SPECIAL TERMINATION AND REDUCTION THEOREM
2004/11/4

OSAMU FUJINO

Abstract. In this paper, we prove (1) Special termination modulo the log MMP for lower dimensional varieties, and (2) the reduction theorem. Furthermore, we explain the log MMP for non-$\mathbb{Q}$-factorial varieties. These results will play a crucial role in Shokurov's proof of pl flips.

Contents

1. Introduction

This paper is a supplement to [S3, Section 2]. First, we give a simple proof of special termination modulo the log MMP for lower dimensional varieties (see Theorem 2.1). Special termination claims that the flipping locus is disjoint from the reduced part of the boundary after finitely many flips. It will be repeatedly used in Shokurov's proof of pl flips [S3]. Next, we explain the reduction theorem: Theorem 3.7. Roughly speaking, the existence of pl flips and special termination imply the existence of all log flips. The reduction theorem is well-known to experts (cf. [FA, Chapter 18]). It grew out of [S1].

Let us recall the two big conjectures in the log MMP.

Date: 2004/11/4.

2000 *Mathematics Subject Classification.* Primary 14C20; Secondary 14E05.

Key words and phrases. log Minimal Model Program, termination of flips, $\mathbb{Q}$-factorial.

Conjecture 1.1 ((Log) Flip Conjecture I: The existence of a (log) flip)**.** *Let* $\varphi\colon (X, B) \longrightarrow W$ *be an extremal flipping contraction of an* n*-dimensional pair, that is,*

1. φ *is small projective and* φ *has only connected fibers,*
2. $-(K_X + B)$ *is* φ*-ample,*
3. $\rho(X/W) = 1$*, and*
4. X *is* $\mathbb{Q}$*-factorial.*

Then there should be a diagram:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] \\ & W & \end{tikzcd}$$

which satisfies the following conditions:

(i) X^+ *is a normal variety,*
(ii) $\varphi^+\colon X^+ \longrightarrow W$ *is small projective, and*
(iii) $K_{X^+} + B^+$ *is* φ^+*-ample, where* B^+ *is the strict transform of* B*.*

Note that to prove Conjecture 1.1 we can assume that B is a $\mathbb{Q}$-divisor, by perturbing B slightly.

Conjecture 1.2 ((Log) Flip Conjecture II: Termination of a sequence of (log) flips)**.** *A sequence of* (*log*) *flips*

$$(X, B) =: (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$

terminates after finitely many steps. Namely, there does not exist an infinite sequence of (*log*) *flips.*

In this paper, we sometimes write as follows: *Assume the log MMP for* $\mathbb{Q}$*-factorial dlt* (*resp. klt*) n*-folds.* This means that the log flip conjectures I and II hold for n-dimensional dlt (resp. klt) pairs. For the details of the log MMP, see [KM, 3.31]. Note that in this paper we run the log MMP only for birational morphisms. Namely, we apply the log MMP to some pair (X, B) over Y, where $f\colon X \longrightarrow Y$ is a projective birational morphism.

We summarize the contents of this paper: In Section 2, we give a simple proof of special termination. In Section 3, we explain the reduction theorem. This section is essentially the same as [FA, Chapter 18]. Finally, in Section 4, we give a remark on the log MMP for non-$\mathbb{Q}$-factorial varieties.

Acknowledgements. Each section was circulated separately as private notes. Section 4 was written in answer to a question of Takagi. I am grateful to the many people who commented on these notes. Parts of this paper were written at the Institute for Advanced Study. I am grateful for its hospitality. I was partially supported by a grant from the National Science Foundation: DMS-0111298. I would like to thank Professor James M$^{\rm c}$Kernan for giving me comments and correcting mistakes in English. I also like to thank Professor Hiromichi Takagi, Doctor Hokuto Uehara, and Dano Kim.

Notation. We use the basic notations and definitions in [KM] freely (see also [F4]). We will work over an algebraically closed field k throughout this paper; my favorite is $k = \mathbb{C}$.

2. Special termination

Special termination is in [S3, Theorem 2.3]. Shokurov gave a sketch of a proof in dimension four in [S3, Section 2]. Here, we give a simple proof, which is based on the ideas of [FA, Chapter 7]. Note that [FA, Chapter 7] grew out of [S1]. The key point of our proof is the *adjunction formula* for dlt pairs, which is explained in [F4, Section 9]. Let us state the main theorem of this section.

Theorem 2.1 (Special Termination)**.** *We assume that the log MMP for $\mathbb{Q}$-factorial dlt pairs holds in dimension $\leq n-1$. Let X be a normal n-fold and B an effective $\mathbb{R}$-divisor such that (X, B) is dlt. Assume that X is $\mathbb{Q}$-factorial. Consider a sequence of log flips starting from $(X, B) = (X_0, B_0)$:*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots ,$$

where $\phi_i \colon X_i \longrightarrow Z_i$ is a contraction of an extremal ray R_i with $(K_{X_i} + B_i) \cdot R_i < 0$, and ${\phi_i}^+ \colon {X_i}^+ = X_{i+1} \longrightarrow Z_i$ is the log flip. Then, after finitely many flips, the flipping locus (and thus the flipped locus) is disjoint from $\lfloor B_i \rfloor$.

Remark 2.2. If B is a $\mathbb{Q}$-divisor in Theorem 2.1, then the log flip conjectures I and II for $\mathbb{Q}$-divisors are sufficient for the proof of the theorem. This is because $\mathcal{S}(\mathbf{b}) \subset \mathbb{Q}$ (see Definition 2.7 below). We note that when we use special termination in Section 3 and [F3], B is a $\mathbb{Q}$-divisor. If B is not a $\mathbb{Q}$-divisor, then we need the log flip conjecture II for $\mathbb{R}$-divisors. For the details, see [S2, 5.2 Theorem].

First, we recall the definition of *flipping* and *flipped* curves.

Definition 2.3. A curve C on X_i is called *flipping* (resp. *flipped*) if $\phi_i(C)$ (resp. $\phi^+_{i-1}(C)$) is a point.

We quickly review *adjunction* for dlt pairs. For the details, see [F4, Section 9].

Proposition 2.4 (cf. [F4, Proposition 9.2])**.** *Let (X, B) be a dlt pair such that $\lfloor B \rfloor = \sum_{i\in I} D_i$, where D_i is a prime divisor on X for every i. Then S is a* center of log canonical singularities (*CLC, for short*) *of the pair (X, B) with* $\mathrm{codim}_X S = k$ *if and only if S is an irreducible component of $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ for some $\{i_1, i_2, \cdots, i_k\} \subset I$. Let S be a CLC of the pair (X, B). Then (S, B_S) is also dlt, where $K_S + B_S = (K_X + B)|_S$. Note that B_S is defined by applying adjunction k times repeatedly.*

Definition 2.5. A morphism $\varphi\colon (X, B) \longrightarrow (X', B')$ of two log pairs is called an *isomorphism of log pairs* if φ is an isomorphism and $\varphi_*(B) = B'$.

We need the following definition since the restriction of a log flip to a higher codimensional CLC is not necessarily a log flip.

Definition 2.6. Let $f\colon V \longrightarrow W$ be a birational contraction with $\dim V \geq 2$. We say that f is type (S) if f is an isomorphism in codimension one. We say that f is type (D) if f contracts at least one divisor. Let

$$V \xrightarrow{\ f\ } W \xleftarrow{\ g\ } U$$

be a pair of birational contractions. We call this type (SD) if f is type (S) and g is type (D). We define (SS), (DS), and (DD) similarly.

Definition 2.7. Let $B = \sum b_j B^j$ be the irreducible decomposition of an $\mathbb{R}$-divisor B. Let $\mathbf{b}$ be the set $\{b_j\}$. We define

$$\mathcal{S}(\mathbf{b}) := \left\{ 1 - \frac{1}{m} + \sum \frac{r_j b_j}{m} \;\middle|\; m \in \mathbb{Z}_{>0}, r_j \in \mathbb{Z}_{\geq 0} \right\}.$$

Let P be a prime divisor on S. Then the coefficient of P in $\{B_S\}$ is an element of $\mathcal{S}(\mathbf{b})$. See [F4, Propositon 9.4]. Before we give the definition of the *difficulty*, let us recall the following useful lemma: [FA, 7.4.4 Lemma]. The proof is obvious.

Lemma 2.8 (cf. [FA, 7.4.4 Lemma])**.** *Fix a sequence of numbers $0 < b_j \leq 1$ and $c > 0$. Then there are only finitely many possible values $m \in \mathbb{Z}_{>0}$ and $r_j \in \mathbb{Z}_{\geq 0}$ such that*

$$1 - \frac{1}{m} + \sum_j \frac{r_j b_j}{m} \leq 1 - c.$$

Definition 2.9 ([FA, 7.5.1 Definition])**.** Let S be a CLC of the dlt pair (X, B). We define

$$d_{\mathbf{b}}(S, B_S) := \sum_{\alpha \in \mathcal{S}(\mathbf{b})} \sharp \left\{ E \;\middle|\; a(E, S, B_S) < -\alpha, \mathrm{Center}_S(E) \not\subset \lfloor B_S \rfloor \right\}.$$

This is a precise version of the *difficulty*. It is obvious that $d_{\mathbf{b}}(S, B_S) < \infty$ by Lemma 2.8. We note that $(U, B_S|_U)$ is klt, where $U = S \setminus \lfloor B_S \rfloor$.

Let us start the proof of Theorem 2.1.

Proof of Theorem 2.1.

Step 1. *After finitely many flips, the flipping locus contains no CLC's.*

Proof. We note that the number of CLC's is finite. If the flipping locus contains a CLC, then the number of CLC's decreases by [FA, (2.28)]. ☐

So we can assume that the flipping locus contains no CLC's of the pair (X_i, B_i) for every i. By this assumption, $\varphi_i \colon X_i \dashrightarrow X_{i+1}$ induces a birational map $\varphi_i|_{S_i} \colon S_i \dashrightarrow S_{i+1}$, where S_i is a CLC of (X_i, B_i) and S_{i+1} is the corresponding CLC of (X_{i+1}, B_{i+1}). We will omit the subscript $|_{S_i}$ if there is no danger of confusion. Before we go to the next step, we prove the following lemma.

Lemma 2.10. *By adjunction, we have*

$$a(E, S_i, B_{S_i}) \leq a(E, S_{i+1}, B_{S_{i+1}}),$$

for every valuation E. In particular,

$$\mathrm{totaldiscrep}(S_i, B_{S_i}) \leq \mathrm{totaldiscrep}(S_{i+1}, B_{S_{i+1}})$$

for every i.

Sketch of the proof. By the resolution lemma (see [F4, Section 5]), we can find a common log resolution

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X_i \arrow[rr, dashed] & & X_{i+1} \end{tikzcd}$$

such that $Y \longrightarrow X_i$ and $Y \longrightarrow X_{i+1}$ are isomorphisms over the generic points of all CLC's. We note that $X_i \dashrightarrow X_{i+1}$ is an isomorphism at the generic point of every CLC's. Apply the negativity lemma to the flipping diagram $X_i \longrightarrow Z_i \longleftarrow X_{i+1}$ and compare discrepancies. Then, by restricting to S_i and S_{i+1}, we obtain the desired inequalities of discrepancies. ☐

Step 2. *Assume that $\varphi_i \colon X_i \dashrightarrow X_{i+1}$ induces an isomorphism of log pairs, for every $(d-1)$-dimensional CLC for every i. Then, after finitely many flips, φ_i induces an isomorphism of log pairs, for every d-dimensional CLC.*

Remark 2.11. The above statement is slightly weaker than Shokurov's claim (B_d). See the proof of special termination 2.3 in [S3].

Remark 2.12. It is obvious that φ_i induces an isomorphism of log pairs for every 0-dimensional CLC. When $d = 1$, Step 2 is obvious by Lemmas 2.8 and 2.10.

So we can assume that $d \geq 2$.

Remark 2.13. Let (S_i, B_{S_i}) be a CLC. Assume that $\varphi_i \colon (S_i, B_{S_i}) \longrightarrow (S_{i+1}, B_{S_{i+1}})$ is an isomorphism of log pairs. Then S_i contains no flipping curves and S_{i+1} contains no flipped curves. This is obvious by applying the negativity lemma to $S_i \longrightarrow T_i \longleftarrow S_{i+1}$, where T_i is the normalization of $\phi_i(S_i)$.

Proposition 2.14. *The inequality $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$ holds. Moreover, if $S_i \longrightarrow T_i \longleftarrow S_i^+ = S_{i+1}$ is type (SD) or (DD), then $d_{\mathbf{b}}(S_i, B_{S_i}) > d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$, where T_i is the normalization of $\phi_i(S_i)$. Note that there exists a $\phi_i^+|_{S_{i+1}}$-exceptional divisor E on S_{i+1}. By adjunction and the negativity lemma,*

$$a(E, S_i, B_{S_i}) < a(E, S_{i+1}, B_{S_{i+1}}) = -\alpha$$

for some $\alpha \in \mathcal{S}(\mathbf{b})$. Therefore, after finitely many flips, $S_i \longrightarrow T_i \longleftarrow S_{i+1}$ is type (SS) or (DS).

Proof. See [FA, 7.5.3 Lemma, 7.4.3 Lemma]. We note that φ_i is an isomorphism of log pairs on $\lfloor B_{S_i} \rfloor$ by assumption. Therefore,

$$\mathrm{Center}_{S_i}(E) \subset \lfloor B_{S_i} \rfloor \text{ if and only if } \mathrm{Center}_{S_{i+1}}(E) \subset \lfloor B_{S_{i+1}} \rfloor.$$

More precisely, if $\mathrm{Center}_{S_i}(E)$ (resp. $\mathrm{Center}_{S_{i+1}}(E)$) is contained in $\lfloor B_{S_i} \rfloor$ (resp. $\lfloor B_{S_{i+1}} \rfloor$), then φ_i is an isomorphism at the generic point of $\mathrm{Center}_{S_i}(E)$ (resp. $\mathrm{Center}_{S_{i+1}}(E)$) by the negativity lemma. Therefore, we obtain $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$, by Lemma 2.10. ☐

So we can assume that every step is type (SS) or (DS) by shifting the index i.

Lemma 2.15. *By shifting the index i, we can assume that $a(E, S_i, B_{S_i}) = a(E, S_{i+1}, B_{S_{i+1}})$ for every i if E is a divisor on both S_i and S_{i+1}.*

Proof. By Lemma 2.10, we have $a(v, S_i, B_{S_i}) \leq a(v, S_{i+1}, B_{S_{i+1}})$ for every valuation v. We note that the coefficient of E is $-a(E, S_i, B_{S_i}) \geq 0$ and that $-a(E, S_i, B_{S_i}) = 1$ or $-a(E, S_i, B_{S_i}) \in \mathcal{S}(\mathbf{b})$. Thus, Lemma 2.8 implies that $-a(E, S_i, B_{S_i})$ becomes stationary after finitely many steps. ☐

Let $f \colon S_0^0 \longrightarrow S_0$ be a $\mathbb{Q}$-factorial dlt model, that is, $(S_0^0, B_{S_0^0})$ is $\mathbb{Q}$-factorial and dlt such that $K_{S_0^0} + B_{S_0^0} = f^*(K_{S_0} + B_{S_0})$. Note that we need the log MMP in dimension d to construct a dlt model. Applying the log MMP to $S_0^0 \longrightarrow T_0$, we obtain a sequence of divisorial contractions and log flips over T_0

$$S_0^0 \dashrightarrow S_0^1 \dashrightarrow \cdots ,$$

and finally a relative log minimal model $S_0^{k_0}$. Since $S_1 \longrightarrow T_0$ is the log canonical model of $S_0^0 \longrightarrow S_0 \longrightarrow T_0$, we have a unique natural morphism $g\colon S_0^{k_0} \longrightarrow S_1$ (see [FA, 2.22 Theorem]). We note that $K_{S_0^{k_0}} + B_{S_0^{k_0}} = g^*(K_{S_1}+B_{S_1})$. Applying the log MMP to $S_1^0 := S_0^{k_0} \longrightarrow S_1 \longrightarrow T_1$ over T_1, we obtain a sequence

$$S_1^0 \dashrightarrow \cdots \dashrightarrow S_1^{k_1} \longrightarrow S_2$$

for the same reason, where $S_1^{k_1}$ is a relative log minimal model of $S_1^0 \longrightarrow S_1 \longrightarrow T_1$. Run the log MMP to $S_2^0 := S_1^{k_1} \longrightarrow S_2 \longrightarrow T_2$. Repeating this procedure, we obtain a sequence of log flips and divisorial contractions. This sequence terminates by the log MMP in dimension d.

Lemma 2.16. *If $S_i \longrightarrow T_i$ or $S_{i+1} \longrightarrow T_i$ is not an isomorphism, then S_i^0 is not isomorphic to $S_i^{k_i}$ over T_i.*

Proof. If $S_i \longrightarrow T_i$ is not an isomorphism, then $K_{S_i^0} + B_{S_i^0}$ is not nef over T_i. So, S_i^0 is not isomorphic to $S_i^{k_i}$ over T_i. If $S_i \longrightarrow T_i$ is an isomorphism, then $K_{S_i^0}+B_{S_i^0}$ is nef over T_i and $S_i^{k_i} = S_i^0$. In particular, S_{i+1} is isomorphic to $S_i \simeq T_i$. □

Thus we obtain the required results.

Remark 2.17. In Step 2, we obtain no information about flipping curves which are not contained in $\lfloor B_i \rfloor$ but which intersect $\lfloor B_i \rfloor$.

Step 3. *After finitely many flips, we can assume that $\lfloor B_i \rfloor$ contains no flipping curves and no flipped curves by* Step 2. *If the flipping locus intersects $\lfloor B_i \rfloor$, then there exists a flipping curve C such that $C \cdot \lfloor B_i \rfloor > 0$. Note that X_i is $\mathbb{Q}$-factorial. Then $\lfloor B_{i+1} \rfloor$ intersects every flipped curve negatively. So $\lfloor B_{i+1} \rfloor$ contains a flipped curve. This is a contradiction.*

Therefore, we finished the proof of Theorem 2.1. □

Remark 2.18. Our proof heavily relies on the adjunction formula for higher codimensional CLC's of a dlt pair. It is treated in [F4, Section 9]. In the final step (Step 3), $\mathbb{Q}$-factoriality plays a crucial role. As explained in [F4], $\mathbb{Q}$-factoriality and the notion of dlt are not analytically local.

Remark 2.19. For recent developments in the termination of 4-fold log flips, see [F2], [F3], and [F5].

3. Reduction theorem

In this section, we prove the reduction theorem [S3, Reduction Theorem 1.2]. It says that the existence of pl flips and the special termination imply the existence of all log flips. Here is the definition of a (*elementary*) *pre limiting contraction*.

Definition 3.1 (Pre limiting contractions)**.** We call $f\colon (X, D) \longrightarrow Z$ a *pre limiting contraction* (*pl contraction*, for short) if

1. (X, D) is a dlt pair,
2. f is small and $-(K_X + D)$ is f-ample, and
3. there exists an irreducible component $S \subset \lfloor D \rfloor$ such that S is f-negative.

Furthermore, if the above f satisfies

4. $\rho(X/Z) = 1$, and
5. X is $\mathbb{Q}$-factorial,

then $f\colon (X, D) \longrightarrow Z$ is called an *elementary* pre limiting contraction (elementary pl contraction, for short).

Caution 3.2. I do not know what is the best definition of a (*elementary*) *pre limiting contraction.* Compare Definition 3.1 with [S3, 1.1] and [FA, 18.6 Definition]. We adopt the above definition in this paper. The reader should check the definition of pl contractions himself, when he reads other papers.

The following is the definition of *log flips* in this section, which is much more general than log flips in Conjecture 1.1.

Definition 3.3 (Log flips)**.** By a *log flip* of f we mean the (K_X+D)-flip of a contraction $f\colon (X, D) \longrightarrow Z$ assuming that

(a) (X, D) is klt,
(b) f is small,
(c) $-(K_X + D)$ is f-nef, and
(d) D is a $\mathbb{Q}$-divisor.

A $(K_X + D)$-flip of f is a *log canonical model* $f^+\colon (X^+, D^+) \longrightarrow Z$ of (X, D) over Z, that is, a diagram

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] \\ & Z & \end{tikzcd}$$

which satisfies the following conditions*:*

(i) X^+ is a normal variety,
(ii) $f^+ \colon X^+ \longrightarrow Z$ is small, projective, and
(iii) $K_{X^+} + D^+$ is f^+-ample, where D^+ is the strict transform of D.

Note that if a log canonical model exists then it is unique.

Remark 3.4. For the definitions of *log minimal models* and *log canonical models*, see [KM, Definition 3.50]. There, they omit "log" for simplicity. So, a log canonical (resp. log minimal) model is called a *canonical* (resp. *minimal*) *model* in [KM].

Let us introduce the notion of *PL-flips*.

Definition 3.5 (PL-flips)**.** A (*elementary*) *pl-flip* is the flip of f, where f is a (elementary) pl contraction as in Definition 3.1. Note that if the flip exists then it is unique up to isomorphism over Z.

We will use the next definition in the proof of the reduction theorem.

Definition 3.6 (Birational transform)**.** Let $f \colon X \dashrightarrow Y$ be a birational map. Let $\{E_i\}$ be the set of exceptional divisors of f^{-1} and D an $\mathbb{R}$-divisor on X. The *birational transform* of D is defined as

$$D_Y := f_*D + \sum E_i.$$

The following is the main theorem of this section. This is essentially the same as [FA, Chapter 18].

Theorem 3.7 (Reduction Theorem)**.** *Log flips exist in dimension n provided that*:

$(PLF)_n^{el}$ *elementary pl-flips exist in dimension n, and*
$(ST)_n$ *special termination holds in dimension n.*

Proof. Let (X, D) be a klt pair and let $f \colon X \longrightarrow Z$ be a contraction as in Definition 3.3. We define $T := f(\mathrm{Exc}(f)) \subset Z$. We may assume that Z is affine without loss of generality.

Step 1. *Let H' be a Cartier divisor on Z such that*

(i) $H := f^*H' = f_*^{-1}H'$ *contains* $\mathrm{Exc}(f)$.
(ii) H' *is reduced and contains* $\mathrm{Sing}(Z)$ *and the singular locus of* $\mathrm{Supp}\, f(D)$.
(iii) *Fix a resolution $\pi \colon Z' \longrightarrow Z$. Let $F_j \subset Z'$ be divisors that generate $N^1(Z'/Z)$. We assume that H' contains $\pi(F_j)$ for every j.* (*This usually implies that H' is reducible.*) *We note that we can assume that* $\mathrm{Supp}\, \pi(F_j)$ *contains no irreducible components of* $\mathrm{Supp}\, f(D)$ *for every j without loss of generality. Therefore, we can assume that H and D have no common irreducible components.*

The main consequence of the last assumption is the following:

(iv) *Let* $h\colon Y \longrightarrow Z$ *be any proper birational morphism such that* Y *is* $\mathbb{Q}$*-factorial. Then the irreducible components of the proper transform of* H' *and the* h*-exceptional divisors generate* $N^1(Y/Z)$.

Step 2. *By Hironaka's desingularization theorem, there is a projective log resolution* $h\colon Y \longrightarrow X \longrightarrow Z$ *for* $(X, D+H)$*, which is an isomorphism over* $Z \setminus H'$.

Then $K_Y + (D+H)_Y$ *is a* $\mathbb{Q}$*-factorial dlt pair, where* $(D+H)_Y$ *is the birational transform of* $D+H$ (*see* Definition 3.6). *Observe that* h^*H' *contains* $h^{-1}(T)$ *and* h^*H' *contains all* h*-exceptional divisors.*

Step 3. *Run the log MMP with respect to* $K_Y + (D+H)_Y$ *over* Z*. We successively construct objects* $(h_i\colon Y_i \longrightarrow Z, (D+H)_{Y_i})$ *such that* $\lfloor (D+H)_{Y_i} \rfloor$ *contains the support of* h_i^*H'*, and every flipping curve for* h_i *is contained in* $\operatorname{Supp} h_i^*H'$*. If* C_i *is a flipping curve, then* $C_i \subset h_i^*H'$ *and* $C_i \cdot h_i^*H' = 0$*. By* Step 1 (iv) *and* Step 2*, there is an irreducible component* $F_i \subset h_i^*H'$ *such that* $C_i \cdot F_i \neq 0$*. Thus a suitable irreducible component of* h_i^*H' *intersects* C_i *negatively. This means that the only flips that we need are elementary* <u>*pl-flips*</u>*. By special termination, we end up with a* $\mathbb{Q}$*-factorial dlt pair* $\overline{h}\colon (\overline{Y}, (D+H)_{\overline{Y}}) \longrightarrow Z$ *such that* $K_{\overline{Y}} + (D+H)_{\overline{Y}}$ *is* $\overline{h}$*-nef.*

Step 4 (cf. [KM, Theorem 7.44]). *This step is called "subtracting* H*". It is independent of the other steps. So we use different notation throughout* Step 4*. Of course, we assume* $(PLF)_n^{el}$ *and* $(ST)_n$ *throughout this step.*

Theorem 3.8 (Subtraction Theorem). *Let* $(X, S+B+H)$ *be an* n*-dimensional* $\mathbb{Q}$*-factorial dlt pair with effective* $\mathbb{Q}$*-divisors* S*,* B*, and* H *such that* $\lfloor S \rfloor = S$*,* $\lfloor B \rfloor = 0$*. Let* $f\colon X \longrightarrow Y$ *be a projective birational morphism. Assume the following*:

(i) $H \equiv_f -\sum b_j S_j$*, where* $b_j \in \mathbb{Q}_{\geq 0}$*, and* S_j *is an irreducible component of* S *for every* j.

(ii) $K_X + S + B + H$ *is* f*-nef.*

Then $(X, S+B)$ *has a log minimal model over* Y.

Proof. We give a proof in the form of several lemmas by running the log MMP over Y guided by H. The notation and the assumptions of Theorem 3.8 are assumed in these lemmas. ☐

Lemma 3.9. *There exists a rational number* $\lambda \in [0,1]$ *such that*

(1) $K_X + S + B + \lambda H$ *is* f*-nef, and*

(2) *if* $\lambda > 0$*, then there exists a* $(K_X + S + B)$*-negative extremal ray* R *over* Y *such that* $R \cdot (K_X + S + B + \lambda H) = 0$.

Proof. This follows from the Cone Theorem. See, for example, [KM, Complement 3.6]. We note that [KM, §3.1] assumes that the pair has only klt singularities. However, the Rationality Theorem holds for dlt pairs. Therefore, [KM, Complement 3.6] is true for dlt pairs. See [KM, Theorem 3.15, Remark 3.16]. ☐

If $\lambda = 0$, then the theorem is proved. Therefore, we assume that $\lambda > 0$ and let $\phi \colon X \longrightarrow V$ be the contraction of R.

Lemma 3.10. *If ϕ contracts a divisor E, then conditions* (i) *and* (ii) *in* Theorem 3.8 *above, still hold if we replace $f \colon X \longrightarrow Y$ with $V \longrightarrow Y$ and B, S, H with ϕ_*B, ϕ_*S, $\lambda\phi_*H$.*

Proof. This is obvious. ☐

Lemma 3.11. *If ϕ is a flipping contraction, then ϕ is an elementary pl contraction* (*see* Definition 3.1). *If $p \colon X \dashrightarrow X^+$ is the flip of ϕ, then conditions* (i) *and* (ii) *above, still hold if we replace $f \colon X \longrightarrow Y$ with $f^+ \colon X^+ \longrightarrow Y$ and B, S, H with p_*B, p_*S, λp_*H.*

Proof. One has to prove that ϕ is an elementary pl contraction. By hypothesis $R \cdot (K_X + S + B + \lambda H) = 0$ and $R \cdot (K_X + S + B) < 0$, thus one sees $R \cdot H > 0$. Hence by condition (i), there exists j_0 such that $R \cdot S_{j_0} < 0$. The latter part is obvious. ☐

Lemma 3.12. *We can apply the above procedure to the new set up in cases* Lemma 3.10 *and* Lemma 3.11 *if $\lambda \neq 0$. After repeating this finitely many times, λ becomes 0, and one obtain a log minimal model of $(X, S + B)$ over Y. In particular,* Theorem 3.8 *holds.*

Proof. It is obvious that Lemma 3.10 does not occur infinitely many times. The flip in Lemma 3.11 is a $(K_X + S + B)$-flip where the flipping curve is contained in S. Hence there cannot be an infinite sequence of such flips by special termination (see Theorem 2.1). The end product is a log minimal model. ☐

Step 5. *We go back to the original setting. Apply* Theorem 3.8 *to $\overline{h} \colon (\overline{Y}, (D+H)_{\overline{Y}}) \longrightarrow Z$, which was obtained in* Step 3. *More precisely, we put $f = \overline{h}$, $X = \overline{Y}$, $Y = Z$, $S+B+H = (D+H)_{\overline{Y}}$, $B = \{(D+H)_{\overline{Y}}\}$, and $H =$ the strict transform of H', and apply* Theorem 3.8. *Then we obtain*

$$\widetilde{h} \colon (\widetilde{Y}, D_{\widetilde{Y}}) \longrightarrow Z$$

such that $\widetilde{Y}$ is $\mathbb{Q}$-factorial, $K_{\widetilde{Y}} + D_{\widetilde{Y}}$ is dlt and $\widetilde{h}$-nef. By the negativity lemma ([KM, Lemma 3.38])*, we can easily check that $\widetilde{h}$ is small and $(\widetilde{Y}, D_{\widetilde{Y}})$ is klt. This is a log minimal model of (X, D) over Z.*

Step 6. *By the base point free theorem over Z, we obtain the log canonical model of the pair (X, D) over Z, which is the required flip.*

Therefore, we have finished the proof of the reduction theorem. □

Corollary 3.13. *In dimension $n \leq 4$, $(PLF)^{el}_n$ implies the existence of all log flips.*

Proof. Special termination $(ST)_n$ holds if $n \leq 4$, since the log MMP is true in dimension ≤ 3. Thus, this corollary is obvious by Theorem 3.7. □

4. A remark on the log MMP

In this section, we explain the log MMP for non-$\mathbb{Q}$-factorial varieties. We need this generalized version of the log MMP in article by Corti and Takagi. For simplicity, we treat only klt pairs and $\mathbb{Q}$-divisors in this section.

Theorem 4.1 (Log MMP for non-$\mathbb{Q}$-factorial varieties)**.** *Assume that the log MMP holds for $\mathbb{Q}$-factorial klt pairs in dimension n. Then the following modified version of the log MMP works for* (*not necessarily $\mathbb{Q}$-factorial*) *klt pairs in dimension n.*

Proof and explanation. Let us start with a projective morphism $f\colon X \longrightarrow Y$, where $X_0 := X$ is a (not necessarily $\mathbb{Q}$-factorial) normal variety, and a $\mathbb{Q}$-divisor $D_0 := D$ on X such that (X, D) is klt. The aim is to set up a recursive procedure which creates intermediate morphisms $f_i\colon X_i \longrightarrow Y$ and divisors D_i. After finitely many steps, we obtain a final object $\widetilde{f}\colon \widetilde{X} \longrightarrow Y$ and $\widetilde{D}$. Assume that we have already constructed $f_i\colon X_i \longrightarrow Y$ and D_i with the following properties:

(i) f_i is projective,
(ii) D_i is a $\mathbb{Q}$-divisor on X_i,
(iii) (X_i, D_i) is klt.

If $K_{X_i} + D_i$ is f_i-nef, then we set $\widetilde{X} := X_i$ and $\widetilde{D} := D_i$. Assume that $K_{X_i} + D_i$ is not f_i-nef. Then we can take a $(K_{X_i} + D_i)$-negative extremal ray R (or, more generally, a $(K_{X_i}+D_i)$-negative extremal face F) of $\overline{NE}(X_i/Y)$. Thus we have a contraction morphism $\varphi\colon X_i \longrightarrow W_i$ over Y with respect to R (or, more generally, with respect to F). If $\dim W_i < \dim X_i$ (in which case we call φ a *Fano contraction*), then we set $\widetilde{X} := X_i$ and $\widetilde{D} := D_i$ and stop the process. If φ is birational, then we put

$$X_{i+1} := \mathrm{Proj}_{W_i} \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_{X_i}(m(K_{X_i} + D_i)),$$

D_{i+1} := the strict transform of φ_*D_i on X_{i+1} and repeat this process. We note that (X_{i+1}, D_{i+1}) is the log canonical model of (X_i, D_i) over W_i and that the existence of log canonical models follows from the log MMP for $\mathbb{Q}$-factorial klt n-folds. If $K_{W_i} + \varphi_*D_i$ is $\mathbb{Q}$-Cartier, then $X_{i+1} \simeq W_i$. So, this process coincides with the usual one if the varieties X_i are $\mathbb{Q}$-factorial. It is not difficult to see that $X_i \longrightarrow W_i \longleftarrow X_{i+1}$ is of type (DS) or (SS) (for the definitions of (DS) and (SS), see Definition 2.6). So, this process always terminates by the same arguments as in Step 2 of the proof of Theorem 2.1 in Section 2. □

We give one example of 3-dimensional non-$\mathbb{Q}$-factorial terminal flips. The reader can find various examples of non-$\mathbb{Q}$-factorial contractions in [F1, Section 4].

Example 4.2 (3-dimensional non-$\mathbb{Q}$-factorial terminal flip)**.** Let e_1, e_2, e_3 form the usual basis of $\mathbb{Z}^3$, and let e_4 be given by

$$e_1 + e_3 = e_2 + e_4,$$

that is, $e_4 = (1, -1, 1)$. We put $e_5 = (a, 1, -r) \in \mathbb{Z}^3$, where $0 < a < r$ and $\gcd(r, a) = 1$. We consider the following fans:

$$\begin{aligned}\Delta_X &= \{\langle e_1, e_2, e_3, e_4\rangle, \langle e_1, e_2, e_5\rangle, \text{and their faces}\},\\ \Delta_W &= \{\langle e_1, e_2, e_3, e_4, e_5\rangle, \text{and its faces}\}, \text{and}\\ \Delta_{X^+} &= \{\langle e_1, e_4, e_5\rangle, \langle e_2, e_3, e_5\rangle, \langle e_3, e_4, e_5\rangle, \text{and their faces}\}.\end{aligned}$$

We put $X := X(\Delta_X), X^+ := X(\Delta_{X^+})$, and $W := X(\Delta_W)$. Then we have a commutative diagram of toric varieties:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] \\ & W & \end{tikzcd}$$

such that

(i) $\varphi\colon X \longrightarrow W$ and $\varphi^+\colon X^+ \longrightarrow W$ are small projective toric morphisms,
(ii) $\rho(X/W) = 1$ and $\rho(X^+/W) = 2$,
(iii) both X and X^+ have only terminal singularities,
(iv) $-K_X$ is φ-ample and K_{X^+} is φ^+-ample, and
(v) X is not $\mathbb{Q}$-factorial, but X^+ is $\mathbb{Q}$-factorial,

Thus, this diagram is a terminal flip. Note that the ampleness of $-K_X$ (resp. K_{X^+}) follows from the convexity (resp. concavity) of the roofs of the maximal cones in Δ_X (resp. Δ_{X^+}). The figure below should help to understand this example.

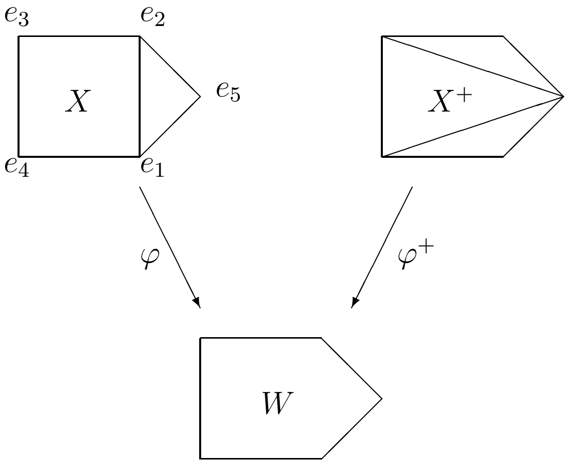

Figure 1

One can check the following properties:

(1) X has one ODP and one quotient singularity,
(2) the flipping locus is $\mathbb{P}^1$ and it passes through the singular points of X, and
(3) the flipped locus is $\mathbb{P}^1 \cup \mathbb{P}^1$ and these two $\mathbb{P}^1$s intersect each other at the singular point of X^+.

This example implies that the relative Picard number may increase after a flip when X is not $\mathbb{Q}$-factorial. So, we do not use the Picard number directly to prove the termination of the log MMP.

References

[F1] O. Fujino, Equivariant completions of toric contraction morphisms, preprint, 2003.
[F2] O. Fujino, Termination of 4-fold canonical flips, Publ. Res. Inst. Math. Sci. **40** (2004), no. 1, 231–237.
[F3] O. Fujino, On termination of 4-fold semi-stable log flips, to appear in Publ. Res. Inst. Math. Sci.
[F4] O. Fujino, What is log terminal ?, in this volume.
[F5] O. Fujino, Addendum to “Termination of 4-fold canonical flips”, to appear in Publ. Res. Inst. Math. Sci.
[FA] J. Kollár et al., *Flips and Abundance for algebraic threefolds*, Astérisque **211**, (1992).
[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties,* Cambridge Tracts in Mathematics, Vol. **134**, 1998.
[S1] V. V. Shokurov, 3-fold log flips, Izv.Ross.Acad.Nauk Ser.Mat.**56** (1992), 105–203 (Russian); English transl., Russian Acad.Sci.Izv.Math.**40** (1993), 95–202.

[S2] V. V. Shokurov, 3-fold log models, Algebraic geometry, 4. J. Math. Sci. **81** (1996), no.3, 2667–2699.

[S3] V. V. Shokurov, Prelimiting flips, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 82–219; translation in Proc. Steklov Inst. Math. 2003, no.1 (240), 75–213

Graduate School of Mathematics, Nagoya University, Chikusa-ku, Naogya, 464-8602 Japan
E-mail address: `fujino@math.nagoya-u.ac.jp`