

**SPECIAL TERMINATION AND REDUCTION
THEOREM
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ABSTRACT. In this paper, we prove (1) Special termination modulo the log MMP for lower dimensional varieties, and (2) Reduction theorem. Moreover, we explain the log MMP for non- \mathbb{Q} -factorial varieties. These results will play crucial roles in Shokurov's proof of pl flips.

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1. INTRODUCTION

This paper is a supplement of [S3, Section 2]. First, we give a simple proof of the special termination modulo the log MMP for lower dimensional varieties (see Theorem 2.1). The special termination claims that the flipping locus is disjoint from the reduced part of the boundary after finitely many flips. It will be repeatedly used in Shokurov's proof of pl flips [S3]. Next, we explain the reduction theorem: Theorem 3.7. Roughly speaking, the existence of pl flips and the special termination imply the existence of all the log flips. This reduction theorem was well-known for experts (cf. [FA, Chapter 18]). It grew out of [S1].

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This is a preliminary version. I appreciate any comments. Please inform me of any errors (fujino@math.nagoya-u.ac.jp) before I will make the final version in September.

Let us recall the two big conjectures in the log MMP.

Conjecture 1.1 ((Log) Flip Conjecture I: The existence of a (log) flip). *Let $\varphi : (X, B) \rightarrow W$ be an extremal flipping contraction of an n -dimensional pair, that is,*

- (1) φ is small projective and φ has only connected fibers,
- (2) $-(K_X + B)$ is φ -ample,
- (3) $\rho(X/W) = 1$, and
- (4) X is \mathbb{Q} -factorial.

Then there should exist the following diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow & \swarrow \\ & W & \end{array}$$

which satisfies the following conditions:

- (i) X^+ is a normal variety,
- (ii) $\varphi^+ : X^+ \rightarrow W$ is small projective, and
- (iii) $K_{X^+} + B^+$ is φ^+ -ample, where B^+ is the strict transform of B .

We note that to prove Conjecture 1.1 we can assume that B is a \mathbb{Q} -divisor by perturbing B slightly.

Conjecture 1.2 ((Log) Flip Conjecture II: The termination of a sequence of (log) flips). *A sequence of (log) flips*

$$(X, B) =: (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$

terminates after finitely many steps. Namely there does not exist an infinite sequence of (log) flips.

In this paper, we sometimes write as follows: *Assume the log MMP for \mathbb{Q} -factorial dlt (resp. klt) n -folds.* This means the log flip conjectures I and II hold for n -dimensional dlt (resp. klt) pairs. For the details of the log MMP, see [KM, 3.31]. Note that in this paper we run the log MMP only for birational morphisms. Namely, we apply the log MMP to some pair (X, B) over Y , where $f : X \rightarrow Y$ is a projective birational morphism.

We summarize the contents of this paper: In Section 2, we give a simple proof of the special termination. In Section 3, we explain the reduction theorem. This section is essentially the same as [FA, Chapter 18]. Finally, in Section 4, we give a remark on the log MMP for non- \mathbb{Q} -factorial varieties.

Acknowledgements. Each section was circulated separately as my private notes. Section 4 was written for an answer to Takagi's question.

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Notation. We use the basic notations and definitions in [KM] freely (see also [F4]). We will work over an algebraically closed field k throughout this paper; my favorite is $k = \mathbb{C}$.

2. SPECIAL TERMINATION

The following special termination is in [S3, Theorem 2.3]. Shokurov wrote a sketch of proof in dimension four in [S3, Section 2]. Here, we give a simple proof, which is based on the idea of [FA, Chapter 7]. Note that [FA, Chapter 7] grew out of [S1]. The key point of our proof is *adjunction formula* for dlt pairs, which is explained in [F4, Section 9]. Let us state the main theorem of this section.

Theorem 2.1 (Special Termination). *We assume that the log MMP for \mathbb{Q} -factorial dlt pairs holds in dimension $\leq n - 1$. Let X be a normal n -fold and B an effective \mathbb{R} -divisor such that (X, B) is dlt. Assume that X is \mathbb{Q} -factorial. Consider a sequence of log flips starting from $(X, B) = (X_0, B_0)$:*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots,$$

where $\phi_i : X_i \rightarrow Z_i$ is a contraction of an extremal ray R_i with $(K_{X_i} + B_i) \cdot R_i < 0$, and $\phi_i^+ : X_i^+ = X_{i+1} \rightarrow Z_i$ is the log flip. Then after finitely many flips, flipping locus (and thus the flipped locus) is disjoint from $\lfloor B_i \rfloor$.

Remark 2.2. If B is a \mathbb{Q} -divisor in Theorem 2.1, then the log flip conjectures I and II for \mathbb{Q} -divisors are sufficient for the proof of the theorem. It is because $\mathcal{S}(\mathbf{b}) \subset \mathbb{Q}$ (see Definition 2.7 below). We note that when we use the special termination in Section 3 and [F3], B is a \mathbb{Q} -divisor. If B is not a \mathbb{Q} -divisor, then we need the log flip conjecture II for \mathbb{R} -divisors. For the details, see [S2, 5.2 Theorem].

First, we recall the definition of *flipping* and *flipped* curves.

Definition 2.3. A curve C on X_i is called *flipping* (resp. *flipped*) if $\phi_i(C)$ (resp. $\phi_{i-1}^+(C)$) is a point.

We quickly review *adjunction* for dlt pairs. For the details, see [F4, Section 9].

Proposition 2.4 (cf. [F4, Proposition 9.2]). *Let (X, B) be a dlt pair such that $\lfloor B \rfloor = \sum_{i \in I} D_i$, where D_i is a prime divisor on X for every i . Then S is a center of log canonical singularities (CLC, for short) of the pair (X, B) with $\text{codim}_X S = k$ if and only if S is an irreducible component of $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ for some $\{i_1, i_2, \dots, i_k\} \subset I$. Let S be a CLC of the pair (X, B) . Then (S, B_S) is also dlt, where $K_S + B_S = (K_X + B)|_S$. We note that B_S is defined by applying adjunction k times repeatedly.*

Next, we introduce the notion of *crepant isomorphism*.

Definition 2.5. Let $\varphi : (X, B) \rightarrow (X', B')$ be a morphism between two dlt pairs. We call φ a *crepant isomorphism* if φ is an isomorphism and $K_X + B = \varphi^*(K_{X'} + B')$.

We need the following definition since the restriction of a log flip to a higher codimensional CLC is not necessarily a log flip.

Definition 2.6. Let $f : V \rightarrow W$ be a birational contraction with $\dim V \geq 2$. We say that f is type (S) if f is an isomorphism in codimension one. We say that f is type (D) if f contracts at least one divisor. Let

$$V \xrightarrow{f} W \xleftarrow{g} U$$

be a pair of birational contractions. We call this type (SD) if f is type (S) and g is type (D) . We can define (SS) , (DS) , and (DD) similarly.

Definition 2.7. Let $B = \sum b_j B^j$ be the irreducible decomposition. We define a finite set of positive numbers $\mathbf{b} = \{b_j\}$. We define

$$\mathcal{S}(\mathbf{b}) := \left\{ 1 - \frac{1}{m} + \sum \frac{r_j b_j}{m} \mid m \in \mathbb{Z}_{>0}, r_j \in \mathbb{Z}_{\geq 0} \right\}.$$

Let P be a prime divisor on S . Then the coefficient of P in B_S is an element of $\mathcal{S}(\mathbf{b})$. See [F4, Proposition 9.4].

Definition 2.8 ([FA, 7.5.1 Definition]). Let S be a CLC of the dlt pair (X, B) . we define

$$d_{\mathbf{b}}(S, B_S) := \sum_{\alpha \in \mathcal{S}(\mathbf{b})} \# \left\{ E \mid a(E, S, B_S) < -\alpha, \text{Center}_S(E) \not\subset \lfloor B_S \rfloor \right\}.$$

This is a precise version of *difficulty*. It is obvious that $d_{\mathbf{b}}(S, B_S) < \infty$. We note that $(U, B_S|_U)$ is klt, where $U = S \setminus \lfloor B_S \rfloor$, and [FA, 7.4.4 Lemma].

Let us go to the proof of Theorem 2.1.

Proof of Theorem 2.1.

Step 1. *After finitely many flips, the flipping locus contains no CLC's.*

Proof. We note that the number of CLC's is finite. If the flipping locus contains a CLC, then the number of CLC's decreases by [FA, (2.28)]. \square

So we can assume that the flipping locus contains no CLC's of the pair (X_i, B_i) for every i . On this assumption, $\varphi_i : X_i \dashrightarrow X_{i+1}$ induces a birational map $\varphi_i|_{S_i} : S_i \dashrightarrow S_{i+1}$, where S_i is a CLC of (X_i, B_i) and S_{i+1} is the corresponding CLC of (X_{i+1}, B_{i+1}) . We will omit the subscript $|_{S_i}$ if there is no danger of confusion.

Remark 2.9. By adjunction, we have

$$a(E, S_i, B_{S_i}) \leq a(E, S_{i+1}, B_{S_{i+1}}),$$

for every valuation E . In particular,

$$\text{totaldiscrep}(S_i, B_{S_i}) \leq \text{totaldiscrep}(S_{i+1}, B_{S_{i+1}})$$

for every i .

Step 2. *Assume that $\varphi_i : X_i \dashrightarrow X_{i+1}$ induces a crepant isomorphism on every $(d-1)$ -dimensional CLC for all i . Then after finitely many flips, φ_i induces a crepant isomorphism on every d -dimensional CLC.*

Remark 2.10. The above statement is slightly weaker than Shokurov's claim (B_d) . See the proof of the special termination 2.3 in [S3].

Remark 2.11. It is obvious that φ_i induces a crepant isomorphism on every 0-dimensional CLC. When $d=1$, Step 2 is obvious by [FA, 7.4.4 Lemma] and Remark 2.9.

So we can assume that $d \geq 2$.

Remark 2.12. Let (S_i, B_{S_i}) be a CLC. Assume that $\varphi_i : (S_i, B_{S_i}) \rightarrow (S_{i+1}, B_{S_{i+1}})$ is a crepant isomorphism. Then S_i contains no flipping curves and S_{i+1} contains no flipped curves. This is obvious by applying the negativity lemma to $S_i \rightarrow T_i \leftarrow S_{i+1}$, where T_i is the normalization of $\phi_i(S_i)$.

Proposition 2.13. *The inequality $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$ holds. Moreover, if $S_i \rightarrow T_i \leftarrow S_{i+1}^+ = S_{i+1}$ is type (SD) or (DD) , then $d_{\mathbf{b}}(S_i, B_{S_i}) > d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$, where T_i is the normalization of $\phi_i(S_i)$. We note that there exists a $\phi_i^+|_{S_{i+1}}$ -exceptional divisor E on S_{i+1} . By adjunction and the negativity lemma,*

$$a(E, S_i, B_{S_i}) < a(E, S_{i+1}, B_{S_{i+1}}) = -\alpha$$

for some $\alpha \in \mathcal{S}(\mathbf{b})$. Therefore, after finitely many flips, $S_i \rightarrow T_i \leftarrow S_{i+1}$ is type (SS) or (DS) .

Proof. See [FA, 7.5.3 Lemma, 7.4.3 Lemma]. We note that φ_i is a crepant isomorphism on $\lfloor B_{S_i} \rfloor$ by the assumption. Therefore,

$$\text{Center}_{S_i}(E) \subset \lfloor B_{S_i} \rfloor \text{ if and only if } \text{Center}_{S_{i+1}}(E) \subset \lfloor B_{S_{i+1}} \rfloor.$$

More precisely, if $\text{Center}_{S_i}(E)$ (resp. $\text{Center}_{S_{i+1}}(E)$) is contained in $\lfloor B_{S_i} \rfloor$ (resp. $\lfloor B_{S_{i+1}} \rfloor$), then φ_i is an isomorphism at the generic point of $\text{Center}_{S_i}(E)$ (resp. $\text{Center}_{S_{i+1}}(E)$) by the negativity lemma. Therefore, we obtain $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$ by Remark 2.9. \square

So we can assume that every step is type (SS) or (DS) by shifting the index i . Let $f : S_0^0 \rightarrow S_0$ be a \mathbb{Q} -factorial dlt model, that is, $(S_0^0, B_{S_0^0})$ is \mathbb{Q} -factorial and dlt such that $K_{S_0^0} + B_{S_0^0} = f^*(K_{S_0} + B_{S_0})$. Note that we need the log MMP in dimension d to construct a dlt model. Applying the log MMP to $S_0^0 \rightarrow T_0$, we obtain a sequence of divisorial contractions and log flips over T_0

$$S_0^0 \dashrightarrow S_0^1 \dashrightarrow \cdots,$$

and finally a relative log minimal model $S_0^{k_0}$. Since $S_1 \rightarrow T_0$ is the lc model of $S_0^0 \rightarrow S_0 \rightarrow T_0$, we have a unique natural morphism $g : S_0^{k_0} \rightarrow S_1$ (see [FA, 2.22 Theorem]). We note that $K_{S_0^{k_0}} + B_{S_0^{k_0}} = g^*(K_{S_1} + B_{S_1})$. Applying the log MMP to $S_1^0 := S_0^{k_0} \rightarrow S_1 \rightarrow T_1$ over T_1 , we obtain a sequence

$$S_1^0 \dashrightarrow \cdots \dashrightarrow S_1^{k_1} \rightarrow S_2$$

by the same reason. Run the log MMP to $S_2^0 := S_1^{k_1} \rightarrow S_2 \rightarrow T_2$. Repeating this procedure, we obtain a sequence of log flips and divisorial contractions. This sequence terminates by the log MMP in dimension d . Thus we obtain the required results.

Remark 2.14. In Step 2, we obtain no information about flipping curves that are not contained in $\lfloor B_i \rfloor$ but intersect with $\lfloor B_i \rfloor$.

Step 3. *After finitely many flips, we can assume that $\lfloor B_i \rfloor$ contains no flipping curves and no flipped curves by Step 2. If the flipping locus intersects $\lfloor B_i \rfloor$, then there exists a flipping curve C such that $C \cdot \lfloor B_i \rfloor > 0$. Note that X_i is \mathbb{Q} -factorial. Then $\lfloor B_{i+1} \rfloor$ intersects with every flipped curve negatively. So $\lfloor B_{i+1} \rfloor$ contains a flipped curve. This is a contradiction.*

Therefore, we finished the proof of Theorem 2.1. \square

Remark 2.15. Our proof heavily relies on adjunction formula for higher codimensional CLC's of a dlt pair. It is treated in [F4, Section 9]. In the final step (Step 3), \mathbb{Q} -factoriality plays crucial roles. As

explained in [F4], \mathbb{Q} -factoriality and the notion of dlt are not analytically local.

Remark 2.16. In [F2], we prove that any sequence of log flips of canonical pairs terminates in dimension four. By combining [F2] with Theorem 2.1, we obtain the termination of 4-fold semi-stable log flips. For the details, see [F2] and [F3].

3. REDUCTION THEOREM

In this section, we prove the reduction theorem [S3, Reduction Theorem 1.2]. It says that the existence of pl flips and the special termination implies the existence of all the log flips. The following is the definition of an (*elementary*) *pre limiting contraction*.

Definition 3.1 (Pre limiting contractions). We call $f : (X, D) \rightarrow Z$ an *pre limiting contraction* (*pl contraction*, for short) if

- (1) (X, D) is a dlt pair,
- (2) f is small and $-(K_X + D)$ is f -ample, and
- (3) there exists an irreducible component $S \subset [D]$ such that S is f -negative.

Furthermore, if the above f satisfies

- (4) $\rho(X/Z) = 1$, and
- (5) X is \mathbb{Q} -factorial,

then $f : (X, D) \rightarrow Z$ is called an *elementary pre limiting contraction* (*elementary pl contraction*, for short).

Caution 3.2. I do not know what is the best definition of (*elementary*) *pre limiting contractions*. Compare Definition 3.1 with [S3, 1.1] and [FA, 18.6 Definition]. We adopt the above definition in this paper. The reader has to check the definition of pl contractions when he reads other papers.

The following is the definition of *log flips* in this section, which is much more general than log flips in Conjecture 1.1.

Definition 3.3 (Log flips). By a *log flip* we mean the $(K_X + D)$ -flip of a contraction $f : (X, D) \rightarrow Z$ assuming that

- (a) (X, D) is klt,
- (b) f is small,
- (c) $-(K_X + D)$ is f -nef, and
- (d) D is a \mathbb{Q} -divisor.

A log flips of f is a *log canonical model* $f^+ : (X^+, D^+) \longrightarrow Z$ of (X, D) over Z , that is, the diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow & \swarrow \\ & Z & \end{array}$$

which satisfies the following conditions:

- (i) X^+ is a normal variety,
- (ii) $f^+ : X^+ \longrightarrow Z$ is small projective, and
- (iii) $K_{X^+} + D^+$ is f^+ -ample, where D^+ is the strict transform of D .

Note that a log canonical model exists then it is unique.

Remark 3.4. For the definitions of *log minimal models* and *log canonical models*, see [KM, Definition 3.50]. There, they omit "log" for simplicity. So, a log canonical (resp. log minimal) model is called a *canonical* (resp. *minimal*) model in [KM].

Let us introduce the notion of *PL-flips*.

Definition 3.5 (PL-flips). A (*elementary*) *pl-flip* is a flip of f , where f is a (elementary) pl contraction as in Definition 3.1. Note that if a flip exists then it is unique up to isomorphisms over Z .

We use the next definition in the proof of the reduction theorem.

Definition 3.6 (Birational transform). Let $f : X \dashrightarrow Y$ be a birational map. Let $\{E_i\}$ be the set of exceptional divisors of f^{-1} and D an \mathbb{R} -divisor on X . The *birational transform* of D is defined as

$$(D)_Y := f_*D + \sum E_i.$$

The following is the main theorem of this section. This is essentially the same as [FA, Chapter 18].

Theorem 3.7 (Reduction Theorem). *Log flips exist in dimension n provided that:*

- $(PLF)_n^{el}$ elementary pl-flips exist in dimension n , and
- $(ST)_n$ special termination holds in dimension n .

Proof. Let (X, D) be a klt pair and let $f : X \longrightarrow Z$ be a contraction as in Definition 3.3. We define $T := f(\text{Exc}(f)) \subset Z$. We can assume that Z is affine without loss of generality.

Step 1. Let H' be a Cartier divisor on Z such that

- (i) $H := f^*H' = f_*^{-1}H'$ contains $\text{Exc}(f)$.
- (ii) H' is reduced and contains $\text{Sing}(Z)$ and the singular locus of $\text{Supp}(D)$.

- (iii) Fix a resolution $\pi : Z' \rightarrow Z$. Let $F_j \subset Z'$ be divisors that generate $N^1(Z'/Z)$. We assume that H' contains $\pi(F_j)$ for every j . (This usually implies that H' is reducible.) We note that we can assume that $\text{Supp}\pi(F_j)$ contains no irreducible components of $\text{Supp}f(D)$ for every j without loss of generality. Therefore, we can assume that H and D have no common irreducible components.

The main consequence of the last assumption is the following:

- (iv) Let $h : Y \rightarrow Z$ be any proper birational morphism such that Y is \mathbb{Q} -factorial. Then the irreducible components of the proper transform of H' and the h -exceptional divisors generate $N^1(Y/Z)$.

Step 2. By Hironaka's desingularization theorem, there is a log resolution $h : Y \rightarrow X \rightarrow Z$ for $(X, D + H)$ that is an isomorphism over $Z \setminus H'$.

Then $K_Y + (D + H)_Y$ is a \mathbb{Q} -factorial dlt pair, where $(D + H)_Y$ is a birational transform of $D + H$ (see Definition 3.6). Observe that $h^*(H')$ contains $h^{-1}(T)$ and $h^*(H')$ contains all h -exceptional divisors.

Step 3. Run the log MMP with respect to $K_Y + (D + H)_Y$ over Z . We successively construct the objects $(h_i : Y_i \rightarrow Z, (D + H)_{Y_i})$ such that $\lfloor (D + H)_{Y_i} \rfloor$ contains the support of h_i^*H' , and every flipping curve for h_i is contained in $\text{Supp}h_i^*H'$. If C_i is a flipping curve, then $C_i \subset h_i^*H'$ and $C_i \cdot h_i^*H' = 0$. By Step 1 (iv) and Step 2, there is an irreducible component $F_i \subset h_i^*H'$ such that $C_i \cdot F_i \neq 0$. Thus a suitable irreducible component of h_i^*H' intersects C_i negatively. This means that all flips that we need are elementary pl-flips. By the special termination, we end up with a \mathbb{Q} -factorial dlt pair $\bar{h} : (\bar{Y}, (D + H)_{\bar{Y}}) \rightarrow Z$ such that $K_{\bar{Y}} + (D + H)_{\bar{Y}}$ is \bar{h} -nef.

Step 4 (cf. [KM, Theorem 7.44]). This step is called “subtracting H ”. It is independent of other steps. So we use the different notation throughout Step 4. Of course, we assume $(PLF)_n^{\text{el}}$ and $(ST)_n$ throughout this step.

Theorem 3.8 (Subtracting Theorem). Let $(X, S + B + H)$ be an n -dimensional \mathbb{Q} -factorial dlt pair with effective \mathbb{Q} -divisors S , B , and H such that $\lfloor S \rfloor = S$, $\lfloor B \rfloor = 0$. Let $f : X \rightarrow Y$ be a projective birational morphism. Assume the following:

- (i) $H \equiv_f - \sum b_j S_j$, where $b_j \in \mathbb{Q}_{\geq 0}$, and S_j is an irreducible component of S for every j .
(ii) $K_X + S + B + H$ is f -nef.

Then $(X, S + B)$ has a log minimal model over Y .

Proof. We give a proof in the form of several lemmas by running the log MMP over Y guided by H . The notation and the assumptions of Theorem 3.8 are assumed in these lemmas. \square

Lemma 3.9. *There exists a rational number $\lambda \in [0, 1]$ such that*

- (1) $K_X + S + B + \lambda H$ is f -nef, and
- (2) if $\lambda > 0$, then there exists a $(K_X + S + B)$ -negative extremal ray R over Y such that $R \cdot (K_X + S + B + \lambda H) = 0$.

Proof. This follows from Cone Theorem. See, for example, [KM, Complement 3.6]. We note that [KM, §3.1] assumes that the pair has only klt singularities. However, Rationality Theorem holds for dlt pairs. Therefore, [KM, Complement 3.6] is true for dlt pairs. See [KM, Theorem 3.15, Remark 3.16]. \square

If $\lambda = 0$, then the theorem is proved. Therefore, we assume that $\lambda > 0$ and let $\phi : X \rightarrow V$ be the contraction of R .

Lemma 3.10. *If ϕ contracts a divisor E , then the above conditions (i) and (ii) in Theorem 3.8 still hold if we replace $f : X \rightarrow Y$ with $V \rightarrow Y$ and B, S, H with $\phi_*B, \phi_*S, \lambda\phi_*H$.*

Proof. It is obvious. We note that ϕ is a divisorial contraction. \square

Lemma 3.11. *If ϕ is a flipping contraction, then ϕ is an elementary pl contraction (see Definition 3.1). If $p : X \dashrightarrow X^+$ is the flip of ϕ , then the above conditions (i) and (ii) still hold if we replace $f : X \rightarrow Y$ with $f^+ : X^+ \rightarrow Y$ and B, S, H with $p_*B, p_*S, \lambda p_*H$.*

Proof. One has to prove that ϕ is an elementary pl contraction. By hypothesis $R \cdot (K_X + S + B + \lambda H) = 0$ and $R \cdot (K_X + S + B) < 0$, one sees $R \cdot H > 0$. Hence by condition (i), there exists j_0 such that $R \cdot S_{j_0} < 0$. The latter part is obvious. \square

Lemma 3.12. *We can apply the above procedure to the new set up in the cases Lemma 3.10 and Lemma 3.11 if $H \neq 0$. After repeating this finitely many times, H becomes 0, and one obtain a log minimal model of $(X, S + B)$ over Y . In particular, Theorem 3.8 holds.*

Proof. It is obvious that Lemma 3.10 does not occur infinitely many times. The flip in Lemma 3.11 is a $(K_X + S + B)$ -flip with the flipping curve on S . Hence it cannot repeat infinitely many times by the special termination (see Theorem 2.1). Therefore, we obtain a log minimal model. \square

Step 5. *We go back to the original setting. Apply Theorem 3.8 to $\bar{h} : (\bar{Y}, (D+H)_{\bar{Y}}) \rightarrow Z$, which was obtained in Step 3. More precisely,*

we put $f = \bar{h}$, $X = \bar{Y}$, $Y = Z$, $S+B+H = (D+H)_{\bar{Y}}$, $B = \{(D+H)_{\bar{Y}}\}$, and $H =$ the strict transform of H' , and apply Theorem 3.8. Then we obtain

$$\tilde{h} : (\tilde{Y}, D_{\tilde{Y}}) \longrightarrow Z$$

such that \tilde{Y} is \mathbb{Q} -factorial, $K_{\tilde{Y}} + D_{\tilde{Y}}$ is dlt and \tilde{h} -nef. By the negativity lemma ([KM, Lemma 3.38]), we can easily check that \tilde{h} is small and $(\tilde{Y}, D_{\tilde{Y}})$ is klt. This is a log minimal model of (X, D) over Z .

Step 6. By the base point free theorem over Z , we obtain the log canonical model of the pair (X, D) over Z , which is the required flip.

Therefore, we finish the proof of the reduction theorem. \square

Corollary 3.13. *In dimension $n \leq 4$, $(PLF)_n^{el}$ implies the existence of all the log flips.*

Proof. The special termination $(ST)_n$ holds if $n \leq 4$ since the log MMP is true in dimension ≤ 3 . Thus, this corollary is obvious by Theorem 3.7. \square

4. A REMARK ON THE LOG MMP

In this section, we explain the log MMP for non- \mathbb{Q} -factorial varieties. We need this generalized version of the log MMP in Takagi's article. For simplicity, we treat only klt pairs and \mathbb{Q} -divisors in this section.

Theorem 4.1 (Log MMP for non- \mathbb{Q} -factorial varieties). *Assume that the log MMP holds for \mathbb{Q} -factorial klt pairs in dimension n . Then the following modified version of the log MMP works for (not necessarily \mathbb{Q} -factorial) klt pairs in dimension n .*

Proof and explanation. Let us start with a projective morphism $f : X \longrightarrow Y$, where $X_0 := X$ is a (not necessarily \mathbb{Q} -factorial) normal variety, and a \mathbb{Q} -divisor $D_0 := D$ on X such that (X, D) is klt. The aim is to set up a recursive procedure which creates intermediate $f_i : X_i \longrightarrow Y$ and D_i . After finitely many steps, we obtain a final objects $\tilde{f} : \tilde{X} \longrightarrow Y$ and \tilde{D} . Assume that we already constructed $f_i : X_i \longrightarrow Y$ and D_i with the following properties:

- (i) f_i is projective,
- (ii) D_i is a \mathbb{Q} -divisor on X_i ,
- (iii) (X_i, D_i) is klt.

If $K_{X_i} + D_i$ is f_i -nef, then we set $\tilde{X} := X_i$ and $\tilde{D} := D_i$. Assume that $K_{X_i} + D_i$ is not f_i -nef. Then we can take a $(K_{X_i} + D_i)$ -negative extremal ray R (or, more generally, a $(K_{X_i} + D_i)$ -negative extremal face F) of

$\overline{NE}(X_i/Y)$. Thus we have a contraction morphism $\varphi : X_i \rightarrow W_i$ over Y with respect to R (or, more generally, with respect to F). If $\dim W_i < \dim X_i$ (in which case we call φ a *Fano contraction*), then we set $\tilde{X} := X_i$ and $\tilde{D} := D_i$ and stop the process. If φ is birational, then we put

$$X_{i+1} := \text{Proj}_{W_i} \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_{X_i}(m(K_{X_i} + D_i)),$$

$D_{i+1} :=$ the strict transform of $\varphi_* D_i$ on X_{i+1} and repeat this process. We note that (X_{i+1}, D_{i+1}) is the log canonical model of (X_i, D_i) over W_i and the existence of the log canonical models follows from the log MMP for \mathbb{Q} -factorial n -folds. If $K_{W_i} + \varphi_* D_i$ is \mathbb{Q} -Cartier, then $X_{i+1} \simeq W_i$. So, this process coincides with the usual one if the varieties X_i are \mathbb{Q} -factorial. It is not difficult to see that $X_i \rightarrow W_i \leftarrow X_{i+1}$ is of type (DS) or (SS) (for the definitions of (DS) and (SS) , see Definition 2.6). Then, this process always terminates by the same arguments as in Step 2 in the proof of Theorem 2.1 in Section 2. \square

We give one example of 3-dimensional non- \mathbb{Q} -factorial terminal flips. The readers can find various examples of non- \mathbb{Q} -factorial contractions in [F1, Section 4].

Example 4.2 (3-dimensional non- \mathbb{Q} -factorial terminal flip). Let e_1, e_2, e_3 form the usual basis of \mathbb{Z}^3 , and let e_4 be given by

$$e_1 + e_3 = e_2 + e_4,$$

that is, $e_4 = (1, -1, 1)$. We put $e_5 = (a, 1, -r) \in \mathbb{Z}^3$, where $0 < a < r$ and $\gcd(r, a) = 1$. We consider the following fans:

$$\begin{aligned} \Delta_X &= \{ \langle e_1, e_2, e_3, e_4 \rangle, \langle e_1, e_2, e_5 \rangle, \text{ and their faces} \}, \\ \Delta_W &= \{ \langle e_1, e_2, e_3, e_4, e_5 \rangle, \text{ and its faces} \}, \text{ and} \\ \Delta_{X^+} &= \{ \langle e_1, e_4, e_5 \rangle, \langle e_2, e_3, e_5 \rangle, \langle e_3, e_4, e_5 \rangle, \text{ and their faces} \}. \end{aligned}$$

We put $X := X(\Delta_X)$, $X^+ := X(\Delta_{X^+})$, and $W := X(\Delta_W)$. Then we have the commutative diagram of toric morphisms:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow & \swarrow \\ & W & \end{array}$$

such that

- (i) $\varphi : X \rightarrow W$ and $\varphi^+ : X^+ \rightarrow W$ are small projective toric morphisms,
- (ii) $\rho(X/W) = 1$ and $\rho(X^+/W) = 2$,
- (iii) both X and X^+ have only terminal singularities,

- (iv) $-K_X$ is φ -ample and K_{X^+} is φ^+ -ample, and
- (v) X is not \mathbb{Q} -factorial, but X^+ is \mathbb{Q} -factorial,

Thus, this diagram is a terminal flip. Note that the ampleness of $-K_X$ (resp. K_{X^+}) follows from the convexity (resp. concavity) of the roofs of the maximal cones in Δ_X (resp. Δ_{X^+}). The figure below helps us to understand this example.

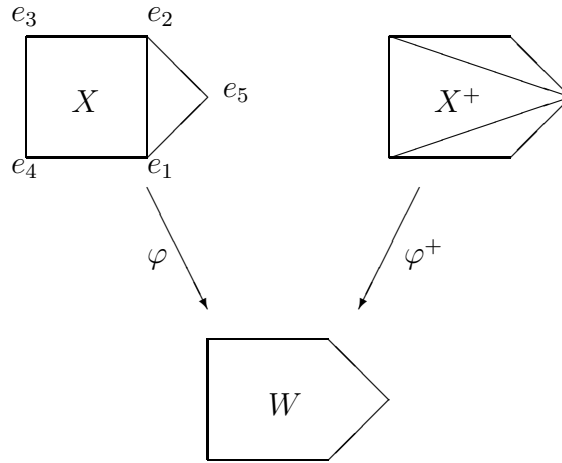


FIGURE 1

We can check the following properties:

- (1) X has one ODP and one quotient singularity,
- (2) the flipping locus is \mathbb{P}^1 and it passes through the singular points of X , and
- (3) the flipped locus is $\mathbb{P}^1 \cup \mathbb{P}^1$ and these two \mathbb{P}^1 s intersect each other at the singular point of X^+ .

This example implies that the relative Picard number may increase by a flip when X is not \mathbb{Q} -factorial. So, we do not use the Picard number directly to prove the termination of the log MMP.

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