LEMMA OF MORIWAKI
(PRIVATE NOTE)

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Abstract. This note is a short memorandum on the finitely generatedness of (relative) log canonical rings.

We will work over $\mathbb{C}$, the complex number field, throughout this note. We freely use the basic notations in [KMM] and [YPG]. The following problem is one of the most famous conjecture in the algebraic geometry.

Problem ($\mathbf{F_n}$)**.** Let V be an n-dimensional non-singular projective variety. Assume that K_V is big. Then the canonical ring

$$R(V, K_V) = \bigoplus_{m \geq 0} H^0(V, \mathcal{O}_V(mK_V))$$

is a finitely generated $\mathbb{C}$-algebra.

The next problem is a log version of the previous one. It is already proved in dimension ≤ 3.

Problem (($\mathbf{F_n}, \mathbf{klt}$) (resp. ($\mathbf{F_n}, \mathbf{lc}$))**.** Let V be an n-dimensional non-singular projective variety and $D = \sum a_i D_i$ an effective $\mathbb{Q}$-divisor on V such that D_i is irreducible and $0 < a_i < 1$ (resp. $0 < a_i \leq 1$) for all i, and $\sum D_i$ is a simple normal crossing divisor. Assume that $K_V + D$ is big. Then the log canonical ring

$$R(V, K_V + D) = \bigoplus_{m \geq 0} H^0(V, \mathcal{O}_V(\llcorner m(K_V + D)\lrcorner))$$

is a finitely generated $\mathbb{C}$-algebra.

A special case of the following lemma was treated in [M]. The title of this note comes from this fact.

Lemma 1 (cf. [M, §4])**.** *Let $f : X \to S$ be a proper surjective morphism between normal varieties with connected fibers. We put $n := \dim X$.*

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Assume that X has only canonical singularities and K_X is f-big. Then the relative canonical ring

$$R(X/S, K_X) = \bigoplus_{m\geq 0} f_*\mathcal{O}_X(mK_X)$$

is a finitely generated $\mathcal{O}_S$-algebra if $(\mathbf{F_n})$ holds.

Proof. Put $\Delta = 0$ in the proof of Lemma 2. □

The next lemma is the main result of this note.

Lemma 2. *Let $f : X \to S$ be a proper surjective morphism between normal varieties with connected fibers. We put $n := \dim X$. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt (resp. lc). Assume that $K_X + \Delta$ is f-big. Then the relative log canonical ring*

$$R(X/S, K_X + \Delta) = \bigoplus_{m\geq 0} f_*\mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner)$$

is a finitely generated $\mathcal{O}_S$-algebra if $(\mathbf{F_n}, \mathbf{klt})$ (resp. $(\mathbf{F_n}, \mathbf{lc})$) holds.

Before we go to the proof, we note one remark.

Remark 3. For a graded ring $R = \bigoplus_{m\geq 0} R_m$ and $k > 0$, the truncated ring $R^{(k)}$ is defined by $R^{(k)} = \bigoplus_{m\geq 0} R_{km}$. Then R is finitely generated if and only if so is $R^{(k)}$. We note that $\mathrm{Proj}R^{(k)} = \mathrm{Proj}R$ by the Veronese embedding.

The following argument is more or less well-known to the experts. See [YPG, §2], [KMM, Theorem 0-3-12], and [M, §4].

Proof. Since the problem is local, we can shrink Z and assume that Z is affine. By compactifying X and Z and by the desingularization theorem, we can further assume that X and Z are projective, X is non-singular, Δ is effective, and $\mathrm{Supp}\Delta$ is a simple normal crossing divisor. Let A be a very ample divisor on Z and $H \in |2rA|$ a general member for $r \gg 0$. We have a double cover $\pi : \widetilde{X} \to X$ ramifying along f^*H. Then $K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta + \frac{1}{2}f^*H)$. Note that $K_X + \Delta + (r-1)f^*A$ is big for $r \gg 0$ (cf. [KMM, Corollary 0-3-4]). Let m_0 be a positive integer such that $m_0(K_X + \Delta + \frac{1}{2}f^*H)$ is Cartier. By $(\mathbf{F_n}, \mathbf{klt})$ (resp. $(\mathbf{F_n}, \mathbf{lc})$), $\bigoplus_{m\geq 0} H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mm_0(K_{\widetilde{X}} + \widetilde{\Delta})))$ is finitely generated. Then

$$\begin{aligned}&(\bigoplus_{m\geq 0} H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mm_0(K_{\widetilde{X}} + \widetilde{\Delta}))))^G \\ &= \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mm_0(K_X + \Delta + \frac{1}{2}f^*H)))\end{aligned}$$

is finitely generated, where $G = \mathrm{Gal}(\widetilde{X}/X)$ is the Galois group. Thus, the relative log canonical model X' over Z exists. Indeed, by assuming that m_0 is sufficient large and divisible, $R(X, K_X + \Delta + \frac{1}{2}f^*H)^{(m_0)}$ is generated by $R(X, K_X+\Delta+\frac{1}{2}f^*H)_{m_0}$ and $|m_0(K_X+\Delta+(r-1)f^*A)| \neq \emptyset$. Then $X' = \mathrm{Proj}\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mm_0(K_X + \Delta + \frac{1}{2}f^*H)))$ and X' is the closure of the image of X by the rational map defined by the complete linear system $|m_0(K_X + \Delta + rf^*A)|$. More precisely, let $g : X'' \to X'$ be the elimination of the indeteminacy of the rational map defined by $|m_0(K_X + \Delta + rf^*A)|$. Let $g' : X'' \to X'$ be the induced morphism and $h : X'' \to Z$ the morphism defined by the complete linear system $|m_0 g^* f^* A|$. Then it is not difficult to see that h factors through X'. Therefore, $\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mm_0(K_X+\Delta))$ is a finitely generated $\mathcal{O}_S$-algebra. We finish the proof. □

Remark 4. If f is birational, then K_X (resp. $K_X + \Delta$) is always f-big. So, $(\mathbf{F_n})$ (resp. $(\mathbf{F_n}, \mathbf{klt})$ or $(\mathbf{F_n}, \mathbf{lc})$) implies the existence of n-dimensional flips (resp. klt or lc log flips). Therefore, the finitely generatedness of (log) canonical rings is much more difficult than the (log) flip conjecture I.

References

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