ON THREE-DIMENSIONAL TERMINAL TORIC SINGULARITIES (PRIVATE NOTE)

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Abstract. We characterize non-$\mathbb{Q}$-factorial affine toric 3-folds with terminal singularities. As applications, we investigate 3-dimensional terminal toric flops and non-$\mathbb{Q}$-factorial terminal toric flips.

In this short note, we characterize non-$\mathbb{Q}$-factorial affine toric 3-folds with terminal singularities. As applications, we investigate 3-dimensional terminal toric flops and non-$\mathbb{Q}$-factorial terminal toric flips. We will use the same notation as in [YPG], which is an excellent exposition on terminal singularities.

Let X be an affine toric 3-fold over an algebraically closed field k. First, let us recall the following well-known theorem of G. K. White, D. Morrison, G. Stevens, V. Danilov, and M. Frumkin (see [YPG, (5.2) Theorem])[1].

Theorem 1. *Assume that X is $\mathbb{Q}$-factorial. Then X is terminal if and only if (up to permutations of (x, y, z) and symmetries of μ_r) $X \simeq \mathbb{A}^3/\mu_r$ of type $\frac{1}{r}(a, -a, 1)$ with a coprime to r, where μ_r is the cyclic group of order r. In particular, if X is Gorenstein and terminal, then X is non-singular.*

The next statement seems to be missing in the literatures[2]. So, we prove it here.

Date: 2004/1/12.

1991 *Mathematics Subject Classification.* Primary 14B05; Secondary 14J30.

I would appreciate any comments. When you use this note in your paper, please inform me.

[1]Sorry, I am not familiar with the history of this theorem (*terminal lemma*). I just copied the names from [YPG].

[2]Hiroshi Sato informed me that Professor Masa-Nori Ishida proved Theorem 2. Sorry, I did not check where the statement is. It may be contained in his paper: On the terminal toric singularities of dimension 3, in *Commutative Algebra*, Karuizawa, Japan, 1982 (S. Goto, ed.), 54–70. Unfortunately, this article seems to be difficult to obtain outside Japan. I can not find it in the library in the IAS nor on MathSciNet. I found that the paper: M. Ishida, and N. Iwashita, Canonical cyclic quotient singularities of dimension three, *Complex analytic singularities*, 135–151,

Theorem 2. *Assume that X is not $\mathbb{Q}$-factorial. Then X is terminal if and only if $X \simeq \operatorname{Spec} k[x, y, z, w]/(xy - zw)$.*[3]

By the above theorems, we classified all the 3-dimensional terminal toric singularities. Related topics are [Fj] and [FS], where we studied the toric Mori theory for *non-$\mathbb{Q}$-factorial* varieties.

Remark. Mori classified 3-dimensional terminal singularities. For the details, see [YPG, (6.1) Theorem]. We do not use his classification table in this paper. We think that we need some arguments to obtain Theorem 2 from Mori's result.

Proof. Let $N = \mathbb{Z}^3$ and $\Delta = \langle e_1, \cdots, e_k \rangle$ the cone in N such that $X = X(\Delta)$, where each e_i is primitive. First, we prove

Claim 1. *If X is non-$\mathbb{Q}$-factorial terminal 3-fold, then $k = 4$.*

Proof of the claim. It is obvious that $k \geq 4$. Since X is $\mathbb{Q}$-Gorenstein, there is a hyperplane $H \subset N$ that contains every e_i. On $H \simeq \mathbb{Z}^2$, e_is span two dimensional convex polygon P. By renumbering e_is, we can assume that they are arranged counter-clockwise. Since $X(\Delta)$ is terminal, all the lattice points in P are e_is. In particular, the triangle on H spanned by e_1, e_2, and e_3 contains only three lattice points e_i $(1 \leq i \leq 3)$ of H. So, after changing the coordinate of H, we can assume that $e_1 = (0, 1), e_2 = (0, 0)$, and $e_3 = (1, 0)$ in $H \simeq \mathbb{Z}^2$. This is an easy consequence of the two dimensional terminal lemma. This means that two dimensional terminal singularities are non-singular. It can be checked easily that $(1, 1) \in P$ since $k \geq 4$. Thus, we obtain that $k = 4$ and $e_4 = (1, 1)$. We finished the proof of the lemma. □

Claim 2. *Assume that X is Gorenstein and terminal. Then X is isomorphic to $\operatorname{Spec} k[x, y, z, w]/(xy - zw)$.*

Proof of the claim. On this assumption, the cones $\langle e_1, e_2, e_3 \rangle$, $\langle e_1, e_2, e_4 \rangle$, $\langle e_1, e_3, e_4 \rangle$, and $\langle e_2, e_3, e_4 \rangle$ define $\mathbb{Q}$-factorial Gorenstein affine toric 3-folds with terminal singularities. By Theorem 1, every cone listed above is non-singular. So, by changing the coordinate of N and easy computations, we can assume that $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and $e_4 = (1, 1, -1)$. In particular, $e_1 + e_2 = e_3 + e_4$. Thus, $X \simeq \operatorname{Spec} k[x, y, z, w]/(xy - zw)$. □

Adv. Stud. Pure Math., **8**, North-Holland, Amsterdam, 1987, treated the similar problem. Though it was not stated explicitly, the main theorem of this paper (Theorem 2) follows immediately from Theorem 3.6 in the above mentioned paper. Since they treat canonical singularities, their proof is much harder than ours. Of course, their results are much more general. Anyway, Theorem 2 is more or less known to experts.

[3] We call this an ODP (ordinary double point).

By the above claim, it is sufficient to prove

Claim 3. *All the non-$\mathbb{Q}$-factorial toric affine 3-folds with terminal singularities are Gorenstein.*

Proof of the claim. We assume that X is not Gorenstein and obtain a contradiction.

Let $\overline{N}$ be the sublattice of N spanned by all the lattice points on H and the origin of N. In $\overline{N}$, $\Delta = \langle e_1, e_2, e_3, e_4 \rangle$ defines a Gorenstein terminal 3-fold. So, we can assume that $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, and $e_4 = (1,1,-1) \in \mathbb{Z}^3 \simeq \overline{N}$ by the proof of Claim 2. First, we consider $\langle e_1, e_2, e_3 \rangle$ in $\overline{N}$ and N. By Theorem 1, we obtain that $N = \overline{N} + \mathbb{Z} \cdot \frac{1}{r}(\alpha, \beta, \gamma)$, where (α, β, γ) is one of the followings: $(a, -a, 1)$, $(a, 1, -a)$, $(-a, a, 1)$, $(-a, 1, a)$, $(1, a, -a)$, $(1, -a, a)$ such that $0 < a < r$ with a coprime to r. Next, we use the terminality of $\langle e_1, e_2, e_4 \rangle$. We consider the linear transform $T : N \longrightarrow N$ such that $Te_1 = e_1$, $Te_2 = e_2$, $Te_4 = e_3$. Then $TN = T\overline{N} + \mathbb{Z} \cdot \frac{1}{r}(\alpha', \beta', \gamma')$, where $(\alpha', \beta', \gamma')$ is one of the followings: $(1+a, 1-a, -1)$, $(0, 1-a, a)$, $(1-a, 1+a, -1)$, $(0, 1+a, -a)$, $(1-a, 0, a)$, $(1+a, 0, -a)$. Note that

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

We treat the first case, that is, $(\alpha', \beta', \gamma') = (1+a, 1-a, -1)$. By the terminal lemma (see [YPG, [(5.4) Theorem]), r divides $(1+a)+(1-a) = 2$ since it does not divide $(1+a)+(-1)$ nor $(1-a)+(-1)$. So, $r = 2$ and $a = 1$. Thus $\frac{1}{r}(\alpha', \beta', \gamma') = \frac{1}{2}(0,0,1)$. It is a contradiction (see Theorem 1). We leave the other cases for the reader's exercise. So, there are no non-Gorenstein non-$\mathbb{Q}$-factorial affine toric 3-folds with terminal singularities. □

Therefore, we completed the proof of the theorem. □

Theorem 2 has a beautiful corollary.

Corollary. *Let*

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

be a 3*-dimensional toric flopping diagram such that* W *is affine. Assume that* X *has only terminal singularities. Then it is the* simplest flop[4]*, where the simplest flop means the flop described in* [Fl, p.49–p.50].

[4]This flop is sometimes called *Atiyah's flop*. I do not know what name is the best.

Proof. By the assumption, W is a non-$\mathbb{Q}$-factorial affine toric 3-fold with terminal singularities. Thus, $X \simeq \operatorname{Spec} k[x, y, z, w]/(xy - zw)$ by Theorem 2. So, the above diagram must be the simplest flop. □

Remark. This corollary describes the behavior of the 3-dimensional terminal flops *locally*. For a *global* example, see [FS].

By the next proposition, we know what 3-dimensional non-$\mathbb{Q}$-factorial terminal flipping contractions are. It is interesting that the flipped variety X^+ is always $\mathbb{Q}$-factorial.

Proposition. *Let*

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

be a 3-dimensional toric flipping diagram such that W is affine. Assume that X has only terminal singularities and $\rho(X/W) = 1$. We assume that the (unique) flipping curve passes through an ODP of X. Then it passes through a quotient singularity[5] *of X, $\rho(X^+/W) = 2$, and X^+ is always $\mathbb{Q}$-factorial.*

Sketch of the proof. First, note that there are no flipping contractions whose flipping curves pass through no singular points[6]. Next, let $Y \longrightarrow X$ be a small resolution of an ODP on X. Then $\rho(Y/W) = 2$. So, there exists another contraction $f : Y \longrightarrow Z$ over W. It is easy to see that f is a flipping contraction such that the (unique) flipping curve passes through at most one singular point. By repeating the above argument, this singularity is not an ODP. So, it is a quotient singularity. Thus, we know that the (unique) flipping curve has to pass through a quotient singularity. Finally, we recommend the reader to draw pictures to understand the latter statements. It is an easy exercise. □

References

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[FS] O, Fujino and H. Sato, An example of toric flops, preprint 2003.

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[M] K. Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.

[5] Here, I exclude non-singular points.

[6] This fact is well-known in the 3-dimensional Minimal Model Program. In the toric category, we can prove it directly and easily (cf. [M, Example-Claim 14-2-5]).

[YPG] M. Reid, Young person's guide to canonical singularities, *Algebraic geometry, Bowdoin, 1985* (*Brunswick, Maine, 1985*), 345–414, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

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