LECTURES ON THE LOG MINIMAL MODEL PROGRAM

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ABSTRACT. We explain the fundamental theorems for the log minimal model program for log canonical pairs.

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1. INTRODUCTION

The aim of this article is to explain the fundamental theorems for the log minimal model program for log canonical pairs. More explicitly, we describe the base point free theorem (see Corollary 4.2) and the rationality theorem (see Theorem 5.1) for log canonical pairs. We note that the cone and contraction theorems (see Theorem 5.4) for log canonical pairs are direct consequences of them.

**Theorem 1.1** (Cone and contraction theorems). Let $(X, B)$ be a projective log canonical pair. Then we have

(i) There are (countably many) integral curves $C_i \subset X$ such that

$$\overline{NE}(X) = \overline{NE}(X)(K_X + B) + \sum \mathbb{R}_{\geq 0}[C_i].$$
For any \( \varepsilon > 0 \) and ample \( \mathbb{Q} \)-divisor \( H \),
\[
\overline{NE}(X) = \overline{NE}(X)_{(K_X + B + \varepsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C].
\]

(ii) Let \( F \subset \overline{NE}(X) \) be a \((K_X + B)\)-negative extremal face. Then there is a unique morphism \( \varphi_F : X \to Z \) such that \((\varphi_F)_*, \mathcal{O}_X \simeq \mathcal{O}_Z\), \( Z \) is projective, and an irreducible curve \( C \subset X \) is mapped to a point by \( \varphi_F \) if and only if \([C] \in F\). The map \( \varphi_F \) is called the contraction of \( F \). Let \( L \) be a line bundle on \( X \) such that \((L \cdot C) = 0\) for every curve \( C \) with \([C] \in F\). Then there is a line bundle \( L_Z \) on \( Z \) such that \( L \simeq \varphi_F^* L_Z \).

This article contains no new statements. However, it must be valuable because there are no good references. We basically follow Ambro’s arguments (see [A, Section 5]) but we change them slightly to clarify the basic ideas and to remove some ambiguities and small mistakes. Our proofs in this paper heavily depend on the results obtained in [F1] because Ambro’s proofs in [A, Section 3] contain various serious gaps. For the details, see [F1]. Note that we only use \( \mathbb{Q} \)-divisors for simplicity. Some of the results can be generalized for \( \mathbb{R} \)-divisors with a little care. We do not treat the relative versions of the fundamental theorems in order to make our arguments transparent. There are no difficulties for the reader to obtain the relative versions once he understands this paper. We hope that this article will make the arguments in [A] clear. This paper will not depreciate but improve [A]. Note that the reader does not have to refer [A] in order to read this article.

We summarize the contents of this paper. In Section 2, we quickly review the torsion-freeness and the vanishing theorem in [F1]. In Section 3, we introduce the notion of \( qlc \) pairs, which is a special case of Ambro’s \textit{quasi-log varieties}, and prove some important and useful lemmas. Section 4 is devoted to the proof of the base point free theorem for \( qlc \) pairs. In Section 5, we give a proof to the rationality theorem for \( lc \) pairs. We note that the rationality theorem directly implies the cone theorem. In the final section: Section 6, we explain some related topics.

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1.1. Notation and Conventions. We will work over the complex number field \( \mathbb{C} \) throughout this paper. But we note that by using the Lefschetz principle, we can extend everything to the case where the
base field is an algebraically closed field of characteristic zero. We will use the following notation and the notation in [KM] freely.

**Notation.** (i) For a $\mathbb{Q}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that $D_i \neq D_j$ for $i \neq j$, we define the round-up $\lceil D \rceil = \sum_{j=1}^{r} \lceil d_j \rceil D_j$ (resp. the round-down $\lfloor D \rfloor = \sum_{j=1}^{r} \lfloor d_j \rfloor D_j$), where for any rational number $x$, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lfloor x \rfloor < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The fractional part $\{D\}$ of $D$ denotes $D - \lfloor D \rfloor$. We define

$$D^{=1} = \sum_{d_j = 1} D_j, \quad D^{<1} = \sum_{d_j < 1} d_j D_j.$$ 

We call $D$ a boundary (resp. subboundary) $\mathbb{Q}$-divisor if $0 \leq d_j \leq 1$ (resp. $d_j \leq 1$) for any $j$. $\mathbb{Q}$-linear equivalence of two $\mathbb{Q}$-divisors $B_1$ and $B_2$ is denoted by $B_1 \sim_{\mathbb{Q}} B_2$.

(ii) For a proper birational morphism $f : X \to Y$, the exceptional locus $\text{Exc}(f) \subset X$ is the locus where $f$ is not an isomorphism.

(iii) Let $X$ be a normal variety and let $B$ be an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution such that $\text{Exc}(f) \cup f^{-1}_* B$ has a simple normal crossing support, where $f^{-1}_* B$ is the strict transform of $B$ on $Y$. We write $K_Y = f^*(K_X + B) + \sum a_i E_i$ and $a(E_i, X, B) = a_i$. We say that $(X, B)$ is lc (resp. klt) if and only if $a_i \geq -1$ (resp. $a_i > -1$) for any $i$. Here, lc (resp. klt) is an abbreviation of log canonical (resp. Kawamata log terminal). Note that the discrepancy $a(E, X, B) \in \mathbb{Q}$ can be defined for any prime divisor $E$ over $X$. Let $(X, B)$ be an lc pair. If $E$ is a prime divisor over $X$ such that $a(E, X, B) = -1$, then the center $c_X(E)$ is called an lc center of $(X, B)$.

## 2. Vanishing and Torsion-Free Theorems

In this section, we quickly review the results in [F1]. First, we fix the notation and the conventions to state Theorem 2.2.

### 2.1 (Global embedded simple normal crossing pairs).

Let $Y$ be a simple normal crossing divisor on a smooth variety $M$ and let $D$ be a $\mathbb{Q}$-divisor on $M$ such that $\text{Supp}(D + Y)$ is simple normal crossing and that $D$ and $Y$ have no common irreducible components. We put $B = D|_Y$ and consider the pair $(Y, B)$. We call $(Y, B)$ a global embedded simple normal crossing pair. Let $\nu : Y^\nu \to Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B)$. A stratum of $(Y, B)$ is an irreducible component of $Y$ or the image of some lc center of $(Y^\nu, \Theta^{=1})$. When $Y$ is smooth and $B$ is a $\mathbb{Q}$-divisor on $Y$ such that $\text{Supp}B$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$ and $D = B \times \mathbb{A}^1$. Then $(Y, B) \simeq (Y \times \{0\}, B \times \{0\})$.
satisfies the above conditions, that is, we can consider \((Y, B)\) to be a global embedded simple normal crossing pair.

The following theorem is a special case of the main result in [F1].

**Theorem 2.2** (Torsion-freeness and vanishing theorem). Let \((Y, B)\) be as above. Assume that \(B\) is a boundary \(\mathbb{Q}\)-divisor. Let \(f : Y \to X\) be a proper morphism and \(L\) a Cartier divisor on \(Y\).

1. Assume that \(H \sim Q L - (K_Y + B)\) is \(f\)-semi-ample. Then every non-zero local section of \(R^q f_* \mathcal{O}_Y(L)\) contains in its support the \(f\)-image of some strata of \((Y, B)\).

2. Assume that \(X\) is projective and \(H \sim Q f^* H'\) for some ample \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(H'\) on \(X\). Then \(H^p(S, R^q f_* \mathcal{O}_Y(L)) = 0\) for any \(p > 0\).

For the proof, which depends on the theory of mixed Hodge structures, see [F1].

3. **Adjunction for qlc varieties**

To carry out \(X\)-method on log canonical pairs, it is better to introduce the notion of *qlc varieties*.

**Definition 3.1** (Qlc varieties). A *qlc variety* is a variety \(X\) with a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(\omega\), and a finite collection \(\{C\}\) of reduced and irreducible subvarieties of \(X\) such that there is a proper morphism \(f : (Y, B_Y) \to X\) from a global embedded simple normal crossing pair as in 2.1 satisfying the following properties:

1. \(f^* \omega \sim Q K_Y + B_Y\) such that \(B_Y\) is a subboundary \(\mathbb{Q}\)-divisor.
2. There is an isomorphism
   \[\mathcal{O}_X \simeq f_* \mathcal{O}_Y \left( \frac{\tau}{(B_Y^{\leq 1})} \right).\]
3. The collection of subvarieties \(\{C\}\) coincides with the image of \((Y, B_Y)\)-strata.

We use the following terminology. The subvarieties \(C\) are the *qlc centers* of \(X\), and \(f : (Y, B_Y) \to X\) is a quasi-log resolution of \(X\). We sometimes simply say that \([X, \omega]\) is a *qlc pair*, or the pair \([X, \omega]\) is *qlc*.

**Remark 3.2.** By the condition (2), we have an isomorphism \(\mathcal{O}_X \simeq f_* \mathcal{O}_Y\). In particular, \(f\) is surjective.

**Proposition 3.3.** Let \((X, B)\) be an lc pair. Then \([X, K_X + B]\) is a qlc pair.
Proof. Let \( f : Y \to X \) be a resolution such that \( K_Y + B_Y = f^*(K_X + B) \) and that \( \text{Supp} B_Y \) is a simple normal crossing divisor. Then \( \mathcal{O}_X \simeq f_* \mathcal{O}_Y (\tau - (B_Y^\leq 1) \gamma) \) because \( \tau - (B_Y^\leq 1) \gamma \) is effective and \( f \)-exceptional. We note that a qlc center \( C \) is \( X \) itself or an lc center of \((X, B)\). \( \square \)

The next lemma is very important and useful to treat qlc pairs.

**Lemma 3.4.** Let \( f : Z \to Y \) be a proper birational morphism between smooth varieties and let \( B_Y \) be a subboundary \( \mathbb{Q} \)-divisor on \( Y \) such that \( \text{Supp} B_Y \) is simple normal crossing. Assume that \( K_Z + B_Z = f^*(K_Y + B_Y) \) and that \( \text{Supp} B_Z \) is simple normal crossing. Then we have

\[
f_* \mathcal{O}_Z (\tau - (B_Z^\leq 1) \gamma) \simeq \mathcal{O}_Y (\tau - (B_Y^\leq 1) \gamma).
\]

Furthermore, let \( S \) be a simple normal crossing divisor on \( Y \) such that \( S \subset \text{Supp} B_Y^\leq 1 \). Let \( T \) be the union of the irreducible components of \( B_Y^\leq 1 \) that are mapped into \( S \) by \( f \). Assume that \( \text{Supp} f_*^{-1} B_Y \cup \text{Exc}(f) \) is simple normal crossing on \( Z \). Then we have

\[
f_* \mathcal{O}_T (\tau - (B_T^\leq 1) \gamma) \simeq \mathcal{O}_S (\tau - (B_S^\leq 1) \gamma),
\]

where \( (K_Z + B_Z)|_T = K_T + B_T \) and \( (K_Y + B_Y)|_S = K_S + B_S \).

**Proof.** By \( K_Z + B_Z = f^*(K_Y + B_Y) \), we obtain

\[
K_Z = f^*(K_Y + B_Y^\leq 1 + \{B_Y\}) + f^*(\cup B_Y^{\leq 1, \cup}) - \cup B_Z^{\leq 1, \cup} - B_Z^\leq 1 - \{B_Z\}.
\]

If \( a(\nu, Y, B_Y^\leq 1 + \{B_Y\}) = -1 \) for a prime divisor \( \nu \) over \( Y \), then we can check that \( a(\nu, Y, B_Y) = -1 \) by using [KM, Lemma 2.45]. Since \( f^*(\cup B_Y^{\leq 1, \cup}) - \cup B_Z^{\leq 1, \cup} \) is Cartier, we can easily see that \( f^*(\cup B_Y^{\leq 1, \cup}) = \cup B_Z^{\leq 1, \cup} + E \), where \( E \) is an effective \( f \)-exceptional divisor. Thus, we obtain

\[
f_* \mathcal{O}_Z (\tau - (B_Z^\leq 1) \gamma) \simeq \mathcal{O}_Y (\tau - (B_Y^\leq 1) \gamma).
\]

Next, we consider

\[
0 \to \mathcal{O}_Z (\tau - (B_Z^\leq 1) \gamma - T) \to \mathcal{O}_Z (\tau - (B_Z^\leq 1) \gamma) \to \mathcal{O}_T (\tau - (B_T^\leq 1) \gamma) \to 0.
\]

Since \( T = f^* S - F \), where \( F \) is an effective \( f \)-exceptional divisor, we can easily see that

\[
f_* \mathcal{O}_Z (\tau - (B_Z^\leq 1) \gamma - T) \simeq \mathcal{O}_Y (\tau - (B_Y^\leq 1) \gamma - S).
\]

We note that

\[
(\tau - (B_Z^\leq 1) \gamma - T) - (K_Z + \{B_Z\} + (B_Z^\leq 1 - T)) = - f^*(K_Y + B_Y).
\]
Therefore, every local section of $R^1f_*\mathcal{O}_Z(\gamma - (B_T^{\leq 1})^\gamma - T)$ contains in its support the $f$-image of some strata of $(Z, \{B_Z\} + B_T^{\leq 1} - T)$ by Theorem 2.2 (1).

**Claim.** No strata of $(Z, \{B_Z\} + B_T^{\leq 1} - T)$ are mapped into $S$ by $f$.

**Proof of Claim.** Assume that there is a stratum $C$ of $(Z, \{B_Z\} + B_T^{\leq 1} - T)$ such that $f(C) \subset S$. Note that $\text{Supp} f^*S \subset \text{Supp} f^{-1}_*B_Y \cup \text{Exc}(f)$ and $\text{Supp} B_T^{\leq 1} \subset \text{Supp} f^{-1}_*B_Y \cup \text{Exc}(f)$. Since $C$ is also a stratum of $(Z, B_T^{\leq 1})$ and $C \subset \text{Supp} f^*S$, there exists an irreducible component $G$ of $B_T^{\leq 1}$ such that $C \subset G \subset \text{Supp} f^*S$. Therefore, by the definition of $T$, $G$ is an irreducible component of $T$ because $f(G) \subset S$ and $G$ is an irreducible component of $B_T^{\leq 1}$. So, $C$ is not a stratum of $(Z, \{B_Z\} + B_T^{\leq 1} - T)$. It is a contradiction.

On the other hand, $f(T) \subset S$. Therefore,

$$f_*\mathcal{O}_T(\gamma - (B_T^{\leq 1})^\gamma) \to R^1f_*\mathcal{O}_S(\gamma - (B_T^{\leq 1})^\gamma - T)$$

is a zero map by the assumption on the strata of $(Z, B_T^{\leq 1} - T)$. Thus,

$$f_*\mathcal{O}_T(\gamma - (B_T^{\leq 1})^\gamma) \simeq \mathcal{O}_S(\gamma - (B_S^{\leq 1})^\gamma).$$

We finish the proof. 

The following theorem is one of the key results for the theory of qlc varieties. It is a consequence of Theorem 2.2.

**Theorem 3.5 (Adjunction and vanishing theorem).** Let $[X, \omega]$ be a qlc pair and let $X'$ be a union of some qlc centers of $[X, \omega]$.

(i) Then $[X', \omega']$ is a qlc pair, where $\omega' = \omega|_{X'}$. Moreover, the qlc centers of $[X', \omega']$ are exactly the qlc centers of $[X, \omega]$ that are included in $X'$.

(ii) Assume that $X$ is projective. Let $L$ be a Cartier divisor on $X$ such that $L - \omega$ is ample. Then $H^q(X, \mathcal{O}_X(L)) = 0$ and $H^q(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for $q > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of $X'$ on $X$. Note that $H^q(X', \mathcal{O}_{X'}(L)) = 0$ for any $q > 0$ because $[X', \omega']$ is a qlc pair by (i) and $L|_{X'} - \omega'$ is ample.

**Proof.** (i) Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Let $M$ be the ambient space of $Y$ and $D$ a subboundary $\mathbb{Q}$-divisor on $M$ such that $B_Y = D|_Y$. By taking blow-ups of $M$, we can assume that the union of all strata of $(Y, B_Y)$ mapped into $X'$, which is denoted by $Y'$, is a union of irreducible components of $Y$ (cf. Lemma 3.4). We put $Y'' = Y - Y'$. We define $(K_Y + B_Y)|_{Y'} = K_{Y'} + B_{Y'}$ and consider $f : (Y', B_{Y'}) \to X'$. We claim that $[X', \omega']$ is a qlc pair, where $\omega' = \omega|_{X'}$, and $f : (Y', B_{Y'}) \to X'$ is a quasi-log resolution. By the
Applying \( f \), we put the support of any non-zero local section of \( \mathcal{O}_{Y'}(-Y') \) on \( Y' \). We consider the following short exact sequence

\[
0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.
\]

We put \( A = \mathcal{O}_{Y''}(-Y') \). Then we have

\[
0 \to \mathcal{O}_{Y''}(A - Y') \to \mathcal{O}_Y(A) \to \mathcal{O}_{Y'}(A) \to 0.
\]

Applying \( f_* \), we obtain

\[
0 \to f_* \mathcal{O}_{Y''}(A - Y') \to \mathcal{O}_X \to f_* \mathcal{O}_{Y'}(A) \to R^1 f_* \mathcal{O}_{Y''}(A - Y') \to \cdots.
\]

The support of any non-zero local section of \( R^1 f_* \mathcal{O}_{Y''}(A - Y') \) can not be contained in \( f(Y') = X' \). Therefore, \( f_* \mathcal{O}_{Y'}(A) \to R^1 f_* \mathcal{O}_{Y''}(A - Y') \) is a zero-map. It is easy to see that \( f_* \mathcal{O}_{Y''}(A - Y') \cong \mathcal{I}_X \), the defining ideal sheaf of \( X' \) on \( X \). Therefore, we obtain that \( \mathcal{O}_X \cong f_* \mathcal{O}_{Y'}(A) \).

The statement for qlc centers is obvious by the construction of the quasi-log resolution. So, we finish the proof of (i).

(ii) Let \( f : (Y, B_Y) \to X \) be a quasi-log resolution as in the proof of (i). Apply Theorem 2.2 (2). Then we obtain \( H^q(X, \mathcal{O}_X(L)) = 0 \) for any \( q > 0 \) because

\[
f^*(L - \omega) \sim_{\mathbb{Q}} f^* L - (K_Y + B_Y)
\]

and \( f_* \mathcal{O}_Y(f^* L + \mathcal{O}_X(L)) \sim \mathcal{O}_X(L) \). We consider \( f : Y'' \to X \). We put \( (K_Y + B_Y)|_{Y''} = K_{Y''} + B_{Y''} \). Then \( f^*(L - \omega) \sim_{\mathbb{Q}} (f^* L - (K_Y + B_Y))|_{Y''} = (f^* L + A - Y')|_{Y''} - (K_{Y''} + B_{Y''} + B_Y - Y'|_{Y''}) \) on \( Y'' \). Note that \( Y'|_{Y''} \) is contained in \( B_{Y''} \). Therefore, \( H^q(X, f_* \mathcal{O}_{Y''}(A - Y') \otimes \mathcal{O}_X(L)) = 0 \) for any \( q > 0 \) by Theorem 2.2 (2). Thus we finish the proof because \( f_* \mathcal{O}_{Y''}(A - Y') \cong \mathcal{I}_X \). □

**Corollary 3.6.** Let \( [X, \omega] \) be a qlc pair and let \( X' \) be an irreducible component of \( X \). Then \( [X', \omega'] \), where \( \omega' = \omega|_{X'} \), is a qlc pair.

**Proof.** It is because \( X' \) is a qlc center of \( [X, \omega] \) by Remark 3.2. □

We use the next definition in Section 4.

**Definition 3.7.** Let \( [X, \omega] \) be a qlc pair. Let \( X' \) be the union of qlc centers of \( X \) that are not mapped onto any irreducible components of \( X \). Then \( X' \) with \( \omega' = \omega|_{X'} \) is a qlc variety by Theorem 3.5 (i). We denote it by \( \text{Nqklt}(X, \omega) \).

We close this section with the following very useful lemma, which seems to be indispensable for the proofs of the base point free theorem and the rationality theorem.
Lemma 3.8. Let \( f : (Y, B_Y) \to X \) be a quasi-log resolution of a qlc pair \( [X, \omega] \). Let \( E \) be a Cartier divisor on \( X \) such that \( \text{Supp} E \) contains no qlc centers of \( [X, \omega] \). By blowing up \( M \), the ambient space of \( Y \), inside \( \text{Supp} f^*E \), we can assume that \((Y, B_Y + f^*E)\) is a global embedded simple normal crossing pair.

**Proof.** First, we take a blow-up of \( M \) along \( f^*E \) and apply Hironaka’s resolution theorem to \( M \). Then we can assume that there exists a Cartier divisor \( F \) on \( M \) such that \( \text{Supp}(F \setminus Y) = \text{Supp} f^*E \). Next, we apply Szabó’s resolution lemma to \( \text{Supp}(D + Y + F) \) on \( M \). Thus, we obtain the desired properties by Lemma 3.4.

\[ \square \]

### 4. Base point free theorem

The next theorem is the main theorem of this section. It is a special case of [A, Theorem 5.1]. This formulation is useful for the inductive treatment of log canonical pairs.

**Theorem 4.1.** Let \([X, \omega]\) be a projective qlc pair and let \( L \) be a nef Cartier divisor on \( X \). Assume that \( qL - \omega \) is ample for some \( q > 0 \). Then \( \mathcal{O}_X(mL) \) is generated by global sections for \( m \gg 0 \), that is, \( \mathcal{O}_X(mL) \) is generated by global sections for any \( m \geq m_0 \).

**Proof.** First, we note that the statement is obvious when \( \dim X = 0 \).

**Claim 1.** We can assume that \( X \) is irreducible.

Let \( X' \) be an irreducible component of \( X \). Then \( X' \) with \( \omega' = \omega|_{X'} \) has a natural qlc structure induced by \([X, \omega]\) by adjunction (see Corollary 3.6). By the vanishing theorem (see Theorem 3.5 (ii)), we have \( H^1(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0 \) for any \( m \geq q \). We consider the following commutative diagram.

\[
\begin{array}{ccc}
H^0(X, \mathcal{O}_X(mL)) \otimes \mathcal{O}_X & \xrightarrow{\alpha} & H^0(X', \mathcal{O}_{X'}(mL)) \otimes \mathcal{O}_{X'} \\
\downarrow & & \downarrow \\
\mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) \\
\end{array}
\]

Since \( \alpha \) is surjective for \( m \geq q \), we can assume that \( X \) is irreducible when we prove this theorem.

**Claim 2.** \( \mathcal{O}_X(mL) \) is generated by global sections around \( \text{Nqklt}(X, \omega) \) for \( m \gg 0 \).

We put \( X' = \text{Nqklt}(X, \omega) \). Then \([X', \omega']\), where \( \omega' = \omega|_{X'} \), is a qlc pair by adjunction (see Theorem 3.5 (i)). By the induction on the
dimension, $\mathcal{O}_{X'}(mL)$ is generated by global sections for $m \gg 0$. By the following commutative diagram:

$$
\begin{array}{ccc}
H^0(X, \mathcal{O}_X(mL)) \otimes \mathcal{O}_X & \xrightarrow{\alpha} & H^0(X', \mathcal{O}_{X'}(mL)) \otimes \mathcal{O}_{X'} \\
\downarrow & & \downarrow \\
\mathcal{O}_X(mL) & \xrightarrow{\alpha} & \mathcal{O}_{X'}(mL)
\end{array}
$$

we know that $\mathcal{O}_X(mL)$ is generated by global sections around $X'$ for $m \gg 0$.

**Claim 3.** $\mathcal{O}_X(mL)$ is generated by global sections on a non-empty Zariski open set for $m \gg 0$.

By Claim 2, we can assume that Nqklt$(X, \omega)$ is empty. If $L$ is numerically trivial, then $H^0(X, \mathcal{O}_X(L)) = H^0(X, \mathcal{O}_X(-L)) = \mathbb{C}$. It is because $h^0(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X) = 1$ by Theorem 3.5 (ii). Therefore, $\mathcal{O}_X(L)$ is trivial. So, we can assume that $L$ is not numerically trivial. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Let $x \in X$ be a general smooth point. Then we can take a $\mathbb{Q}$-divisor $D$ such that $\text{mult}_x D > \dim X$ and that $D \sim_{\mathbb{Q}} (q + r)L - \omega$ for some $r > 0$ (see [KM, 3.5 Step 2]). By blowing up $M$, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair by Lemma 3.8. We note that any stratum of $(Y, B_Y)$ is mapped onto $X$ by the assumption. By the construction of $D$, we can find a positive rational number $c < 1$ such that $B_Y + cf^*D$ is a subboundary and some stratum of $(Y, B_Y + cf^*D)$ does not dominate $X$. Then the pair $[X, \omega + cD]$ is qlc and $f : (Y, B_Y + cf^*D) \to X$ is a quasi-log resolution. We note that $q'L - (\omega + cD)$ is ample, where $q' = q + cr$. By the construction, Nqklt$(X, \omega + cD)$ is non-empty. Therefore, by applying Claim 2 to $[X, \omega + cD]$, $\mathcal{O}_X(mL)$ is generated by global sections around Nqklt$(X, \omega + cD)$ for $m \gg 0$. So, we finish the proof of Claim 3.

Let $p$ be a prime number and let $l$ be a large integer. Then $|p^lL| \neq \emptyset$ by Claim 3 and $|p^lL|$ is free around Nqklt$(X, \omega)$ by Claim 2.

**Claim 4.** If the base locus $\text{Bs}|p^lL|$ (with reduced scheme structure) is not empty, then $\text{Bs}|p^lL|$ is not contained in $\text{Bs}|p^{l'}L|$ for $l' \gg l$.

Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. We take a general member $D \in |p^lL|$. We note that $|p^lL|$ is free around the Nqklt$(X, \omega)$. Thus, $f^*D$ intersects any strata of $(Y, B_Y)$ transversally over $X \setminus \text{Bs}|p^lL|$ by Bertini. By taking blow-ups of $M$, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair. See the proof of Lemma 3.8. We take the maximal positive rational number $c$ such that $B_Y + cf^*D$ is a subboundary. Then $f : (Y, B_Y + cf^*D) \to X$ is a
quasi-log resolution of \([X, \omega'] = \omega + cD]\). Note that \([X, \omega']\) has a qlc center \(C\) that intersects \(B|p'L|\) by the construction. By the induction, \(\mathcal{O}_C(mL)\) is generated by global sections for \(m \gg 0\). We can lift the sections of \(\mathcal{O}_C(mL)\) to \(X\) for \(m \ge q + cp^l\) by Theorem 3.5 (ii). Then we obtain that \(\mathcal{O}_X(mL)\) is generated by global sections around \(C\) for \(m \gg 0\). Therefore, \(B|p'^lL|\) is strictly smaller than \(B|p'L|\) for \(l' \gg l\).

**Claim 5.** \(\mathcal{O}_X(mL)\) is generated by global sections for \(m \gg 0\).

By Claim 4 and the noetherian induction, \(\mathcal{O}_X(p'L)\) and \(\mathcal{O}_X(p''L)\) is generated by global sections for large \(l\) and \(l'\), where \(p\) and \(p'\) are prime numbers and they are relatively prime. So, \(\mathcal{O}_X(mL)\) is generated by global sections for any \(m \ge m_0\).

The next corollary is obvious by Theorem 4.1 and Proposition 3.3.

**Corollary 4.2** (Base point free theorem). Let \((X, B)\) be a projective lc pair and let \(L\) be a nef Cartier divisor on \(X\). Assume that \(qL - (K_X + B)\) is ample for some \(q > 0\). Then \(\mathcal{O}_X(mL)\) is generated by global sections for \(m \gg 0\).

5. Cone theorem

In this section, we treat the cone theorem for log canonical pairs.

5.1. **Rationality theorem.** Here, we prove the rationality theorem for log canonical pairs. It implies the cone theorem for log canonical pairs.

**Theorem 5.1** (Rationality theorem). Let \((X, B)\) be a projective lc pair such that \(a(K_X + B)\) is Cartier for a positive integer \(a\). Let \(H\) be an ample Cartier divisor on \(X\). Assume that \(K_X + B\) is not nef. We put

\[
r = \max\{t \in \mathbb{R} : H + t(K_X + B) \text{ is nef}\}.
\]

Then \(r\) is a rational number of the form \(u/v\) (\(u, v \in \mathbb{Z}\)) where \(0 < v < a(\dim X + 1)\).

Before we go to the proof, we recall the following lemmas.

**Lemma 5.2** (cf. [KM, Lemma 3.19]). Let \(P(x, y)\) be a non-trivial polynomial of degree \(\le n\) and assume that \(P\) vanishes for all sufficiently large integral solutions of \(0 < ay - rx < \varepsilon\) for some fixed positive integer \(a\) and positive \(\varepsilon\) for some \(r \in \mathbb{R}\). Then \(r\) is rational, and in reduced form, \(r\) has denominator \(\le a(n + 1)/\varepsilon\).

For the proof, see [KM, Lemma 3.19].
Lemma 5.3 (cf. [KM, 3.4 Step 2]). Let \([Y, \omega]\) be a projective qlc pair and let \(\{D_i\}\) be a finite collection of Cartier divisors. Consider the Hilbert polynomial

\[
P(u_1, \ldots, u_k) = \chi(Y, \mathcal{O}_Y(\sum_{i=1}^k u_i D_i)).
\]

Suppose that for some values of the \(u_i\), \(\sum_{i=1}^k u_i D_i\) is nef and \(\sum_{i=1}^k u_i D_i - \omega\) is ample. Then \(P(u_1, \ldots, u_k)\) is not identically zero by the base point free theorem for qlc pairs (see Theorem 4.1) and the vanishing theorem (see Theorem 3.5 (ii)), and its degree is \(\leq \dim Y\).

Note that the arguments in [KM, 3.4 Step 2] work for our setting.

Proof of Theorem 5.1. Without loss of generality, we can assume that \(H\) is very ample. We put \(\omega = K_X + B\) for simplicity. For each \((p, q) \in \mathbb{Z}^2\), let \(L(p, q)\) denote the base locus of the linear system \(|M(p, q)|\) on \(X\) (with reduced scheme structure), where \(M(p, q) = pH + qa\omega\). By the definition, \(L(p, q) = X\) if and only if \(|M(p, q)| = \emptyset\).

Claim 1 (cf. [KM, Claim 3.20]). For \((p, q)\) sufficiently large and \(0 < aq - rp < \varepsilon\), \(L(p, q)\) is the same subset of \(X\). We call this subset \(L_0\). We let \(I \subset \mathbb{Z} \times \mathbb{Z}\) be the set of \((p, q)\) for which \(0 < aq - rp < 1\) and \(L(p, q) = L_0\). We note that \(I\) contains all sufficiently large \((p, q)\) with \(0 < aq - rp < 1\).

For the proof, see [KM, Claim 3.20].

Claim 2. We assume that \(r\) is not rational or that \(r\) is rational and has denominator \(> a(n+1)/\varepsilon\) in reduced form, where \(n = \dim X\). For each qlc center \(C\) of \([X, \omega]\), there exists \((p, q)\) such that \((p, q)\) sufficiently large and \(0 < aq - rp < \varepsilon\) and that \(\mathcal{O}_C(M(p, q))\) is generated by global sections on some non-empty Zariski open set.

We note that \(M(p, q) - \omega = pH + (qa - 1)\omega\). If \(aq - rp < 1\), then \(M(p, q) - \omega\) is ample. Let \(C\) be a qlc center of \([X, \omega]\). Then \(P(p, q) = \chi(C, \mathcal{O}_C(M(p, q)))\) is a non-zero polynomial of degree at most \(\dim C \leq \dim X\) by Lemma 5.3. By Lemma 5.2, \(P(p, q) \neq 0\) for \(0 < aq - rp < \varepsilon\) with \((p, q)\) sufficiently large. By the ampleness of \(M(p, q) - \omega\), \(\chi(C, \mathcal{O}_C(M(p, q))) = h^0(C, \mathcal{O}_C(M(p, q)))\). So, we obtain Claim 2.

Note that \(\mathcal{O}_X(M(p, q))\) is not generated by global sections because \(M(p, q)\) is not nef. Therefore, \(L_0 \neq \emptyset\). Let \(D_1, \ldots, D_n\) be general members of \(|M(p_0, q_0)|\) with \((p_0, q_0) \in I\). Then \(K_X + B + \sum_{i=1}^n D_i\) is not lc at any generic points of any irreducible components of \(L_0\). Let
Let \( 0 < c < 1 \) be the maximal rational number such that \( K_X + B + c \sum_{i=1}^n D_i \) is lc. Thus, the lc pair \((X, B + c \sum_{i=1}^n D_i)\) has some lc centers contained in \( L_0 \). Let \( C \) be an lc center contained in \( L_0 \). We consider \( \omega' = \omega + c \sum_{i=1}^n D_i \sim Q \ cnq_0 H + (1 + cnq_0 a) \omega \). Thus we have \( p H + qa \omega - \omega' \sim Q \ (p - cnq_0) H + (qa - (1 + cnq_0 a)) \omega \). If \( p \) and \( q \) are large enough and \( aq - rp \leq aq_0 - rp_0 \), then \( p H + qa \omega - \omega' \) is ample. It is because

\[
(p - cnq_0) H + (qa - (1 + cnq_0 a)) \omega = (p - (1 + cn) p_0) H + (qa - (1 + cn) q_0 a) \omega + p_0 H + (q_0 a - 1) \omega.
\]

Suppose that \( r \) is not rational. There must be arbitrarily large \((p, q)\) such that \( 0 < aq - rp < \varepsilon = aq_0 - rp_0 \) and \( H^0(C, \mathcal{O}_C(M(p, q))) \neq 0 \) by applying Claim 2 to \([X, \omega']\). It is because \( M(p, q) - \omega' \) is ample by \( aq - rp < aq_0 - rp_0 \) and \( P(p, q) = \chi(C, \mathcal{O}_C(M(p, q))) \) is a non-trivial polynomial of degree at most \( \dim C \) by Lemma 5.3. By the vanishing theorem,

\[
H^0(X, \mathcal{O}_X(M(p, q))) \to H^0(C, \mathcal{O}_C(M(p, q)))
\]

is surjective by the ampleness of \( M(p, q) - \omega' \). Thus \( C \) is not contained in \( L(p, q) \). Therefore, \( L(p, q) \) is a proper subset of \( L(p_0, q_0) = L_0 \), giving the desired contradiction. So now we know that \( r \) is rational. We next suppose that the assertion of the theorem concerning the denominator of \( r \) is false. Choose \((p_0, q_0) \in I \) such that \( aq_0 - rp_0 \) is the maximum, say it is equal to \( d/v \). In this strip \( 0 < aq - rp \leq d/v \), \( \chi(C, \mathcal{O}_C(M(p, q))) = h^0(C, \mathcal{O}_C(M(p, q))) \) because \( M(p, q) - \omega' \) is ample. There exists \((p, q)\) in the strip \( 0 < aq - rp < 1 \) with \( \varepsilon = 1 \) for which \( h^0(C, \mathcal{O}_C(M(p, q))) \neq 0 \) by applying Claim 2 to \([X, \omega']\). Note that \( aq - rp \leq d/v = aq_0 - rp_0 \) holds automatically for \((p, q) \in I \). Since \( H^0(X, \mathcal{O}_X(M(p, q))) \to H^0(C, \mathcal{O}_C(M(p, q))) \) is surjective by the ampleness of \( M(p, q) - \omega' \), we obtain the desired contradiction by the same reason as above. So, we finish the proof. \( \square \)

### 5.2. Cone and contraction theorems

Let us state the main theorem of this section.

**Theorem 5.4** (Cone and contraction theorems). Let \((X, B)\) be a projective lc pair. Then we have

(i) There are (countably many) rational curves \( C_j \subset X \) such that \( 0 < -(K_X + B) \cdot C_j \leq 2 \dim X \), and

\[
\mathcal{NE}(X) = \overline{\mathcal{NE}(X)}_{(K_X+B)\geq 0} + \sum_{j \geq 0} \mathbb{R}_{\geq 0}[C_j].
\]
(ii) For any $\varepsilon > 0$ and ample $\mathbb{Q}$-divisor $H$,

\[
\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + B + \varepsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j].
\]

(iii) Let $F \subset \overline{\text{NE}}(X)$ be a $(K_X + B)$-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ such that $(\varphi_F)_* \mathcal{O}_X \simeq \mathcal{O}_Z$, $Z$ is projective, and an irreducible curve $C \subset X$ is mapped to a point by $\varphi_F$ if and only if $[C] \in F$. The map $\varphi_F$ is called the contraction of $F$.

(iv) Let $F$ and $\varphi_F$ be as in (iii). Let $L$ be a line bundle on $X$ such that $(L \cdot C) = 0$ for every curve $C$ with $[C] \in F$. Then there is a line bundle $L_Z$ on $Z$ such that $L \simeq \varphi_F^* L_Z$.

**Proof.** The estimate $\leq 2 \dim X$ in (i) can be proved by Kawamata’s argument in [K] with the aid of [BCHM]. For the detail, see [F2]. The other statements in (i) and (ii) are formal consequences of the rationality theorem. For the proof, see [KM, Theorem 3.15]. The statements (iii) and (iv) are obvious by Corollary 4.2 and the statements (i) and (ii). See Steps 7 and 9 in [KM, 3.3]. \qed

6. Related topics

In this paper, we did not prove Theorem 2.2, which is a key result for the study of lc pairs. For the proof, see [F1]. In [F2], we gave a proof of the existence of fourfold lc flips and proved the base point free theorem of Reid–Fukuda type for lc pairs. The base point free theorem for lc pairs was generalized in [F3], where we obtained Kollár’s effective base point free theorem for lc pairs. In [F4], we proved the effective base point free theorem of Angehrn–Siu type for lc pairs. Recently, we introduced the notion of non-lc ideal sheaves and proved the restriction theorem (see [F5]). It implies the inversion of adjunction on log canonicity for normal divisors. Note that everything depends on the results in [F1].

**References**


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