, Math. Phys. Stud., **21**, Kluwer Acad. Publ., Dordrecht, 2000.

[LV] Lambrechts, P. and Volić, I., *Formality of the little N-disks operad*, Mem. Amer. Math. Soc. **230** (2014), no. 1079.

[LM96a] Le, T.Q.T. and Murakami, J.; *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Compositio Math. **102** (1996), no. 1, 41-64.

[LM96b] Le, T.Q.T. and Murakami, J., *Kontsevich's integral for the Kauffman polynomial*, Nagoya Math. J. **142** (1996), 39–65.

[P] Piunikhin, S.; *Combinatorial expression for universal Vassiliev link invariant*, Comm. Math. Phys. **168** (1995), no. 1, 1-22.

[RW] Rossi, C. and Willwacher, T., *P. Etingof's conjecture about Drinfeld associators*, preprint `arXiv:1404.2047`.

[SW] Ševera, P. and Willwacher, T., *Equivalence of formalities of the little discs operad*, Duke Math. J. **160** (2011), no. 1, 175–206.

[Si] Sinha, Dev P., *Manifold-theoretic compactifications of configuration spaces*, Selecta Math. (N.S.) **10** (2004), no. 3, 391–428.

[Ta] Tamarkin, D. E.; *Formality of chain operad of little discs*, Lett. Math. Phys. **66** (2003), no. 1-2, 65–72.

[To] Torossian, C., *Sur la conjecture combinatoire de Kashiwara-Vergne*, J. Lie Theory **12** (2002), no. 2, 597–616.

[W] Willwacher, T., *M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra*, Invent. Math. **200** (2015), no. 3, 671–760.

Graduate School of Mathematics, Nagoya University, Chikusa-ku, Furo-cho, Nagoya, 464-8602, Japan

Email address: `furusho@math.nagoya-u.ac.jp`