WHAT IS LOG TERMINAL ?
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Abstract. In this paper, we explain the subtleties of various kinds of log terminal singularities. We focus on the notion of divisorial log terminal singularities, which seems to be the most useful one. We explain Szabó's resolution lemma, the notion of log resolution, adjunction formula for divisorial log terminal pairs, and so on. We also collect miscellaneous results and examples on singularities of pairs in the log MMP that help us understand log terminal singularities.

Contents

1. What is log terminal?

This paper is a guide to go around the world of *log terminal singularities*. The main purpose is to attract the reader's attention to the

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subtleties of various kinds of log terminal singularities. Needless to say, my opinion is not necessarily the best. We hope that this paper will help the reader understand *log terminal*. Almost all the results in this paper are known to experts but were not accessible for non-experts. Note that this paper is not self-contained and contains only sketches of proofs. For systematic treatments of singularities in the log MMP, see, for example, [KM, Section 2.3]. We require that the reader is familiar with the basic properties of singularities of pairs.

In the log MMP, there are too many variants of *log terminal*. We sometimes have troubles when we treat log terminal singularities. We already have four bibles on the log MMP: [KMM], [FA], [KM], and [Ma]. It is unpleasant for us that each bible adopted different definitions of log terminal and *log resolutions*. Historically, Shokurov introduced various kinds of log terminal singularities in his famous paper [Sh1, §1]. However, we do not mention [Sh1] anymore for simplicity. We only treat the above four bibles. Before we come to the subject, we note [Ma].

Remark 1.1. In [Ma, Chapter 4], Matsuki explains various kinds of singularities in details. Unfortunately, he confused *normal crossing divisors* with *simple normal crossing divisors* (see Definition 2.8 below) in the definition of dlt (see Definition 7.1) and so on. Therefore, when we read [Ma], we have to replace *normal crossings* with *simple normal crossings* in [Ma, Definition 4-3-2 (2")]. See also Remarks 7.6 and 10.4.

We summarize the contents of this paper: Sections 2 and 3 are preliminaries. We recall well-known definitions and fix some notations. In Section 4, we define the notion of *divisorial log terminal singularities*, which is one of the most important log terminal singularities. In Section 5, we treat Szabó's resolution lemma, which is very important in the log MMP. Section 6 was suggested by Mori. Here, we explain that Szabó's resolution lemma is not true for normal crossing divisors by using the Whitney umbrella. Section 7 deals with *log resolutions*. Here, we explain subtleties of various kind of log terminal singularities. In Section 8, we collects examples that help us understand singularities of pairs. In Section 9, we describe *adjunction formula* for dlt pairs. It will play important roles in the log MMP. We need it in [F4]. Finally, section 10 collects miscellaneous comments.

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Notation. The set of integers (resp. rational numbers, real numbers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$). We will work over an algebraically closed field k of characteristic zero; my favorite is $k = \mathbb{C}$.

2. Preliminaries on $\mathbb{Q}$-divisors

Before we introduce singularities of pairs, let us recall the basic definitions about $\mathbb{Q}$-divisors.

Definition 2.1 ($\mathbb{Q}$-Cartier divisor)**.** Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X, that is, $d_i \in \mathbb{Q}$ and D_i is a prime divisor on X for every i. Then D is $\mathbb{Q}$-*Cartier* if there exists a positive integer m such that mD is a Cartier divisor.

Definition 2.2 (Boundary and subboundary)**.** Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X, where $d_i \in \mathbb{Q}$ and D_i are mutually prime Weil divisors. If $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i, then we call D a *boundary* (resp. *subboundary*).

The following $\mathbb{Q}$-factoriality sometimes plays crucial roles in the log MMP.

Definition 2.3 ($\mathbb{Q}$-factoriality)**.** A normal variety X is said to be $\mathbb{Q}$-*factorial* if every prime divisor D on X is $\mathbb{Q}$-Cartier.

We treat one example to understand $\mathbb{Q}$-factoriality.

Example 2.4 (cf. [Ka, p.140])**.** We consider

$$X := \{(x, y, z, w) \in \mathbb{C}^4 \mid xy + zw + z^3 + w^3\}.$$

Claim. The variety X is $\mathbb{Q}$-factorial. More precisely, X is factorial, that is,

$$R := \mathbb{C}[x, y, z, w]/(xy + zw + z^3 + w^3)$$

is a UFD.

Proof. By Nagata's lemma (see [Mu, p.196]), it is sufficient to check that $x \cdot R$ is a prime ideal of R and $R[1/x]$ is a UFD. It is an easy exercise. □

Note that the $\mathbb{Q}$-factoriality is not an analytically local condition.

Claim. Let X^{an} be the underlying analytic space of X. Then X^{an} is not *analytically* $\mathbb{Q}$-factorial at $(0, 0, 0, 0)$.

Proof. We consider a germ of X^{an} around the origin. Then X^{an} is local analytically isomorphic to $(xy - uv = 0) \subset \mathbb{C}^4$. So, X^{an} is not $\mathbb{Q}$-factorial since two divisors $(x = u = 0)$ and $(y = v = 0)$ intersect at a single point. Note that two $\mathbb{Q}$-Cartier divisors must intersect each other in codimension one. □

We recall an important property of $\mathbb{Q}$-factorial varieties, which is much more useful than we expect. For the proof, see [Ko].

Proposition 2.5 (cf. [Ko, VI.1, 1.5 Theorem]). *Let $f : X \longrightarrow Y$ be a birational morphism between normal varieties. Assume that Y is $\mathbb{Q}$-factorial. Then the exceptional locus* $\mathrm{Exc}(f)$ *is of pure codimension one in X.*

We write the next definition for the reader's convenience. We only use the round down of $\mathbb{Q}$-divisors in this paper.

Definition 2.6 (Operations of $\mathbb{Q}$-divisors). Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X, where d_i are rational numbers and D_i are mutually prime Weil divisors. We define

$$\begin{aligned} \lfloor D \rfloor &:= \sum \lfloor d_i \rfloor D_i, \text{ the } \textit{round down} \text{ of } D, \\ \lceil D \rceil &:= \sum \lceil d_i \rceil D_i = -\lfloor -D \rfloor, \text{ the } \textit{round up} \text{ of } D, \\ \{D\} &:= \sum \{d_i\} D_i = D - \lfloor D \rfloor, \text{ the } \textit{fractional part} \text{ of } D, \end{aligned}$$

where for $r \in \mathbb{R}$, we define $\lfloor r \rfloor := \max\{t \in \mathbb{Z}; t \leq r\}$.

Remark 2.7. In some literatures, for example, [KMM], $[D]$ (resp. $\langle D \rangle$) denotes $\lfloor D \rfloor$ (resp. $\{D\}$). The round down $\lfloor D \rfloor$ is sometimes called the *integral part* of D.

We define (simple) normal crossing divisors, which will play important roles in the following sections.

Definition 2.8 (Normal crossings and simple normal crossings). Let X be a smooth variety. A reduced effective divisor D is said to be a *simple normal crossing divisor* (resp. *normal crossing divisor*) if for each closed point p of X, a local defining equation of D at p can be written as $f = z_1 \cdots z_{j_p}$ in $\mathcal{O}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{X,p}$), where $\{z_1, \cdots, z_{j_p}\}$ is a part of a regular system of parameters.

Remark 2.9. The notion of normal crossing divisors is local for the étale topology (cf. [A, Section 2]). When $k = \mathbb{C}$, it is also local for the classical topology. On the other hand, the notion of simple normal crossing divisors is not local for the étale topology.

Remark 2.10. Let D be a normal crossing divisor. Then D is a simple normal crossing divisor if and only if each irreducible component of D is smooth.

Remark 2.11. Someone uses the word *normal crossing* to represent *simple* normal crossing. For example, a normal crossing divisor in [BEV] is a simple normal crossing divisor in our sense. See [BEV, Definition 2.1]. So, we recommend the reader to check the definition of (simple) normal crossing divisors whenever he reads papers on the log MMP.

3. Singularities of pairs

In this section, we quickly review the definitions of singularities which we use in the log MMP. For details, see, for example, [KM, §2.3]. First, we define the canonical divisor.

Definition 3.1 (Canonical divisor)**.** Let X be a normal variety with $\dim X = n$. The *canonical divisor* K_X is defined so that its restriction to the smooth part of X is a divisor of a regular n-form. The reflexive sheaf of rank one $\omega_X := \mathcal{O}_X(K_X)$ corresponding to K_X is called the *canonical sheaf*.

Next, let us recall the definitions of the singularities for pairs.

Definition 3.2 (Discrepancies and singularities of pairs)**.** Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X, where D_i are distinct and irreducible such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \longrightarrow X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, X, D) \in \mathbb{Q}$. This $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We define

$$\mathrm{discrep}(X, D) := \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\}.$$

From now on, we assume that D is a boundary. We say that (X, D) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{cases} \quad \text{if} \quad \mathrm{discrep}(X, D) \quad \begin{cases} > 0, \\ \geq 0, \\ > -1 \quad \text{and} \quad \lfloor D \rfloor = 0, \\ > -1, \\ \geq -1. \end{cases}$$

Here klt is an abbreviation for *Kawamata log terminal*, plt for *purely log terminal*, and lc for *log-canonical*.

Remark 3.3. In [KM, Definition 2.34], D is not a *boundary* but only a *subboundary*. In some literatures, (X, D) is called *sub lc* (resp. *sub plt*, etc.) if $\operatorname{discrep}(X, D) \geq -1$ (resp. > -1, etc.) and D is only a *subboundary*.

4. Divisorial log terminal

Let X be a smooth variety and D a reduced simple normal crossing divisor on X. Then (X, D) is lc. Furthermore, it is not difficult to see that (X, D) is plt if and only if every connected component of D is irreducible. We would like to define some kind of log terminal singularities that contain the above pair (X, D). So, we need a new notion of *log terminal*.

Definition 4.1 (Divisorial log terminal)**.** Let (X, D) be a pair where X is a normal variety and D is a boundary. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that (X, D) is *dlt* or *divisorial log terminal* if and only if there is a closed subset $Z \subset X$ such that

(1) $X \setminus Z$ is smooth and $D|_{X \setminus Z}$ is a simple normal crossing divisor.
(2) If $f : Y \longrightarrow X$ is a birational and $E \subset Y$ is an irreducible divisor such that $\operatorname{center}_X E \subset Z$, then $a(E, X, D) > -1$.

So, the following example is obvious.

Example 4.2. If X is a smooth variety and D is a reduced simple normal crossing divisor on X, then the pair (X, D) is dlt.

The above definition of dlt is [KM, Definition 2.37], which is useful for many applications. However, it has a quite different flavor from other *log terminal singularities*. We will explain the relationships between dlt and other log terminal singularities in the following sections.

5. Resolution Lemma

We think that one of the most useful log terminal singularities is *divisorial log terminal* (dlt, for short), which was introduced by Shokurov (see [FA, (2.13.3)]). We defined it in Definition 4.1 above. By Szabó's work [Sz], the notion of dlt coincides with that of *weakly Kawamata log terminal* (wklt, for short). For the definition of wklt, see [FA, (2.13.4)]. This fact is non-trivial and based on the deep results about desingularization theorem. For the details, see the original fundamental paper [Sz]. The key result is Szabó's resolution lemma [Sz, Resolution Lemma]. The following is a weak version of Resolution Lemma, but it contains the essential part of Szabó's result and is sufficient for applications. For the precise statement, see [Sz, Resolution Lemma] or [BEV, Section 7]. By combining Theorem 5.1 with the usual desingularization arguments, we can recover the original Resolution Lemma without any difficulties. This means that, first, we use Hironaka's desingularization theorem suitably, next, we apply Theorem 5.1 below, then we can recover Szabó's results. The details are left to the reader as an easy exercise (see the proof of Resolution Lemma in [Sz]). Note Example 5.4 below.

Theorem 5.1. *Let X be a smooth variety and D a reduced divisor. Then there exists a proper birational morphism $f : Y \longrightarrow X$ with the following properties*:

1. *f is a composition of blowing ups of smooth subvarieties,*
2. *Y is smooth,*
3. *$f_*^{-1}D \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor, where $f_*^{-1}D$ is the strict transform of D on Y, and*
4. *f is an isomorphism over U, where U is the largest open set of X such that the restriction $D|_U$ is a simple normal crossing divisor on U.*

Note that f is projective and the exceptional locus $\mathrm{Exc}(f)$ *is of pure codimension one in Y since f is a composition of blowing ups.*

Remark 5.2. Recently, it is reproved by the new canonical desingularization algorithm. See [BEV, Theorem 7.11]. Note that in [BEV] normal crossing means simple normal crossing in our sense. See Remark 2.11 and Remark 7.4 below.

Remark 5.3. Szabó's results depend on Hironaka's paper [H], which is very hard to read. Thus, we recommend the reader to see [BEV] for proofs. Now there are many papers on desingularization theorems. Sorry, I do not know which is the best.

The following example says that Szabó's resolution lemma (and Theorem 5.1) is not true if we replace simple normal crossing with normal crossing. We will treat this example in detail in the next section.

Example 5.4. Let $X := \mathbb{C}^3$ and D the Whitney umbrella, that is, $W = (x^2 - zy^2 = 0)$. Then W is a normal crossing divisor outside the origin. In this case, we can not make W a normal crossing divisor only by blowing ups of smooth subvarieties over the origin.

Sketch of the proof. This is an exercise of how to calculate blow ups of smooth centers. If we blow up W finitely many times along smooth subvarieties over the origin, then we will find that the strict transform of

W always has a pinch point, where a pinch point means a singular point that is local analytically isomorphic to $0 \in (x^2 - zy^2 = 0) \subset \mathbb{C}^3$. □

Theorem 5.1 and Hironaka's desingularization imply the following corollary. It is useful for proving vanishing theorems and so on (see also Remark 6.11 below).

Corollary 5.5. *Let X be a non-complete smooth variety and D a simple normal crossing divisor on X. Then there exists a compactification $\overline{X}$ of X and a simple normal crossing divisor $\overline{D}$ on $\overline{X}$ such that $\overline{D}|_X = D$. Furthermore, if X is quasi-projective, then we can make $\overline{X}$ projective.*

6. Whitney umbrella

We will work over $k = \mathbb{C}$ throughout this section. First, we define *normal crossing varieties.*

Definition 6.1 (Normal crossing variety)**.** Let X be a variety. We say that X is *normal crossing* at x if and only if

$$\widehat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[x_1, x_2, \cdots, x_l]]/(x_1 x_2 \cdots x_k)$$

for some $k \leq l$. If X is normal crossing at any point, we call X a *normal crossing variety.*

Remark 6.2. It is obvious that a normal crossing divisor (see Definition 2.8) is a normal crossing variety. By [A, Corollary (2.6)], X is normal crossing at x if and only if $x \in X$ is locally isomorphic to $0 \in (x_1 x_2 \cdots x_k = 0) \subset \mathbb{C}^l$ for the étale (or classical) topology. So, let U be a small open neighborhood (by the classical topology) of X around x and U' the normalization of U. Then each irreducible component V of U' is smooth and $V \longrightarrow U$ is an embedding.

Next, we introduce the notion of *WU singularities.*

Definition 6.3 (WU singularity)**.** Let X be a variety and x a closed point of X, and $p : X' \longrightarrow X$ the normalization. If there exist a smooth irreducible curve $C' \subset X'$ and a point $x' \in \overline{C' \times_X C' \setminus \Delta_{C'}} \cap \Delta_{C'} \cap p^{-1}(x)$, where $\Delta_{C'}$ is the diagonal of $C' \times_X C'$, then we say that X has a WU singularity at x, where WU is an abbreviation of Whitney Umbrella.

Example 6.4. Let $W = (x^2 - zy^2 = 0) \subset \mathbb{C}^3$ be the Whitney umbrella. Then the normalization of W is $\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[u, v]$ such that the normalization map $\mathbb{C}^2 \longrightarrow W$ is given by $(u, v) \longmapsto (uv, u, v^2)$. Therefore, the line $(u = 0) \subset \mathbb{C}^2$ maps onto $(x = y = 0) \subset W$. Thus the origin is a WU singularity. Note that W is normal crossing outside the origin.

We give one more example.

Example 6.5. Let $V = (z^3 - x^2yz - x^4 = 0) \subset \mathbb{C}^3$. Then V is singular along the y-axis. By blowing up $\mathbb{C}^3$ along the y-axis, we obtain the normalization $p : V' \longrightarrow V$ such that V' is smooth and there exists a smooth curve C' on V' that maps onto the y-axis with the mapping degree two. It can be checked easily that the origin $(0, 0, 0)$ is a WU singularity of V.

Remark 6.6. Let $x \in X$ be a WU singularity. We shrink X around x (by the classical topology). Then there exists an isomorphism $\sigma : C' \longrightarrow C'$ with finite order such that $\sigma \neq id_{C'}$, $\sigma(x') = x'$, and $p = p \circ \sigma$ on C'. When X is the Whitney umbrella, σ corresponds to the graph $\overline{C' \times_X C' \setminus \Delta_{C'}}$ and the order of σ is two.

Lemma 6.7. *Let $x \in X$ be a WU singularity. Then X is not normal crossing at x.*

Proof. Assume that X is normal crossing at x. Let X'_1 be the irreducible component of X' containing C'. Since $X'_1 \longrightarrow X$ is injective in a neighborhood of x', $C' \times_X C' = \Delta_{C'}$ near x'. It is a contradiction. □

The following theorem is the main theorem of this section.

Theorem 6.8. *Let $x \in X$ be a WU singularity and $f : Y \longrightarrow X$ is a proper birational morphism such that $f : f^{-1}(X \setminus \{x\}) \longrightarrow X \setminus \{x\}$ is an isomorphism. Then Y has a WU singularity.*

Proof. Let C', x' be as in Definition 6.3, σ as in Remark 6.6. Let $q : Y' \longrightarrow Y$ be the normalization. Then there exists a proper birational morphism $f' : Y' \longrightarrow X'$. By the assumption, $Y \longrightarrow X$ is an isomorphism over $p(C') \setminus \{x\}$. Thus $Y' \longrightarrow X'$ is an isomorphism over $C' \setminus p^{-1}(x)$. The embedding $C' \subset X'$ induces an embedding $C' \subset Y'$, and $p = p \circ \sigma$ implies $q = q \circ \sigma$. Therefore, Definition 6.3 implies that $g(x') \in Y$ is a WU singularity. □

Proposition 6.9. *Let $x \in X$ be a WU singularity and Z a normal crossing variety. Then there are no proper birational morphisms $g : X \longrightarrow Z$ such that $p(C') \not\subset \mathrm{Exc}(g)$, where p, C' are as in* Definition 6.3.

Proof. Assume that there exists a proper birational morphism as above. We put $C := g(p(C'))$. Then the mapping degree of $C' \longrightarrow C$ is greater than one by the definition of WU singularities. On the other hand, $C' \longrightarrow C$ factors through the normalization Z' of Z. Thus, the mapping degree of $C' \longrightarrow C$ is one. This is a contradiction. $\square$

The next corollary follows from Theorem 6.8 and Proposition 6.9.

Corollary 6.10. *There are no proper birational maps* (*that is, birational maps such that the first and the second projections from the graph are proper*) *between the Whitney umbrella* W *and a normal crossing variety* V *that induce* $W \setminus \{0\} \simeq V \setminus E$*, where* E *is a closed subset of* V*.*

Therefore, we obtain

Remark 6.11. Corollary 5.5 does not hold for normal crossing divisors.

7. What is a log resolution?

We often use the words *good resolution* or *log resolution* without defining them precisely. It sometimes causes some serious problems. We will define our log resolution later (see Definition 7.3). Let us recall another definition of dlt, which is equivalent to Definition 4.1. We do not prove the equivalence of Definition 4.1 and Definition 7.1 in this paper. However, it is not difficult if we understand how to use Theorem 5.1. For the details, see [Sz, Divisorial Log Terminal Theorem].

Definition 7.1 (Divisorial log terminal)**.** Let X be a normal variety and D a boundary on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. There exists a *log resolution* $f : Y \longrightarrow X$ such that $a(E, X, D) > -1$ for every f-exceptional divisor E. Then we say that (X, D) is *dlt* or *divisorial log terminal*.

7.2. There are three questions about the above definition.

- f is projective?
- the exceptional locus $\mathrm{Exc}(f)$ is of pure codimension one?
- $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a *simple normal crossing* divisor or only a *normal crossing* divisor?

In [KM, Notaion 0.4 (10)], they assume that $\mathrm{Exc}(f)$ is of pure codimension one and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a simple normal crossing divisor. We note that, in [FA, 2.9 Definition], $\mathrm{Exc}(f)$ is not necessarily of pure codimenison one. So, the definition of lt in [FA, (2.13.1)] is the same as Definition 7.1 above, but lt in the sense of [FA] is different from dlt. See Remark 7.5 and Examples 8.3 and 9.3 below. The difference exists in the definition of log resolutions!

Our definition of a log resolution is the following, which is [KM, Notaion 0.4 (10)]. By Hironaka, log resolutions exist for varieties over a field of characteristic zero (see [BEV]).

Definition 7.3 (Log resolution)**.** Let X be a variety and D a $\mathbb{Q}$-divisor on X. A *log resolution* of (X, D) is a *proper* birational morphism $f : Y \longrightarrow X$ such that Y is smooth, $\mathrm{Exc}(f)$ is a *divisor* and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a *simple* normal crossing divisor.

Remark 7.4. In the definition of the log resolution in [BEV, Definition 7.10], they do not assume that the exceptional locus $\mathrm{Exc}(\mu)$ is of pure codimension one. However, if μ is a composition of blowing ups, then $\mathrm{Exc}(\mu)$ is always of pure codimension one.

We note lt in the sense of [FA] again. We do not repeat the definition of lt in [FA, (2.13.1)] since it is not so useful.

Remark 7.5 (lt in the sense of [FA])**.** If we do not assume that $\mathrm{Exc}(f)$ is a divisor in Definition 7.3, then Definition 7.1 is the definition of *lt* in the sense of [FA] (see [FA, (2.13.1)]).

Remark 7.6 (Analytically local?)**.** We assume that $k = \mathbb{C}$. Then the notion of terminal, canonical, klt, plt, and lc, is not only algebraically local but also analytically local. For the precise statement, see [Ma, Proposition 4-4-4]. However, the notion of dlt is not analytically local. It is because the notion of simple normal crossing divisors is not analytically local. So, [Ma, Exercise 4-4-5] is incorrect. To obtain an analytically local notion of log terminal singularities, we remove the word: *simple* from Definition 4.1 (2). However, this new notion of log terminal singularities seems to be useless. Consider the pair $(\mathbb{C}^3, W)$, where W is the Whitney umbrella (see Sections 5 and 6).

We note that, by Szabó's resolution lemma, we do not need the projectivity of f in the definition of dlt. It is because the log resolution f in Definition 7.1 can be taken to be a composition of blowing ups by Hironaka's desingularization and Theorem 5.1 (see also Definition 4.1, [KM, Proposition 2.40 and Theorem 2.44], and [Sz, Divisorial Log Terminal Theorem]). We summarize;

Proposition 7.7. *The log resolution f in* Definition 7.1 *can be taken to be a composition of blowing ups of smooth centers. In particular, there exists an effective f-anti-ample divisor whose support coincides with* $\mathrm{Exc}(f)$*. Thus, the notion of dlt coincides with that of wklt* (*see* [FA, (2.13.4)]).

So, we can omit the notion of wklt in the log MMP. In [KMM], they adopted normal crossing divisors instead of simple normal crossing divisors. So there is a difference between wklt and *weak log-terminal.* We note that wklt is weak log-terminal in the sense of [KMM, Definition 0-2-10 (2)], but weak log-terminal is not necessarily wklt. See Section

8, especially, Example 8.1. In my experience, dlt, which is equivalent to wklt, is easy to treat and useful for inductive arguments, but weak log-terminal is very difficult to use. We think that [KM, Corollary 5.50] makes dlt useful. For the usefulness of dlt, see [F1], [F2], [F3], and [F4]. See also Example 8.1, Remark 8.2, and Section 9. We summarize;

Conclusion 7.8. *The notion of dlt coincides with that of wklt by* [Sz] (*see* Proposition 7.7)*. In particular, dlt is weak log-terminal in the sense of* [KMM]*. Therefore, we can freely apply the results that were proved for weak log-terminal pairs in* [KMM] *to dlt pairs. We note*

$$\textit{klt} \Longrightarrow \textit{plt} \Longrightarrow \textit{dlt} \Longleftrightarrow \textit{wklt} \Longrightarrow \textit{weak log-terminal} \Longrightarrow \textit{lc}.$$

For other characterizations of dlt, see [Sz, Divisorial Log Terminal Theorem], which is an exercise of Theorem 5.1. See also [KM, Proposition 2.40, and Theorem 2.44]. The notion of dlt is natural by the following proposition.

Proposition 7.9 (cf. [KM, Proposition 5.51])**.** *Let (X, D) be a dlt pair. Then, any irreducible component of $\lfloor D \rfloor$ is irreducible* (*resp.* $\lfloor D \rfloor = 0$) *if and only if (X, D) is plt* (*resp. klt*)*.*

Thus, dlt is a natural generalization of plt. By the way, we have a question about lt in the sense of [FA].

Question 7.10. Does [FA, (2.16.2)] hold without the projectivity assumption? That is, the notion of $\mathbb{Q}$-factorial lt in the sense of [FA] is equivalent to that of $\mathbb{Q}$-factorial dlt?

One solution of the above question is that we do not use lt in the sense of [FA]. In my experience, this is one of the best solutions.

Conclusion 7.11. *It is better not to use lt in the sense of* [FA]*.* Examples 8.4 *and* 9.3 *imply that the existence of small resolution causes many unexpected phenomena. If the varieties are $\mathbb{Q}$-factorial, then there are no small resolution by* Proposition 2.5*. However, in this case, we do not know if we need to assume the resolution f is projective or not* (*see* Proposition 7.7 *and* Question 7.10)*.*

8. Examples

In this section, we collect some examples. The following example says that weak log-terminal is not necessarily wklt. We omit the definition of weak log-terminal since we do not use it in this paper. See [KMM, Definition 0-2-10].

Example 8.1 (Simple normal crossing vs normal crossing)**.** Let X be a smooth surface and D a nodal curve on X. Then the pair (X, D) is not wklt but weak log-terminal.

The next fact is crucial for inductive arguments.

Remark 8.2. Let (X, D) be a dlt (resp. weak log-terminal) pair and S an irreducible component of $\lfloor D \rfloor$. Then $(S, \mathrm{Diff}(D-S))$ is dlt (resp. not necessarily weak log-terminal), where the $\mathbb{Q}$-divisor $\mathrm{Diff}(D-S)$ on S is defined by the following equation:

$$(K_X + D)|_S = K_S + \mathrm{Diff}(D - S).$$

This is a so-called *adjunction formula*.

We will treat adjunction formula for dlt pairs in detail in Section 9. In Example 8.1, $S := \lfloor D \rfloor$ is not normal. This makes weak log-terminal difficult to use for inductive arguments. The next example explains that we have to assume that $\mathrm{Exc}(f)$ is a divisor in Definition 7.1.

Example 8.3 (Small resolution)**.** Let $X := (xy - uv = 0) \subset \mathbb{C}^4$. It is well-known that this X is a toric variety. We take the torus invariant divisor D; the complement of the big torus. Then (X, D) is not dlt but lt in the sense of [FA]. Especially, it is lc. We note that there exists a small resolution.

The following is a variant of the above example.

Example 8.4 ([FA, 17.5.2 Example])**.** Let $X := (xy - uv = 0) \subset \mathbb{C}^4$ and

$$D = (x = u = 0) + (y = v = 0) + \frac{1}{2}\sum_{i=1}^{4}(x + 2^i u = y + 2^{-i} v = 0).$$

If we put

$$F = \sum_{i=1}^{2}(x + 2^i u = 0) + \sum_{i=3}^{4}(y + 2^{-i} v = 0),$$

then $2D = F \cap X$. Thus, $2(K_X + D)$ is Cartier since X is Gorenstein. We can check that (X, D) is lt in the sense of [FA] by blowing up $\mathbb{C}^4$ along the ideal (x, u). In particular (X, D) is lc. The divisor $\lfloor D \rfloor$ is two planes intersecting at a single point. Thus it is not S_2. So, (X, D) is not dlt.

Remark 8.5. If (X, D) is dlt, then $\lfloor D \rfloor$ is seminormal and S_2 by [FA, 17.5 Corollary].

Remark 8.6. Example 8.4 says that [FA, (16.9.1)] is not true. The problem is that S does not necessarily satisfy Serre's condition S_2.

9. Adjunction for dlt pairs

To treat pairs effectively, we have to understand *adjunction*. The adjunction is explained nicely in [FA, Chapter 16]. We recommend the reader to read it. In this section, we treat adjunction formula only for dlt pairs. Let us recall the definition of center of log canonical singularities.

Definition 9.1 (Center of lc singularities)**.** Let X be a normal variety and D a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there exists a proper birational morphism from a normal variety $\mu : Y \longrightarrow X$ and a prime divisor E on Y with the discrepancy coefficient $a(E, X, D) \leq -1$ such that $\mu(E) = W$.

The next proposition is the adjunction for a higher codimensional center of log canonical singularities of a dlt pair. We use it in [F4]. For the definition of the *different* Diff, see [FA, 16.6 Proposition].

Proposition 9.2 (Adjunction for dlt pairs)**.** *Let (X, D) be a dlt pair. We put $S = \lfloor D \rfloor$ and let $S = \sum_{i \in I} S_i$ be the irreducible decomposition of S. Then, W is a center of log canonical singularities for the pair (X, D) with* $\operatorname{codim}_X W = k$ *if and only if W is an irreducible component of $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$ for some $\{i_1, i_2, \cdots, i_k\} \subset I$. By adjunction, we obtain*

$$K_{S_{i_1}} + \operatorname{Diff}(D - S_{i_1}) = (K_X + D)|_{S_{i_1}},$$

and $(S_{i_1}, \operatorname{Diff}(D - S_{i_1}))$ is dlt. Note that S_{i_1} is normal, W is a center of log canonical singularities for the pair $(S_{i_1}, \operatorname{Diff}(D - S_{i_1}))$, $S_{i_j}|_{S_{i_1}}$ is a reduced part of $\operatorname{Diff}(D - S_{i_1})$ *for $2 \leq j \leq k$, and W is an irreducible component of $(S_{i_2}|_{S_{i_1}}) \cap (S_{i_3}|_{S_{i_1}}) \cap \cdots \cap (S_{i_k}|_{S_{i_1}})$. By applying adjunction k times repeatedly, we obtain a $\mathbb{Q}$-divisor Δ on W such that*

$$(K_X + D)|_W = K_W + \Delta$$

and (W, Δ) is dlt.

Sketch of the proof. Note that S_{i_1} is normal by [KM, Corollary 5.52], and [FA, 17.2 Theorem] and Definition 4.1 imply that $(S_{i_1}, \operatorname{Diff}(D - S_{i_1}))$ is dlt. The other statements are obvious. □

The above proposition makes dlt more valuable than other kinds of *log terminal singularities*.

Example 9.3. Let (X, D) be as in Example 8.4. Recall that (X, D) is lt in the sense of [FA] but not dlt. Then it is not difficult to see that the centers of log canonical singularities for the pair (X, D) are as follows: the origin $(0, 0, 0, 0) \in X$, two Weil divisors $(x = u = 0)$ and $(y = v = 0)$ on X. So, there are no one dimensional center of log canonical singularities.

The final proposition easily follows from the above proposition: Proposition 9.2. It will play crucial roles in the proof of the special termination (see [F4]).

Proposition 9.4. *Let (X, D) be as in* Proposition 9.2. *We write $D = \sum d_j D_j$, where $d_j \in \mathbb{Q}$ and D_j is a prime divisor on X. Let P be a divisor on W. Then the coefficient of P is 0, 1, or $1 - \frac{1}{m} + \sum \frac{r_j d_j}{m}$ for suitable non-negative integers r_js and positive integer m. Note that the coefficient of P is 1 if and only if P is a center of log canonical singularities for the pair (X, D).*

Sketch of the proof. Apply [FA, 16.7 Lemma] k times repeatedly as in Proposition 9.2. We note that [FA, 7.4.3 Lemma]. □

10. Miscellaneous comments

In this section, we collect some comments.

10.1 ($\mathbb{R}$-divisors)**.** In the previous sections, we only use $\mathbb{Q}$-divisors for simplicity. We note that almost all the definitions and results are generalized for $\mathbb{R}$-divisors by a little effort. In Shokurov's proof of PL flips (see [Sh2]), $\mathbb{R}$-divisors appear naturally and is indispensable. Sorry, we do not pursue $\mathbb{R}$-generalizations here anymore. However, if the reader understands the results in this paper, then $\mathbb{R}$-generalizations are good exercises.

10.2 (Comments on the four bibles)**.** We give miscellaneous comments on the four bibles.

- [KMM] is the oldest bible for the log MMP. The notion of *log-terminal* in [KMM, Definition 0-2-10] is equivalent to that of klt.
- [FA] is the only one bible that treats $\mathbb{R}$-divisors and *differents* (see [FA, Chapters 2 and 16]). In Chapter 2, five log terminal singularities, that is, klt, plt, dlt, wklt, and lt, were introduced according to Shokurov [Sh1].
- [KM] seems to be the best bible for singularities of pairs in the log MMP. In the definitions of singularities of pairs, they assume that D is only a subboundary (see [KM, Definition 2.34]

and Remark 3.3). We have to take care of this fact. We note lt in [KM, Definition 2.34 (3)]. If $D = 0$ in Definition 3.2, then the notions klt, plt, and dlt coincide (see also Proposition 7.9) and they say that X has *log terminal* (abbreviated to *lt*) singularities.

Remark 10.3 (Error)**.** There is an error in [KM, Lemma 5.17 (2)]. We can construct a counterexample easily. We put $X = \mathbb{P}^2$, $\Delta =$ a line on X, and $|H| = |\mathcal{O}_X(1)|$. Then we have

$$-1 = \text{discrep}(X, \Delta + H_g) \neq \min\{0, \text{discrep}(X, \Delta)\} = 0,$$

since $\text{discrep}(X, \Delta) = 0$.

- The latest bible [Ma] explains singularities in details (see Chapter 4 in [Ma]). However, as we pointed out before (see Remark 1.1), Matsuki confused normal crossings with simple normal crossings. In the definition of lt (see [Ma, Definition 4-3-2]), he assumed that the resolution is *projective*. So, lt in [Ma] is slightly different from lt in [FA]. Therefore, $\mathbb{Q}$-factorial lt in the sense of [Ma] is equivalent to $\mathbb{Q}$-factorial dlt (see [FA, (2.16.2)], Question 7.10, and Conclusion 7.11).

Remark 10.4 (Comment by Matsuki)**.** In page 178, line 8–9, "by blowing up only over the locus where $\sigma^{-1}(D) \cup \text{Exc}(\sigma)$ is not a normal crossing divisor, we obtain..." is incorrect. See Example 5.4 and Section 6.

Conclusion 10.5. *It is very difficult to understand subtleties of various kinds of log terminal singularities. My idea in this paper is not necessarily the best. We recommend the reader to check definitions by himself.*

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