# WHAT IS LOG TERMINAL ? 2004/11/4

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Abstract. In this paper, we explain the subtleties of the various kinds of log terminal singularities. We focus on the notion of divisorial log terminal singularities, which seems to be the most useful one. Most significantly for us, divisorial log terminal singularities behave very well under adjunction, see Section 9. We also explain Szabó's resolution lemma and the notion of log resolution. The results in Section 6 seem to be lacking in the literature. Finally we collect miscellaneous results and examples on singularities of pairs in the log MMP to better understand log terminal singularities.

Contents



## 1. What is log terminal?

This paper is a guide to go around the world of *log terminal singularities*. The main purpose is to attract the reader's attention to the subtleties of the various kinds of log terminal singularities. Needless to

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say, my opinion is not necessarily the best. We hope that this paper will help the reader understand the definition of *log terminal*. Almost all the results in this paper are known to experts, and perhaps only to them. Note that this paper is not self-contained. For systematic treatments of singularities in the log MMP, see, for example, [KM, Section 2.3]. We expect that the reader is familiar with the basic properties of singularities of pairs.

In the log MMP, there are too many variants of *log terminal*. This sometimes causes trouble when we treat log terminal singularities. We already have four bibles on the log MMP: [KMM], [FA], [KM], and [Ma]. It is unpleasant that each bible adopted different definitions of log terminal and even of *log resolutions*. Historically, Shokurov introduced various kinds of log terminal singularities in his famous paper [Sh1, §1]. However, we do not mention [Sh1] anymore for simplicity. We only treat the above four bibles. Before we come to the subject, we note:

**Remark 1.1.** In [Ma, Chapter 4], Matsuki explains various kinds of singularities in details. Unfortunately he made the mistake of applying Theorem 5.1 to normal crossing divisors, whilst it is only valid with simple normal crossing divisors. Accordingly, when we read [Ma] we have to replace *normal crossings* with *simple normal crossings* in the definition of dlt and so forth. See Definitions 2.8, 7.1, Remarks 7.6, 10.8, and ($2''$) of [Ma, Definition 4-3-2].

We summarize the contents of this paper: Sections 2 and 3 are preliminaries. We recall well-known definitions and fix some notation. In Section 4, we define the notion of *divisorial log terminal singularities*, which is one of the most important notions of log terminal singularities. In Section 5, we treat Szabó's resolution lemma, which is very important in the log MMP. Section 6 was suggested by Mori. Here, we explain that Szabó's resolution lemma is not true for normal crossing divisors by using the Whitney umbrella. Section 7 deals with *log resolutions*. Here, we explain subtleties of various kinds of log terminal singularities. In Section 8, we collect examples to help the reader understand singularities of pairs. In Section 9, we describe the *adjunction formula* for dlt pairs, which plays an important role in the log MMP. We need it in [F4]. Finally, Section 10 collects some miscellaneous comments.

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**Notation.** The set of integers (resp. rational numbers, real numbers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$). We will work over an algebraically closed field $k$ of characteristic zero; my favorite is $k = \mathbb{C}$.

## 2. Preliminaries on $\mathbb{Q}$-divisors

Before we introduce singularities of pairs, let us recall the basic definitions about $\mathbb{Q}$-divisors.

**Definition 2.1** ($\mathbb{Q}$-Cartier divisor)**.** Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety $X$, that is, $d_i \in \mathbb{Q}$ and $D_i$ is a prime divisor on $X$ for every $i$. Then $D$ is $\mathbb{Q}$-*Cartier* if there exists a positive integer $m$ such that $mD$ is a Cartier divisor.

**Definition 2.2** (Boundary and subboundary)**.** Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety $X$, where $d_i \in \mathbb{Q}$ and $D_i$ are mutually prime Weil divisors. If $0 \le d_i \le 1$ (resp. $d_i \le 1$) for every $i$, then we call $D$ a *boundary* (resp. *subboundary*).

$\mathbb{Q}$-factoriality often plays a crucial role in the log MMP.

**Definition 2.3** ($\mathbb{Q}$-factoriality)**.** A normal variety $X$ is said to be $\mathbb{Q}$-*factorial* if every prime divisor $D$ on $X$ is $\mathbb{Q}$-Cartier.

We give one example to understand $\mathbb{Q}$-factoriality.

**Example 2.4** (cf. [Ka2, p.140])**.** We consider

$$X := \{(x, y, z, w) \in \mathbb{C}^4 \mid xy + zw + z^3 + w^3 = 0\}.$$

**Claim.** The variety $X$ is $\mathbb{Q}$-factorial. More precisely, $X$ is factorial, that is,

$$R := \mathbb{C}[x, y, z, w]/(xy + zw + z^3 + w^3)$$

is a UFD.

*Proof.* By Nagata's lemma (see [Mu, p.196]), it is sufficient to check that $x \cdot R$ is a prime ideal of $R$ and $R[1/x]$ is a UFD. This is an easy exercise. $\square$

Note that $\mathbb{Q}$-factoriality is not an analytically local condition.

**Claim.** Let $X^{an}$ be the underlying analytic space of $X$. Then $X^{an}$ is not *analytically* $\mathbb{Q}$-factorial at $(0,0,0,0)$.

*Proof.* We consider a germ of $X^{an}$ around the origin. Then $X^{an}$ is local analytically isomorphic to $(xy-uv=0)\subset\mathbb{C}^4$. So, $X^{an}$ is not $\mathbb{Q}$-factorial since the two divisors $(x=u=0)$ and $(y=v=0)$ intersect at a single point. Note that two $\mathbb{Q}$-Cartier divisors must intersect each other in codimension one. □

We recall an important property of $\mathbb{Q}$-factorial varieties, which is much more useful than one might first expect. For the proof, see [Ko].

**Proposition 2.5** (cf. [Ko, VI.1, 1.5 Theorem])**.** *Let $f\colon X\longrightarrow Y$ be a birational morphism between normal varieties. Assume that $Y$ is $\mathbb{Q}$-factorial. Then the exceptional locus* $\mathrm{Exc}(f)$ *is of pure codimension one in $X$.*

We give the next definition for the reader's convenience. We only use the round down of $\mathbb{Q}$-divisors in this paper.

**Definition 2.6** (Operations on $\mathbb{Q}$-divisors)**.** Let $D=\sum d_iD_i$ be a $\mathbb{Q}$-divisor on a normal variety $X$, where $d_i$ are rational numbers and $D_i$ are mutually prime Weil divisors. We define

$$\begin{aligned}\lfloor D\rfloor &:= \sum\lfloor d_i\rfloor D_i, \text{ the } \textit{round down} \text{ of } D,\\ \lceil D\rceil &:= \sum\lceil d_i\rceil D_i=-\lfloor -D\rfloor, \text{ the } \textit{round up} \text{ of } D,\\ \{D\} &:= \sum\{d_i\}D_i=D-\lfloor D\rfloor, \text{ the } \textit{fractional part} \text{ of } D,\end{aligned}$$

where for $r\in\mathbb{R}$, we define $\lfloor r\rfloor:=\max\{t\in\mathbb{Z}; t\le r\}$.

**Remark 2.7.** In some of the literature, for example, [KMM], $[D]$ (resp. $\langle D\rangle$) denotes $\lfloor D\rfloor$ (resp. $\{D\}$). The round down $\lfloor D\rfloor$ is sometimes called the *integral part* of $D$.

We define (simple) normal crossing divisors, which will play an important role in the following sections.

**Definition 2.8** (Normal crossings and simple normal crossings)**.** Let $X$ be a smooth variety. A reduced effective divisor $D$ is said to be a *simple normal crossing divisor* (resp. *normal crossing divisor*) if for each closed point $p$ of $X$, a local defining equation of $D$ at $p$ can be written as $f=z_1\cdots z_{j_p}$ in $\mathcal{O}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{X,p}$), where $\{z_1,\cdots,z_{j_p}\}$ is a part of a regular system of parameters.

**Remark 2.9.** The notion of normal crossing divisors is local for the étale topology (cf. [A, Section 2]). When $k=\mathbb{C}$, it is also local for the classical topology. On the other hand, the notion of simple normal crossing divisors is not local for the étale topology.

**Remark 2.10.** Let $D$ be a normal crossing divisor. Then $D$ is a simple normal crossing divisor if and only if each irreducible component of $D$ is smooth.

**Remark 2.11.** Some authorities use the word *normal crossing* to represent *simple* normal crossing. For example, a normal crossing divisor in [BEV] is a simple normal crossing divisor in our sense. See [BEV, Definition 2.1]. So, we recommend that the reader check the definition of (simple) normal crossing divisors whenever he read a paper on the log MMP.

## 3. Singularities of pairs

In this section, we quickly review the definitions of singularities which we use in the log MMP. For details, see, for example, [KM, §2.3]. First, we define the canonical divisor.

**Definition 3.1** (Canonical divisor)**.** Let $X$ be a normal variety with $\dim X = n$. The *canonical divisor* $K_X$ is any Weil divisor whose restriction to the smooth part of $X$ is a divisor of a regular $n$-form. The reflexive sheaf of rank one $\omega_X := \mathcal{O}_X(K_X)$ corresponding to $K_X$ is called the *canonical sheaf*.

Next, let us recall the various definitions of the singularities of pairs.

**Definition 3.2** (Discrepancies and singularities of pairs)**.** Let $X$ be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on $X$, where $D_i$ are distinct and irreducible such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \longrightarrow X$ be a proper birational morphism from a normal variety $Y$. Then we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, X, D) \in \mathbb{Q}$. $a(E, X, D)$ is called the *discrepancy* of $E$ with respect to $(X, D)$. We define

$$\operatorname{discrep}(X, D) := \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\}.$$

From now on, we assume that $D$ is a boundary. We say that $(X, D)$ is

$$\begin{cases}\text{terminal}\\ \text{canonical}\\ \text{klt}\\ \text{plt}\\ \text{lc}\end{cases} \quad \text{if} \quad \operatorname{discrep}(X, D) \quad \begin{cases} > 0, \\ \geq 0, \\ > -1 \quad \text{and} \quad \lfloor D \rfloor = 0, \\ > -1, \\ \geq -1.\end{cases}$$

Here klt is an abbreviation for *Kawamata log terminal*, plt for *purely log terminal*, and lc for *log-canonical*.

**Remark 3.3.** In [KM, Definition 2.34], $D$ is not a *boundary* but only a *subboundary*. In some of the literature, $(X, D)$ is called *sub lc* (resp. *sub plt*, etc.) if $\operatorname{discrep}(X, D) \geq -1$ (resp. $> -1$, etc.) and $D$ is only a *subboundary*.

**Remark 3.4** (Log discrepancies)**.** We put $a_\ell(E, X, D) = 1 + a(E, X, D)$ and call it the *log discrepancy*. We define

$$\operatorname{logdiscrep}(X, D) = 1 + \operatorname{discrep}(X, D).$$

In some formulas, log discrepancies behave much better than discrepancies. However, we do not use log discrepancies in this paper.

## 4. Divisorial log terminal

Let $X$ be a smooth variety and $D$ a reduced simple normal crossing divisor on $X$. Then $(X, D)$ is lc. Furthermore, it is not difficult to see that $(X, D)$ is plt if and only if every connected component of $D$ is irreducible. We would like to define some kind of log terminal singularities that contain the above pair $(X, D)$. So, we need a new notion of *log terminal*.

**Definition 4.1** (Divisorial log terminal)**.** Let $(X, D)$ be a pair where $X$ is a normal variety and $D$ is a boundary. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that $(X, D)$ is *dlt* or *divisorial log terminal* if and only if there is a closed subset $Z \subset X$ such that

(1) $X \setminus Z$ is smooth and $D|_{X \setminus Z}$ is a simple normal crossing divisor.
(2) If $f\colon Y \longrightarrow X$ is birational and $E \subset Y$ is an irreducible divisor such that $\mathrm{Center}_X E \subset Z$, then $a(E, X, D) > -1$.

So, the following example is obvious.

**Example 4.2.** If $X$ is a smooth variety and $D$ is a reduced simple normal crossing divisor on $X$, then the pair $(X, D)$ is dlt.

The above definition of dlt is [KM, Definition 2.37], which is useful for many applications. However, it has a quite different flavor from the other definitions of *log terminal singularities*. We will explain the relationship between the definition of dlt and the other definitions of log terminal singularities in the following sections.

## 5. Resolution Lemma

We think that one of the most useful log terminal singularities is *divisorially log terminal* (dlt, for short), which was introduced by Shokurov (see [FA, (2.13.3)]). We defined it in Definition 4.1 above. By Szabó's work [Sz], the notion of dlt coincides with that of *weakly Kawamata log terminal* (wklt, for short). For the definition of wklt, see [FA, (2.13.4)]. This fact is non-trivial and based on deep results about the desingularization theorem. For the details, see the original fundamental paper [Sz]. The key result is Szabó's Resolution Lemma [Sz, Resolution Lemma]. The following is a weak version of the Resolution Lemma, but it contains the essential part of Szabó's result and is sufficient for applications. For the precise statement, see [Sz, Resolution Lemma] or [BEV, Section 7]. By combining Theorem 5.1 with the usual desingularization arguments, we can recover the original Resolution Lemma without any difficulties. Explicitly, first we use Hironaka's desingularization theorem suitably, next we apply Theorem 5.1 below, and then we can recover Szabó's results. The details are left to the reader as an easy exercise (see the proof of Resolution Lemma in [Sz]). Note Example 5.4 below.

**Theorem 5.1.** *Let $X$ be a smooth variety and $D$ a reduced divisor. Then there exists a proper birational morphism $f\colon Y \longrightarrow X$ with the following properties*:

(1) *$f$ is a composition of blow ups of smooth subvarieties,*
(2) *$Y$ is smooth,*
(3) *$f^{-1}_*D\cup \mathrm{Exc}(f)$ is a simple normal crossing divisor, where $f^{-1}_*D$ is the strict transform of $D$ on $Y$, and*
(4) *$f$ is an isomorphism over $U$, where $U$ is the largest open set of $X$ such that the restriction $D|_U$ is a simple normal crossing divisor on $U$.*

*Note that $f$ is projective and the exceptional locus* $\mathrm{Exc}(f)$ *is of pure codimension one in $Y$ since $f$ is a composition of blowing ups.*

**Remark 5.2.** Recently, this was reproved by the new canonical desingularization algorithm. See [BEV, Theorem 7.11]. Note that in [BEV] normal crossing means simple normal crossing in our sense. See Remark 2.11 and Remark 7.4 below.

**Remark 5.3.** Szabó's results depend on Hironaka's paper [H], which is very hard to read. Thus, we recommend that the reader consult [BEV] for proofs. Now there are many papers on desingularization theorems. Sorry, I do not know which is the best.

The following example says that Szabó's resolution lemma (and Theorem 5.1) is not true if we replace simple normal crossing with normal crossing. We will treat this example in detail in the next section.

**Example 5.4.** Let $X := \mathbb{C}^3$ and $D$ the Whitney umbrella, that is, $W = (x^2 - zy^2 = 0)$. Then $W$ is a normal crossing divisor outside the origin. In this case, we can not make $W$ a normal crossing divisor only by blowing ups of smooth subvarieties over the origin.

*Sketch of the proof.* This is an exercise of how to calculate blow ups of smooth centers. If we blow up $W$ finitely many times along smooth subvarieties over the origin, then we will find that the strict transform of $W$ always has a pinch point, where a pinch point means a singular point that is local analytically isomorphic to $0 \in (x^2 - zy^2 = 0) \subset \mathbb{C}^3$. $\square$

Theorem 5.1 and Hironaka's desingularization imply the following corollary. It is useful for proving relative vanishing theorems and so on (see also Remark 6.11 below).

**Corollary 5.5.** *Let $X$ be a non-complete smooth variety and $D$ a simple normal crossing divisor on $X$. Then there exists a compactification $\overline{X}$ of $X$ and a simple normal crossing divisor $\overline{D}$ on $\overline{X}$ such that $\overline{D}|_X = D$. Furthermore, if $X$ is quasi-projective, then we can make $\overline{X}$ projective.*

## 6. Whitney umbrella

We will work over $k = \mathbb{C}$ throughout this section. First, we define *normal crossing varieties.*

**Definition 6.1** (Normal crossing variety)**.** Let $X$ be a variety. We say that $X$ is *normal crossing* at $x$ if and only if

$$\widehat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[x_1, x_2, \cdots, x_l]]/(x_1 x_2 \cdots x_k)$$

for some $k \leq l$. If $X$ is normal crossing at every point, we call $X$ a *normal crossing variety.*

**Remark 6.2.** It is obvious that a normal crossing divisor (see Definition 2.8) is a normal crossing variety. By [A, Corollary (2.6)], $X$ is normal crossing at $x$ if and only if $x \in X$ is locally isomorphic to $0 \in (x_1 x_2 \cdots x_k = 0) \subset \mathbb{C}^l$ for the étale (or classical) topology. So, let $U$ be a small open neighborhood (in the classical topology) of $X$ around $x$ and $U'$ the normalization of $U$. Then each irreducible component $V$ of $U'$ is smooth and $V \longrightarrow U$ is an embedding.

Next, we introduce the notion of *WU singularities*.

**Definition 6.3** (WU singularity)**.** Let $X$ be a variety and $x$ a closed point of $X$, and $p\colon X' \longrightarrow X$ the normalization. If there exist a smooth irreducible curve $C' \subset X'$ and a point

$$x' \in \overline{C' \underset{X}{\times} C' \setminus \Delta_{C'}} \cap \Delta_{C'} \cap p^{-1}(x),$$

where $\Delta_{C'}$ is the diagonal of $C' \underset{X}{\times} C'$, then we say that $X$ has a WU singularity at $x$, where WU is an abbreviation of Whitney Umbrella.

**Example 6.4.** Let $W = (x^2 - zy^2 = 0) \subset \mathbb{C}^3$ be the Whitney umbrella. Then the normalization of $W$ is $\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[u, v]$ such that the normalization map $\mathbb{C}^2 \longrightarrow W$ is given by $(u, v) \longmapsto (uv, u, v^2)$. Therefore, the line $(u = 0) \subset \mathbb{C}^2$ maps onto $(x = y = 0) \subset W$. Thus the origin is a WU singularity. Note that $W$ is normal crossing outside the origin.

We give one more example.

**Example 6.5.** Let $V = (z^3 - x^2yz - x^4 = 0) \subset \mathbb{C}^3$. Then $V$ is singular along the $y$-axis. By blowing up $\mathbb{C}^3$ along the $y$-axis, we obtain the normalization $p\colon V' \longrightarrow V$. Note that $V'$ is smooth and that there is a smooth curve $C'$ on $V'$ which double covers the $y$-axis. It can be checked easily that the origin $(0, 0, 0)$ is a WU singularity of $V$.

**Remark 6.6.** Let $x \in X$ be a WU singularity. We shrink $X$ around $x$ (in the classical topology). Then there exists an isomorphism $\sigma\colon C' \longrightarrow C'$ of finite order such that $\sigma \neq id_{C'}$, $\sigma(x') = x'$, and $p = p \circ \sigma$ on $C'$. When $X$ is the Whitney umbrella, $\sigma$ corresponds to the graph $\overline{C' \underset{X}{\times} C' \setminus \Delta_{C'}}$ and the order of $\sigma$ is two.

**Lemma 6.7.** *Let $x \in X$ be a WU singularity. Then $X$ is not normal crossing at $x$.*

*Proof.* Assume that $X$ is normal crossing at $x$. Let $X'_1$ be the irreducible component of $X'$ containing $C'$. Since $X'_1 \longrightarrow X$ is injective in a neighborhood of $x'$, $C' \underset{X}{\times} C' = \Delta_{C'}$ near $x'$. This is a contradiction. □

The following theorem is the main theorem of this section.

**Theorem 6.8.** *Let $x \in X$ be a WU singularity and $f\colon Y \longrightarrow X$ be a proper birational morphism such that $f\colon f^{-1}(X \setminus \{x\}) \longrightarrow X \setminus \{x\}$ is an isomorphism. Then $Y$ has a WU singularity.*

*Proof.* Let $C'$, $x'$ be as in Definition 6.3, $\sigma$ as in Remark 6.6. Let $q\colon Y' \longrightarrow Y$ be the normalization. Then there exists a proper birational morphism $f'\colon Y' \longrightarrow X'$. By assumption, $Y \longrightarrow X$ is an isomorphism over $p(C') \setminus \{x\}$. Thus $Y' \longrightarrow X'$ is an isomorphism over $C' \setminus p^{-1}(x)$. The embedding $C' \subset X'$ induces an embedding $C' \subset Y'$, and $p = p \circ \sigma$ implies $q = q \circ \sigma$. Therefore, Definition 6.3 implies that $q(x') \in Y$ is a WU singularity. □

**Proposition 6.9.** *Let $x \in X$ be a WU singularity and $Z$ a normal crossing variety. Then there are no proper birational morphisms $g\colon X \longrightarrow Z$ such that $p(C') \not\subset \operatorname{Exc}(g)$, where $p$, $C'$ are as in* Definition 6.3.

*Proof.* Assume that there exists a proper birational morphism as above. We put $C := g(p(C'))$. Then the mapping degree of $C' \longrightarrow C$ is greater than one by the definition of WU singularities. On the other hand, $C' \longrightarrow C$ factors through the normalization $Z'$ of $Z$. Thus, the mapping degree of $C' \longrightarrow C$ is one. This is a contradiction. □

The next corollary follows from Theorem 6.8 and Proposition 6.9.

**Corollary 6.10.** *There are no proper birational maps* (*that is, birational maps such that the first and the second projections from the graph are proper*) *between the Whitney umbrella $W$ and a normal crossing variety $V$ that induce $W \setminus \{0\} \simeq V \setminus E$, where $E$ is a closed subset of $V$.*

Therefore, we obtain

**Remark 6.11.** Corollary 5.5 does not hold for normal crossing divisors.

## 7. What is a log resolution?

We often use the words *good resolution* or *log resolution* without defining them precisely. This sometimes causes some serious problems. We will give our definition of log resolution later (see Definition 7.3). Let us recall another definition of dlt, which is equivalent to Definition 4.1. We do not prove the equivalence of Definition 4.1 and Definition 7.1 in this paper. Note that it is an easy consequence of Theorem 5.1. For the details, see [Sz, Divisorial Log Terminal Theorem].

**Definition 7.1** (Divisorial log terminal)**.** Let $X$ be a normal variety and $D$ a boundary on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. There exists a *log resolution* $f\colon Y \longrightarrow X$ such that $a(E, X, D) > -1$ for every $f$-exceptional divisor $E$. Then we say that $(X, D)$ is *dlt* or *divisorial log terminal.*

**7.2.** There are three questions about the above definition.

- Is $f$ projective?
- Is the exceptional locus $\mathrm{Exc}(f)$ of pure codimension one?
- Is $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ a *simple normal crossing* divisor or only a *normal crossing* divisor?

In [KM, Notation 0.4 (10)], they assume that $\mathrm{Exc}(f)$ is of pure codimension one and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a simple normal crossing divisor. We note that, in [FA, 2.9 Definition], $\mathrm{Exc}(f)$ is not necessarily of pure codimenison one. So, the definition of lt in [FA, (2.13.1)] is the same as Definition 7.1 above, but lt in the sense of [FA] is different from dlt. See Remark 7.5 and Examples 8.3 and 9.3 below. The difference exists in the definition of log resolutions!

Our definition of a log resolution is the same as [KM, Notation 0.4 (10)]. By Hironaka, log resolutions exist for varieties over a field of characteristic zero (see [BEV]).

**Definition 7.3** (Log resolution)**.** Let $X$ be a variety and $D$ a $\mathbb{Q}$-divisor on $X$. A *log resolution* of $(X, D)$ is a *proper* birational morphism $f\colon Y \longrightarrow X$ such that $Y$ is smooth, $\mathrm{Exc}(f)$ is a *divisor* and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a *simple* normal crossing divisor.

**Remark 7.4.** In the definition of the log resolution in [BEV, Definition 7.10], they do not assume that the exceptional locus $\mathrm{Exc}(\mu)$ is of pure codimension one. However, if $\mu$ is a composition of blowing ups, then $\mathrm{Exc}(\mu)$ is always of pure codimension one.

**Remark 7.5** (lt in the sense of [FA])**.** If we do not assume that $\mathrm{Exc}(f)$ is a divisor in Definition 7.3, then Definition 7.1 is the definition of *lt* in the sense of [FA] (see [FA, (2.13.1)]).

**Remark 7.6** (Analytically local?)**.** We assume that $k = \mathbb{C}$. Then the notion of terminal, canonical, klt, plt, and lc, is not only algebraically local but also analytically local. For the precise statement, see [Ma, Proposition 4-4-4]. However, the notion of dlt is not analytically local. It is because the notion of simple normal crossing divisors is not analytically local. So, [Ma, Exercise 4-4-5] is incorrect. To obtain an analytically local notion of log terminal singularities, we must remove the word *simple* from Definition 4.1 (2). However, this new notion of log terminal singularities seems to be useless. Consider the pair $(\mathbb{C}^3, W)$, where $W$ is the Whitney umbrella (see Sections 5 and 6).

We note that, by Szabó's resolution lemma, we do not need the projectivity of $f$ in the definition of dlt. It is because the log resolution $f$ in Definition 7.1 can be taken to be a composition of blowing ups by Hironaka's desingularization and Theorem 5.1 (see also Definition 4.1, [KM, Proposition 2.40 and Theorem 2.44], and [Sz, Divisorial Log Terminal Theorem]). We summarize:

**Proposition 7.7.** *The log resolution $f$ in* Definition 7.1 *can be taken to be a composition of blow ups of smooth centers. In particular, there exists an effective $f$-anti-ample divisor whose support coincides with* $\mathrm{Exc}(f)$. *Thus, the notion of dlt coincides with that of wklt* (*see* [FA, (2.13.4)]).

So, we can omit the notion of wklt in the log MMP. In [KMM], they adopted normal crossing divisors instead of simple normal crossing divisors. So there is a difference between wklt and *weak log-terminal*. We note that any wklt singularity is a weak log-terminal singularity in the sense of [KMM, Definition 0-2-10 (2)], but the converse does not always hold. See Section 8, especially, Example 8.1. In my experience, dlt, which is equivalent to wklt, is easy to treat and useful for inductive arguments, but weak log-terminal is very difficult to use. We think that [KM, Corollary 5.50] makes dlt useful. For the usefulness of dlt, see [F1], [F2], [F3], and [F4]. See also Example 8.1, Remark 8.2, and Section 9. We summarize:

**Conclusion 7.8.** *The notion of dlt coincides with that of wklt by* [Sz] (*see* Proposition 7.7). *In particular, a dlt singularity is automatically a weak log-terminal singularity in the sense of* [KMM]. *Therefore, we can freely apply the results that were proved for weak log-terminal pairs in* [KMM] *to dlt pairs. We note*

$$\textit{klt} \Longrightarrow \textit{plt} \Longrightarrow \textit{dlt} \Longleftrightarrow \textit{wklt} \Longrightarrow \textit{weak log-terminal} \Longrightarrow \textit{lc}.$$

For other characterizations of dlt, see [Sz, Divisorial Log Terminal Theorem], which is an exercise of Theorem 5.1. See also [KM, Proposition 2.40, and Theorem 2.44]. The notion of dlt is natural by the following proposition.

**Proposition 7.9** (cf. [KM, Proposition 5.51])**.** *Let $(X, D)$ be a dlt pair. Then every connected component of $\lfloor D \rfloor$ is irreducible* (*resp.* $\lfloor D \rfloor = 0$) *if and only if $(X, D)$ is plt* (*resp. klt*).

Thus, dlt is a natural generalization of plt.

**Conclusion 7.10.** *Lt in the sense of* [FA] *seems to be useless.* Examples 8.4 *and* 9.3 *imply that the existence of a small resolution causes many unexpected phenomena. We note that if the varieties are $\mathbb{Q}$-factorial, then there are no small resolutions by* Proposition 2.5. *Therefore, $\mathbb{Q}$-factorial lt in the sense of* [FA] *is equivalent to $\mathbb{Q}$-factorial dlt.*

## 8. Examples

In this section, we collect some examples. The following example says that weak log-terminal is not necessarily wklt. We omit the definition of weak log-terminal since we do not use it in this paper. See [KMM, Definition 0-2-10].

**Example 8.1** (Simple normal crossing vs normal crossing)**.** Let $X$ be a smooth surface and $D$ a nodal curve on $X$. Then the pair $(X, D)$ is not wklt but it is weak log-terminal.

The next fact is crucial for inductive arguments.

**Remark 8.2.** Let $(X, D)$ be a dlt (resp. weak log-terminal) pair and $S$ an irreducible component of $\lfloor D \rfloor$. Then $(S, \mathrm{Diff}(D-S))$ is dlt (resp. not necessarily weak log-terminal), where the $\mathbb{Q}$-divisor $\mathrm{Diff}(D-S)$ on $S$ is defined by the following equation:

$$(K_X + D)|_S = K_S + \mathrm{Diff}(D - S).$$

This is a so-called *adjunction formula*.

We will treat the adjunction formula for dlt pairs in detail in Section 9. In Example 8.1, $S := \lfloor D \rfloor$ is not normal. This makes weak log-terminal difficult to use for inductive arguments. The next example explains that we have to assume that $\mathrm{Exc}(f)$ is a divisor in Definition 7.1.

**Example 8.3** (Small resolution)**.** Let $X := (xy - uv = 0) \subset \mathbb{C}^4$. It is well-known that $X$ is a toric variety. We take the torus invariant divisor $D$, the complement of the big torus. Then $(X, D)$ is not dlt but it is lt in the sense of [FA]. Especially, it is lc. Note that there is a small resolution.

The following is a variant of the above example.

**Example 8.4** ([FA, 17.5.2 Example])**.** Let $X := (xy - uv = 0) \subset \mathbb{C}^4$ and

$$D = (x = u = 0) + (y = v = 0) + \frac{1}{2}\sum_{i=1}^{4}(x + 2^i u = y + 2^{-i} v = 0).$$

If we put

$$F=\sum_{i=1}^{2}(x+2^i u=0)+\sum_{i=3}^{4}(y+2^{-i}v=0),$$

then $2D=F\cap X$. Thus, $2(K_X+D)$ is Cartier since $X$ is Gorenstein. We can check that $(X,D)$ is lt in the sense of [FA] by blowing up $\mathbb{C}^4$ along the ideal $(x,u)$. In particular $(X,D)$ is lc. The divisor $\lfloor D\rfloor$ is two planes intersecting at a single point. Thus it is not $S_2$. So, $(X,D)$ is not dlt. See Remark 8.5 below.

**Remark 8.5.** If $(X,D)$ is dlt, then $\lfloor D\rfloor$ is seminormal and $S_2$ by [FA, 17.5 Corollary].

**Remark 8.6.** Example 8.4 says that [FA, (16.9.1)] is not true. The problem is that $S$ does not necessarily satisfy Serre's condition $S_2$.

## 9. Adjunction for dlt pairs

To treat pairs effectively, we have to understand *adjunction*. Adjunction is explained nicely in [FA, Chapter 16]. We recommend it to the reader. In this section, we treat adjunction formula only for dlt pairs. Let us recall the definition of center of log canonical singularities.

**Definition 9.1** (Center of lc singularities)**.** Let $X$ be a normal variety and $D$ a $\mathbb{Q}$-divisor on $X$ such that $K_X+D$ is $\mathbb{Q}$-Cartier. A subvariety $W$ of $X$ is said to be a *center of log canonical singularities* for the pair $(X,D)$, if there exists a proper birational morphism from a normal variety $\mu\colon Y\longrightarrow X$ and a prime divisor $E$ on $Y$ with discrepancy $a(E,X,D)\le -1$ such that $\mu(E)=W$.

The next proposition is adjunction for a higher codimensional center of log canonical singularities of a dlt pair. We use it in [F4]. For the definition of the *different* Diff, see [FA, 16.6 Proposition].

**Proposition 9.2** (Adjunction for dlt pairs)**.** *Let $(X,D)$ be a dlt pair. We put $S=\lfloor D\rfloor$ and let $S=\sum_{i\in I}S_i$ be the irreducible decomposition of $S$. Then, $W$ is a center of log canonical singularities for the pair $(X,D)$ with* $\mathrm{codim}_X W=k$ *if and only if $W$ is an irreducible component of $S_{i_1}\cap S_{i_2}\cap\cdots\cap S_{i_k}$ for some $\{i_1,i_2,\cdots,i_k\}\subset I$. By adjunction, we obtain*

$$K_{S_{i_1}}+\mathrm{Diff}(D-S_{i_1})=(K_X+D)|_{S_{i_1}},$$

*and $(S_{i_1},\mathrm{Diff}(D-S_{i_1}))$ is dlt. Note that $S_{i_1}$ is normal, $W$ is a center of log canonical singularities for the pair $(S_{i_1},\mathrm{Diff}(D-S_{i_1}))$, $S_{i_j}|_{S_{i_1}}$ is a reduced part of* $\mathrm{Diff}(D-S_{i_1})$ *for $2\le j\le k$, and $W$ is an irreducible* *component of* $(S_{i_2}|_{S_{i_1}}) \cap (S_{i_3}|_{S_{i_1}}) \cap \cdots \cap (S_{i_k}|_{S_{i_1}})$*. By applying adjunction* $k$ *times repeatedly, we obtain a* $\mathbb{Q}$*-divisor* $\Delta$ *on* $W$ *such that*

$$(K_X + D)|_W = K_W + \Delta$$

*and* $(W, \Delta)$ *is dlt.*

*Sketch of the proof.* Note that $S_{i_1}$ is normal by [KM, Corollary 5.52], and [FA, 17.2 Theorem] and Definition 4.1 imply that $(S_{i_1}, \mathrm{Diff}(D - S_{i_1}))$ is dlt. The other statements are obvious. $\square$

The above proposition is one reason why dlt is more valuable than the other kinds of *log terminal singularities*.

**Example 9.3.** Let $(X, D)$ be as in Example 8.4. Recall that $(X, D)$ is lt in the sense of [FA] but not dlt. It is not difficult to see that the centers of log canonical singularities for the pair $(X, D)$ are as follows: the origin $(0, 0, 0, 0) \in X$ and the two Weil divisors $(x = u = 0)$ and $(y = v = 0)$ on $X$. So, there are no one dimensional center of log canonical singularities.

The final proposition easily follows from the above proposition: Proposition 9.2. It will play a crucial role in the proof of the special termination (see [F4]).

**Proposition 9.4.** *Let* $(X, D)$ *be as in* Proposition 9.2. *We write* $D = \sum d_j D_j$*, where* $d_j \in \mathbb{Q}$ *and* $D_j$ *is a prime divisor on* $X$*. Let* $P$ *be a divisor on* $W$*. Then the coefficient of* $P$ *is* 0*,* 1*, or* $1 - \frac{1}{m} + \sum \frac{r_j d_j}{m}$ *for suitable non-negative integers* $r_j$*s and positive integer* $m$*. Note that the coefficient of* $P$ *is* 1 *if and only if* $P$ *is a center of log canonical singularities for the pair* $(X, D)$*.*

*Sketch of the proof.* Apply [FA, 16.7 Lemma] $k$ times repeatedly as in Proposition 9.2 and then apply [FA, 7.4.3 Lemma]. $\square$

## 10. Miscellaneous comments

In this section, we collect some comments.

**10.1** ($\mathbb{R}$-divisors)**.** In the previous sections, we only use $\mathbb{Q}$-divisors for simplicity. We note that almost all the definitions and results are generalized to $\mathbb{R}$-divisors with a little effort. In Shokurov's proof of PL flips (see [Sh2]), $\mathbb{R}$-divisors appear naturally and are indispensable. Sorry, we do not pursue $\mathbb{R}$-generalizations here anymore. However, if the reader understands the results in this paper, then the natural $\mathbb{R}$-generalizations are good exercises.

**10.2** (Comments on the four bibles)**.** We give miscellaneous comments on the four bibles.

- [KMM] is the oldest bible of the log MMP. The notion of *log-terminal* in [KMM, Definition 0-2-10] is equivalent to that of klt. We make two remarks.

  **Remark 10.3.** We note the following comment, noted by Matsuki in [Ma, Remark 14-2-7]. In [KMM] at the end of Example 5-2-5 there is a slightly misleading statement: "The morphisms given in Example 5-2-4 and 5-2-5 are the only contractions of flipping type from $\mathbb{Q}$-factorial terminal toric varieties of dimension 3 by the theorem of White-Frumkin." This is, however, true only under the assumption that the extremal rational curve contains only one singular point. See also Remark 10.9 below.

  **Remark 10.4.** Theorem 6-1-6 in [KMM] is [Ka1, Theorem 4.3]. We use the same notation as in the proof of Theorem 4.3 in [Ka1]. By [Ka1, Theorem 3.2], ${}'E_1^{p,q} \longrightarrow {}''E_1^{p,q}$ are zero for all $p$ and $q$. This just implies that

  $$\mathrm{Gr}^p H^{p+q}(X, \mathcal{O}_X(-\ulcorner L \urcorner)) \longrightarrow \mathrm{Gr}^p H^{p+q}(D, \mathcal{O}_D(-\ulcorner L \urcorner))$$

  are zero for all $p$ and $q$. Kawamata points out that we need one more Hodge theoretic argument to conclude that

  $$H^i(X, \mathcal{O}_X(-\ulcorner L \urcorner)) \longrightarrow H^i(D, \mathcal{O}_D(-\ulcorner L \urcorner))$$

  are zero for all $i$.

- [FA] is the only bible that treats $\mathbb{R}$-divisors and *differents* (see [FA, Chapters 2 and 16]). In Chapter 2, five log terminal singularities, that is, klt, plt, dlt, wklt, and lt, were introduced according to Shokurov [Sh1]. Alexeev pointed out that [FA, (4.12.2.1)] is wrong. The following example contradicts [FA, (4.12.1.3), (4.12.2.1)].

  **Example 10.5.** Let $X = \mathbb{P}^2$, $B = \frac{2}{3}L$, where $L$ is a line on $X$. Let $P$ be any point on $L$. First, blow up $X$ at $P$. Then we obtain an exceptional divisor $E_P$ such that $a(E_P, X, B) = \frac{1}{3}$. Let $L'$ be the strict transform of $L$. Next, take a blow-up at $L' \cap E_P$. Then we obtain an exceptional divisor $F_P$ whose discrepancy $a(F_P, X, B) = \frac{2}{3}$. On the other hand, it is easy to see that $\mathrm{discrep}(X, B) = \frac{1}{3}$. Thus, $\min\{1, 1 + \mathrm{discrep}(X, B)\} = 1$.

**Remark 10.6.** By this example, [F5, Lemma 2.1], which is the same as [FA, (4.12.2.1)], is incorrect. For the details about the discrepancy lemma, see [F7].

- [KM] seems to be the best bible for singularities of pairs in the log MMP. In the definitions of singularities of pairs, they assume that $D$ is only a subboundary (see [KM, Definition 2.34] and Remark 3.3). One must be aware of this fact.

  Consider the definition of lt in [KM, Definition 2.34 (3)]. If $D = 0$ in Definition 3.2, then the notions klt, plt, and dlt coincide (see also Proposition 7.9) and they say that $X$ has *log terminal* (abbreviated to *lt*) singularities.

**Remark 10.7.** There is an error in [KM, Lemma 5.17 (2)]. We can construct a counterexample easily. We put $X = \mathbb{P}^2$, $\Delta =$ a line on $X$, and $|H| = |\mathcal{O}_X(1)|$. Then we have

$$-1 = \operatorname{discrep}(X, \Delta + H_g) \neq \min\{0, \operatorname{discrep}(X, \Delta)\} = 0,$$

since $\operatorname{discrep}(X, \Delta) = 0$.

- The latest bible [Ma] explains singularities in details (see Chapter 4 in [Ma]). However, as we pointed out before (see Remark 1.1), Matsuki made a mistake.

  In the definition of lt (see [Ma, Definition 4-3-2]), he assumed that the resolution is *projective*. So, lt in [Ma] is slightly different from lt in [FA]. See Conclusion 7.10 above.

**Remark 10.8** (Comment by Matsuki)**.** On page 178, line 8–9,“by blowing up only over the locus where $\sigma^{-1}(D) \cup \operatorname{Exc}(\sigma)$ is not a normal crossing divisor, we obtain...” is incorrect. See Example 5.4 and Section 6.

**Remark 10.9** (Toric Mori theory)**.** In [KMM, §5-2] and [Ma, Chapter 14], toric varieties are investigated from the Mori theoretic viewpoint. Toric Mori theory originates from Reid's beautiful paper [R]. Chapter 14 in [Ma] corrects some minor errors in [R]. In [KMM] and [Ma], the toric Mori theory is formulated for toric projective morphism $f\colon X \longrightarrow S$. We note that $X$ is always assumed to be *complete*. So, the statement at the end of [Ma, Proposition 14-1-5] is nonsensical. Matuski wrote: “In the relative setting for statement (ii), such a vector $v'_i$ may not exist at all. If that is the case, then the two $(n-1)$-dimensional cones $w_{i,n}$ and $w_{i,n+1}$ are on the boundary of $\Delta$.” However, $\Delta$ has no boundary since $\Delta$ is a complete fan in [Ma]. For the details of the toric Mori theory for the case when $X$ is not complete, see [FS], [F6], and [Sa].

**Conclusion 10.10.** *Some care should be exercised when using the various notions of log terminal and we recommend that the reader check the definitions very carefully.*

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